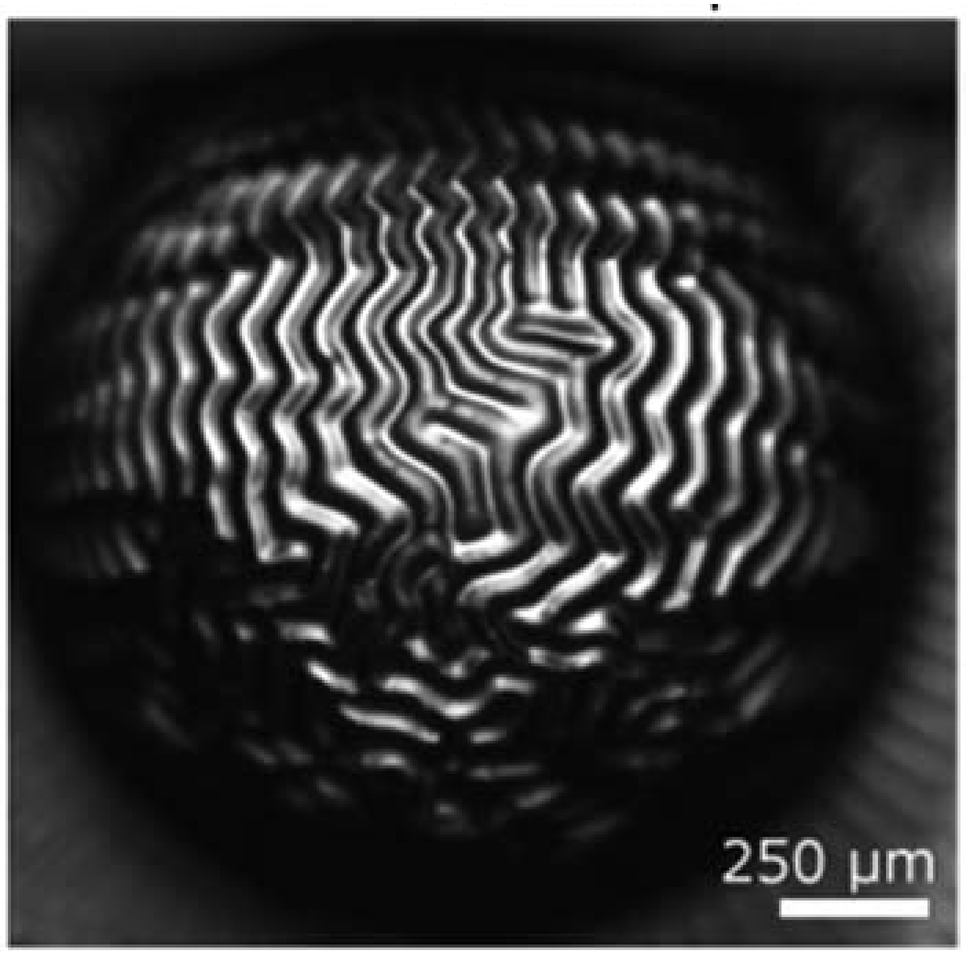
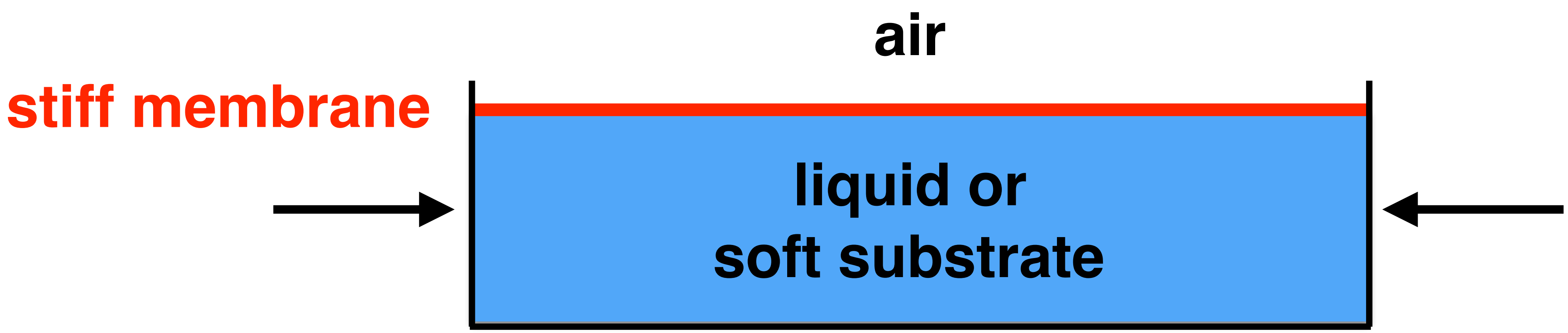
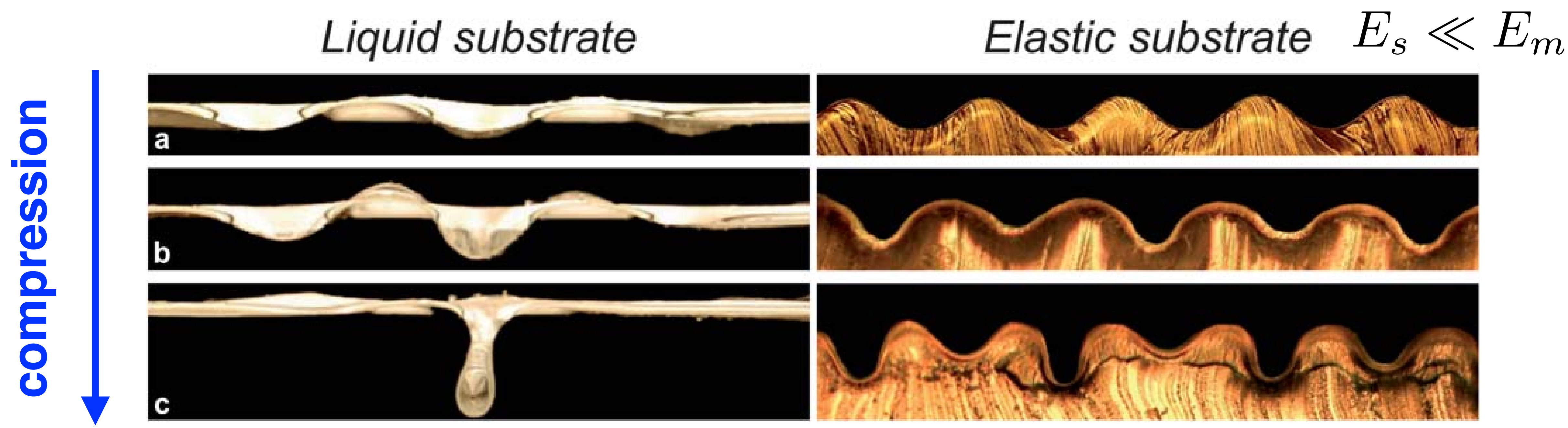
MAE 545: Lecture 19 (12/2)

Wrinkled surfaces



Compression of stiff thin membranes on liquid and soft elastic substrates





10 μm thin sheet of polyester on water

$\lambda_0 \sim 1.6\text{cm}$

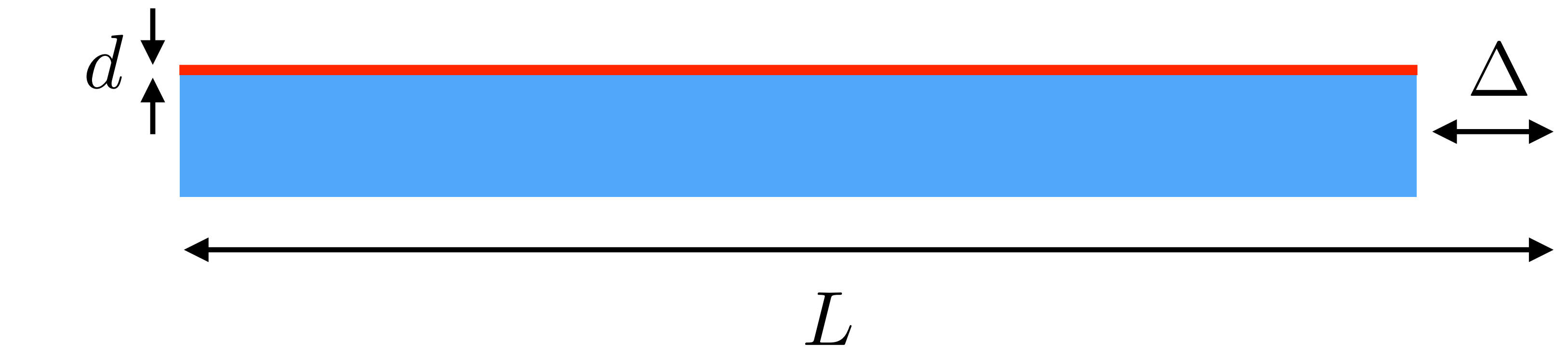
~10 μm thin PDMS (stiffer) sheet on PDMS (softer) substrate

$\lambda_0 \sim 70\mu\text{m}$

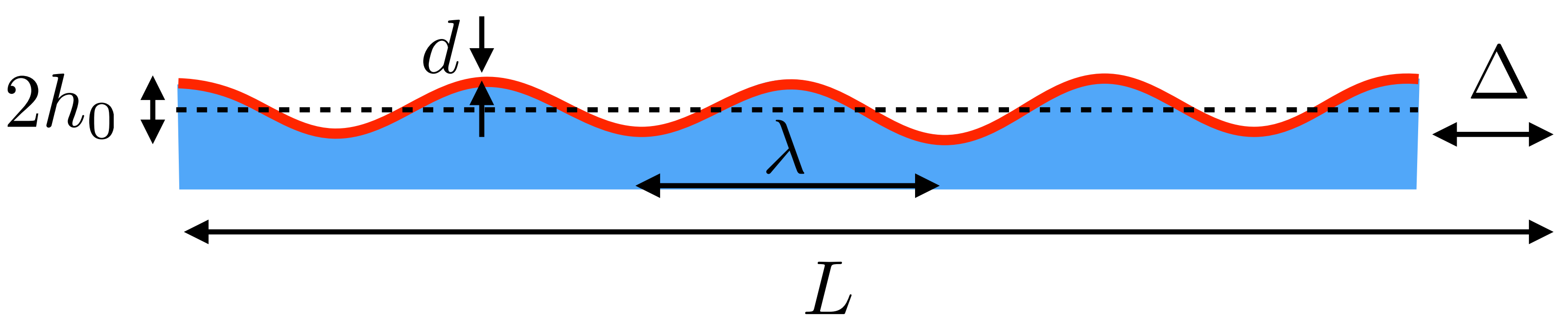
L. Pocivavsek et al., Science 320, 912 (2008)

F. Brau et al., Soft Matter 9, 8177 (2013)

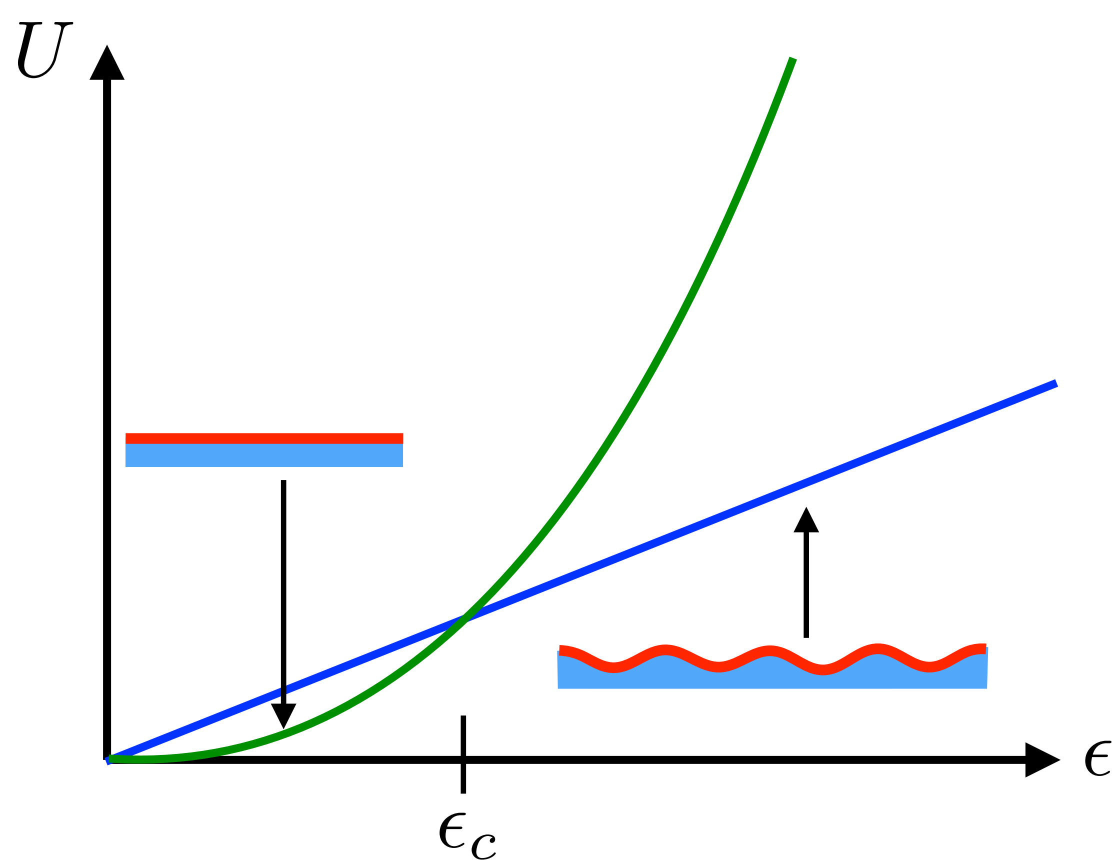
Compression of stiff thin membranes on liquid substrates



$$U_c \sim A \times E_m d \times \epsilon^2$$



$$U_b, U_p \sim A\epsilon\sqrt{E_m d^3 \rho g}$$



wrinkles are stable above the critical strain

$$\epsilon > \epsilon_c \sim \sqrt{\frac{\rho g d}{E_m}}$$

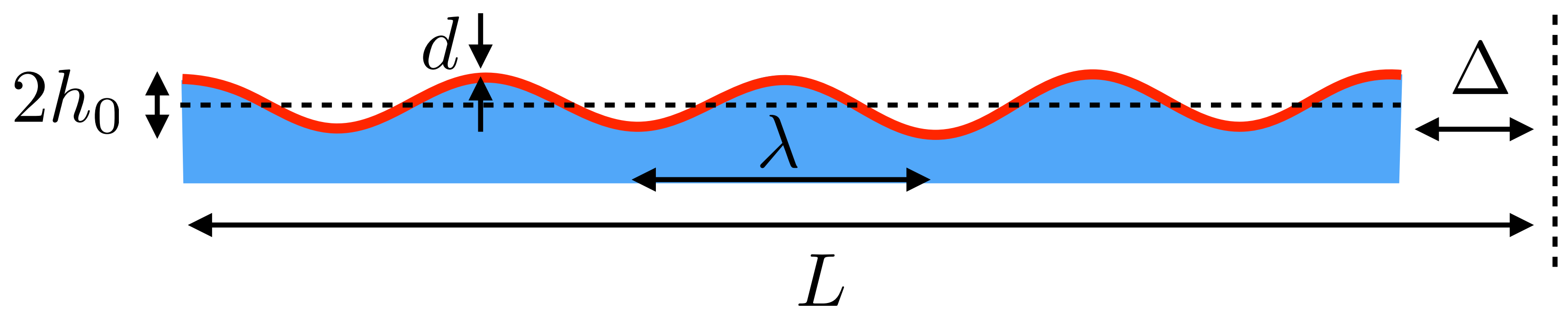
wavelength of wrinkles

$$\lambda \sim \left(\frac{E_m d^3}{\rho g}\right)^{1/4}$$

amplitude of wrinkles at the critical strain

$$h_0^* \sim \lambda\sqrt{\epsilon_c} \sim d$$

Compression of stiff thin membranes on liquid substrates

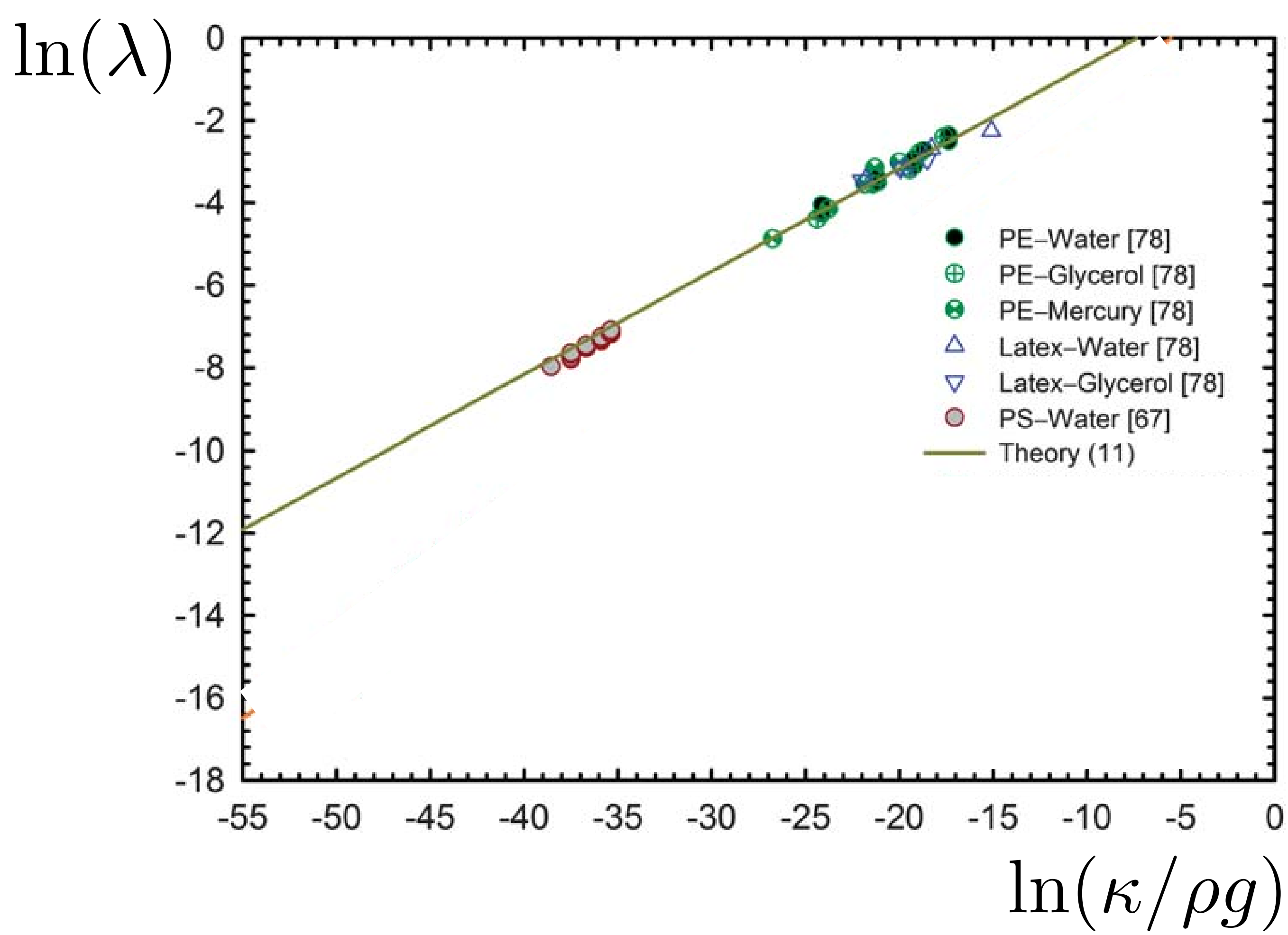


scaling analysis

$$\lambda \sim \left(\frac{E_m d^3}{\rho g}\right)^{1/4}$$

exact result

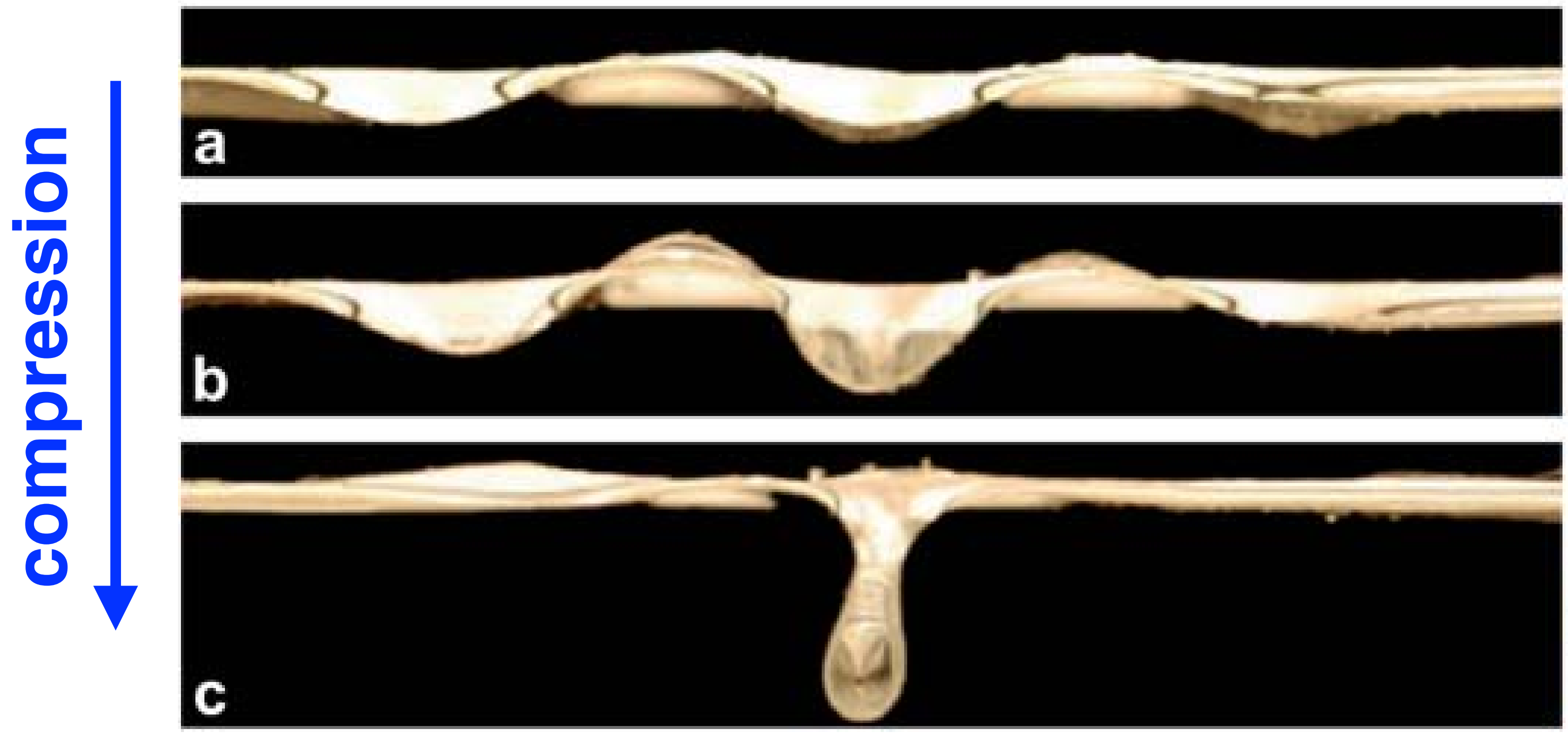
$$\lambda = 2\pi \left(\frac{\kappa}{\rho g}\right)^{1/4}$$



F. Brau et al., Soft Matter 9, 8177 (2013)

Compression of stiff thin membranes on liquid substrates

How to go beyond the simple scaling analysis to determine the nonlinear post-buckling behavior?



Find shape profile h(s) that minimizes total energy

$$U_b + U_p = W \int_0^L \frac{ds}{2} \left[\frac{\kappa h''^2}{(1+h'^2)^3} + \rho g h^2 \sqrt{1 - h'^2} \right]$$

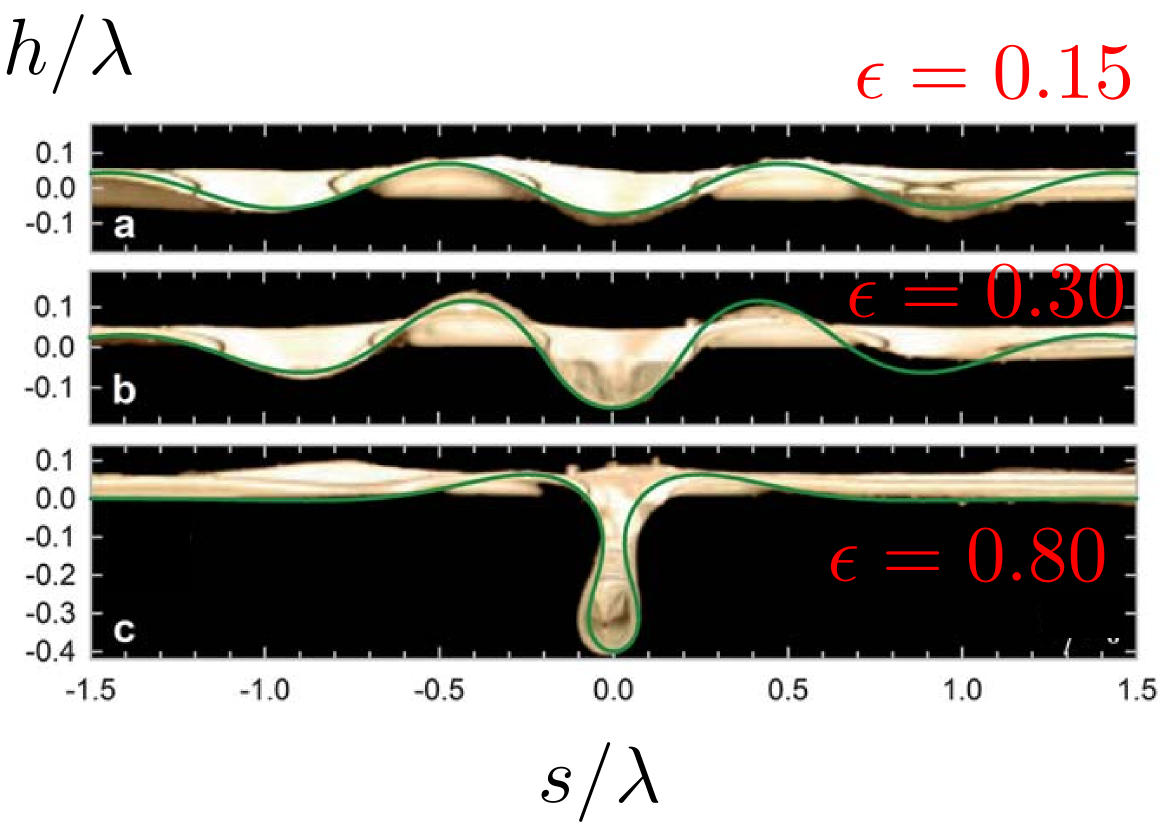
subject to constraint

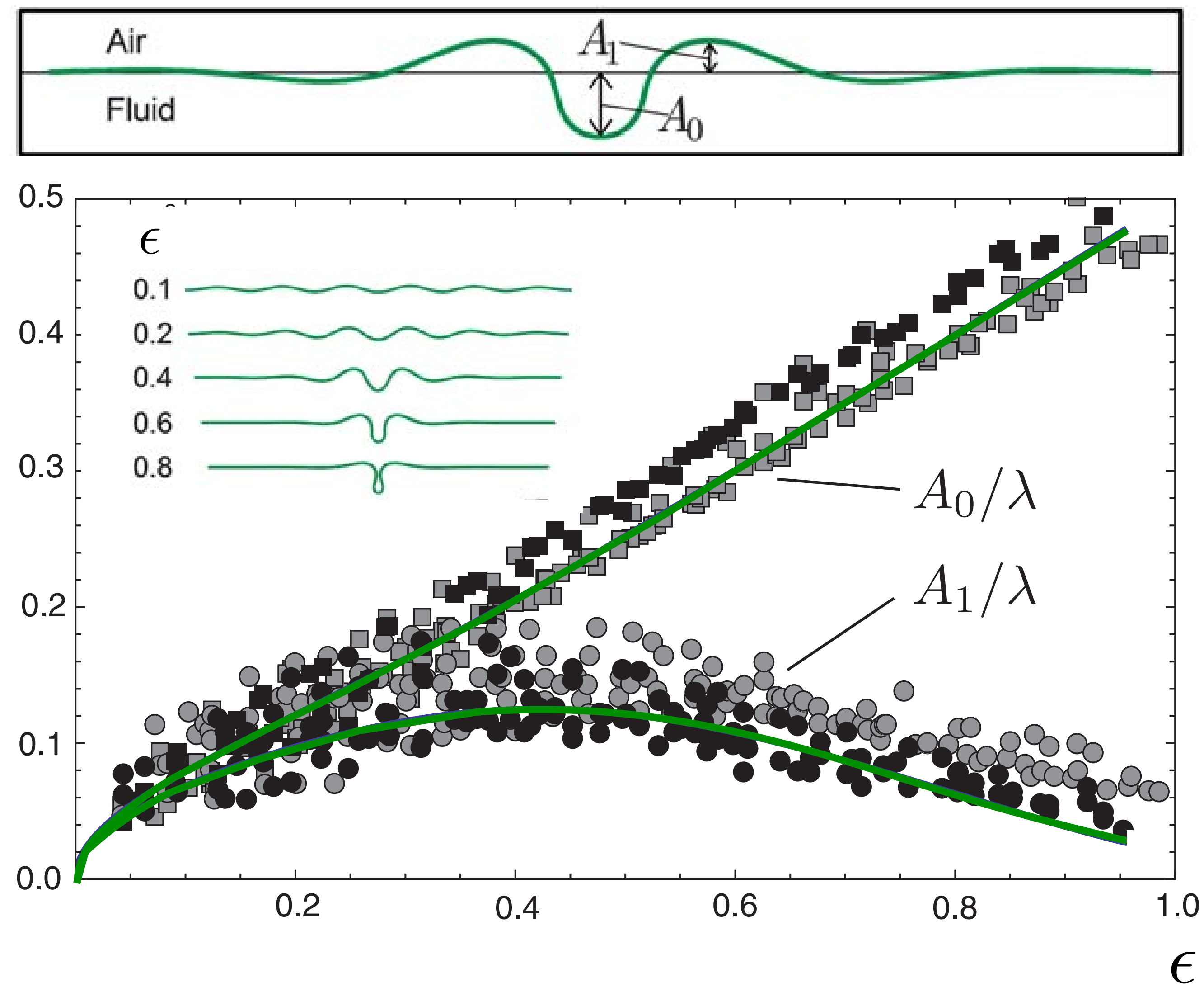
$$L - \Delta = \int_0^L ds \sqrt{1 - h'^2}$$

F. Brau et al., Soft Matter 9, 8177 (2013)

Compression of stiff thin membranes on liquid substrates

Comparison between theory (infinite membrane) and experiment

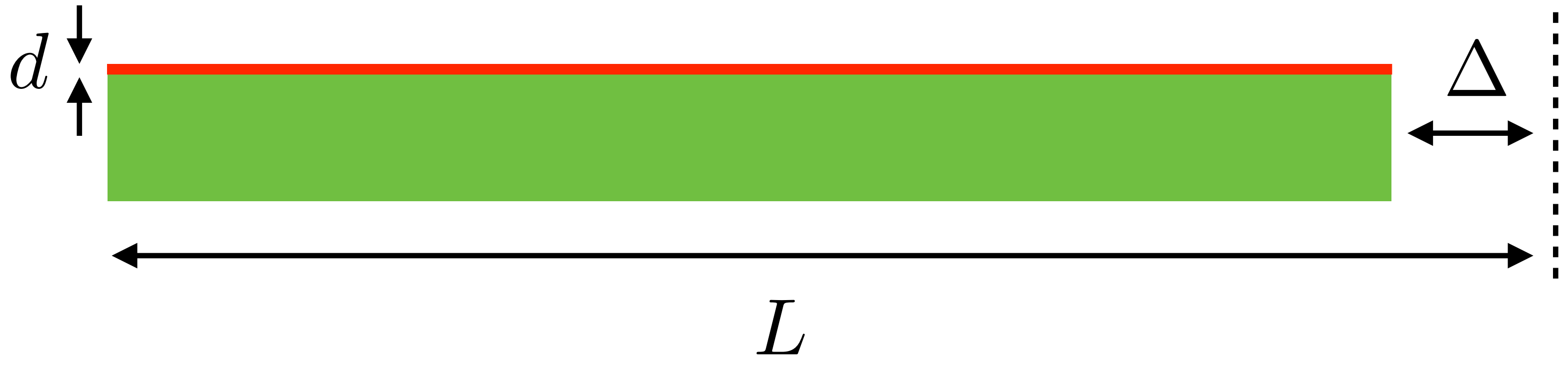




L. Pocivavsek et al., Science 320, 912 (2008)

F. Brau et al., Soft Matter 9, 8177 (2013)

Compression of stiff thin membranes on soft elastic substrates



compression energy of thin membrane

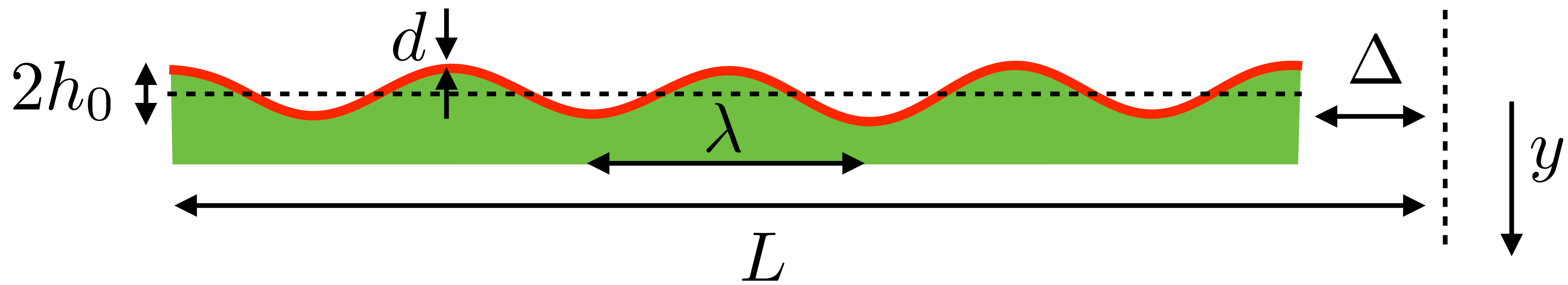
$$U_c \sim A \times E_m d \times \epsilon^2$$

membrane area	membrane 3D Young's modulus	strain	substrate 3D Young's modulus
$A = WL$	E_m	$\epsilon = \dfrac{\Delta}{L}$	E_s

Compression of stiff thin membranes on soft elastic substrates

assumed profile

$h(s) = h_0 \cos(2\pi s/\lambda)$



amplitude of wrinkles

$$h_0 = \frac{\lambda}{\pi}\sqrt{\frac{\Delta}{L}} = \frac{\lambda\sqrt{\epsilon}}{\pi}$$

deformation of the soft substrate decays exponentially away from the surface

$$h(s,y) \approx h_0 \cos(2\pi s/\lambda) e^{-y/\lambda}$$

bending energy of stiff membrane

$$U_b \sim A \times \kappa \times \frac{1}{R^2} \sim A \times E_m d^3 \times \frac{h_0^2}{\lambda^4} \sim \frac{A E_m d^3 \epsilon}{\lambda^2}$$

deformation energy of soft substrate

$$U_s \sim V \times E_s \times \epsilon_s^2 \sim A\lambda \times E_s \times \frac{h_0^2}{\lambda^2} \sim A E_s \lambda \epsilon$$

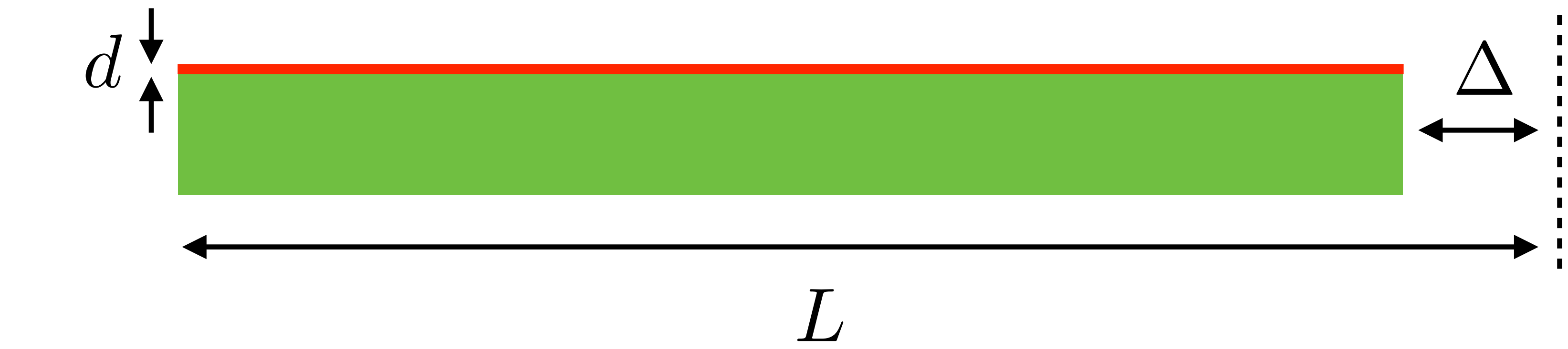
minimize total energy (U_b+U_s) with respect to λ →

$$\lambda \sim d \left(\frac{E_m}{E_s}\right)^{1/3}$$

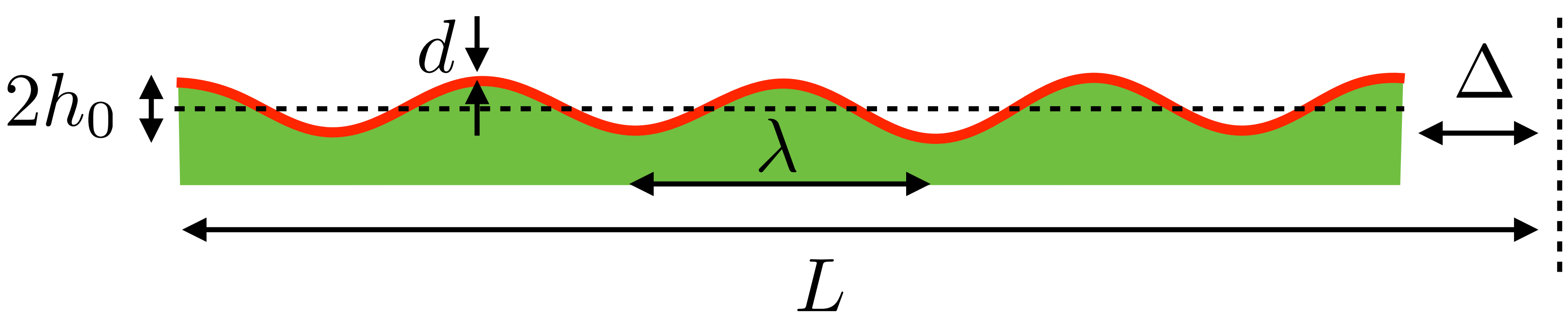
→

$$U_b, U_s \sim A d\epsilon \left(E_s^2 E_m\right)^{1/3}$$

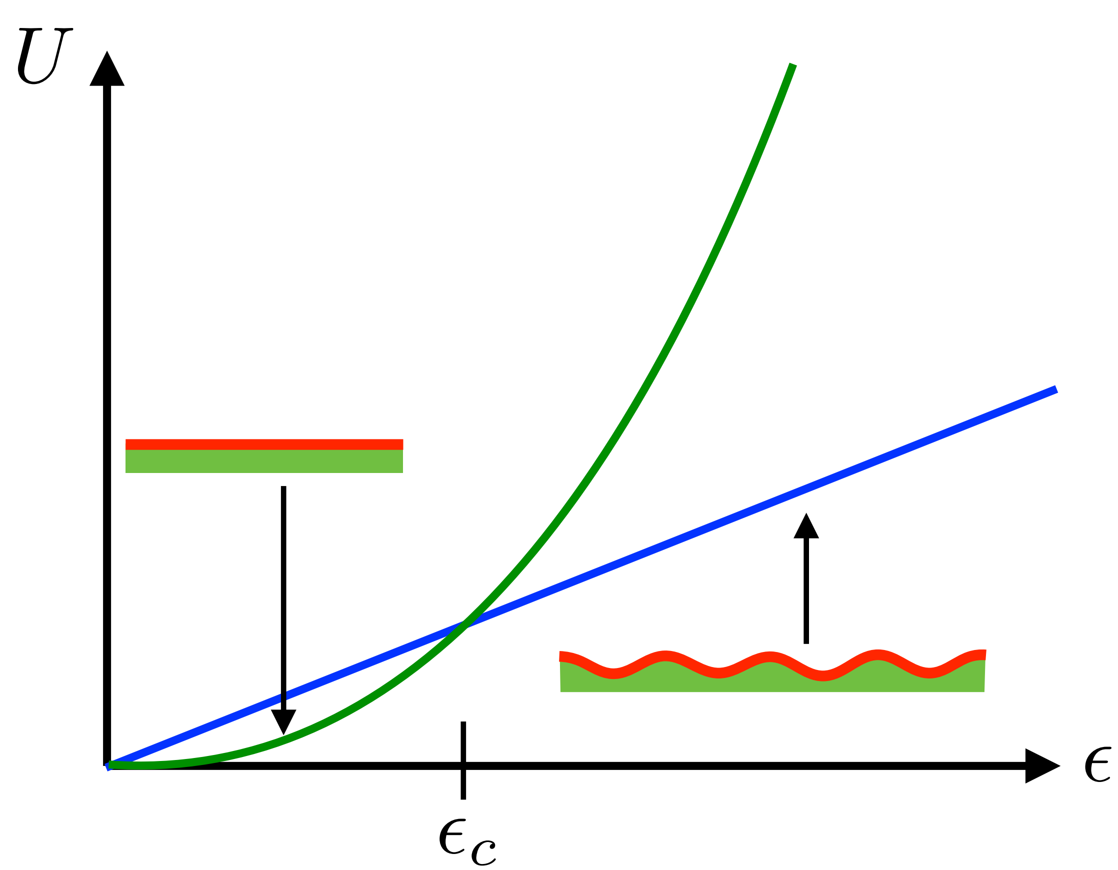
Compression of stiff thin membranes on soft elastic substrates



$$U_c \sim A \times E_m d \times \epsilon^2$$



$$U_b, U_s \sim A d \epsilon \left(E_s^2 E_m\right)^{1/3}$$



wrinkles are stable for large strains

$$\epsilon > \epsilon_c \sim \left(\frac{E_s}{E_m}\right)^{2/3}$$

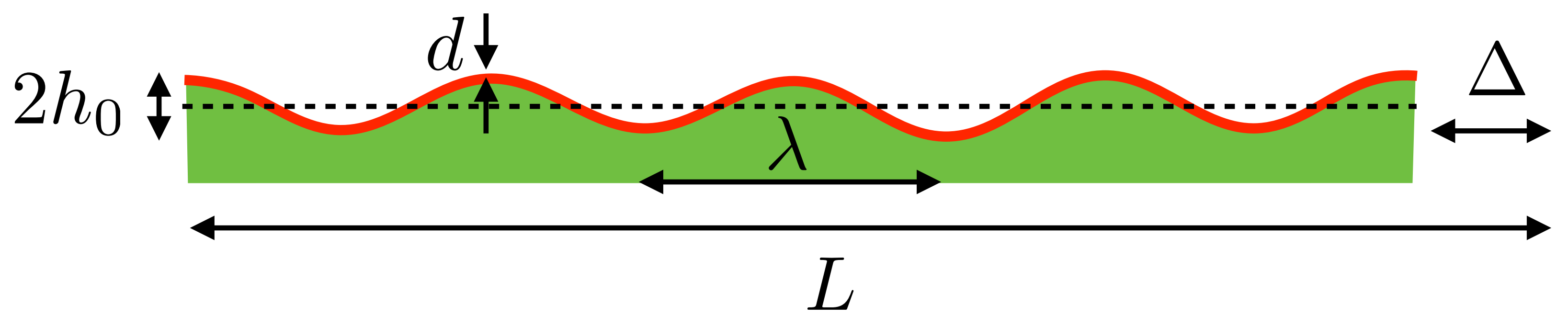
wavelength of wrinkles

$$\lambda \sim d \left(\frac{E_m}{E_s}\right)^{1/3}$$

amplitude of wrinkles at the critical strain

$$h_0^* \sim \lambda \sqrt{\epsilon_c} \sim d$$

Compression of stiff thin membranes on liquid and soft elastic substrates

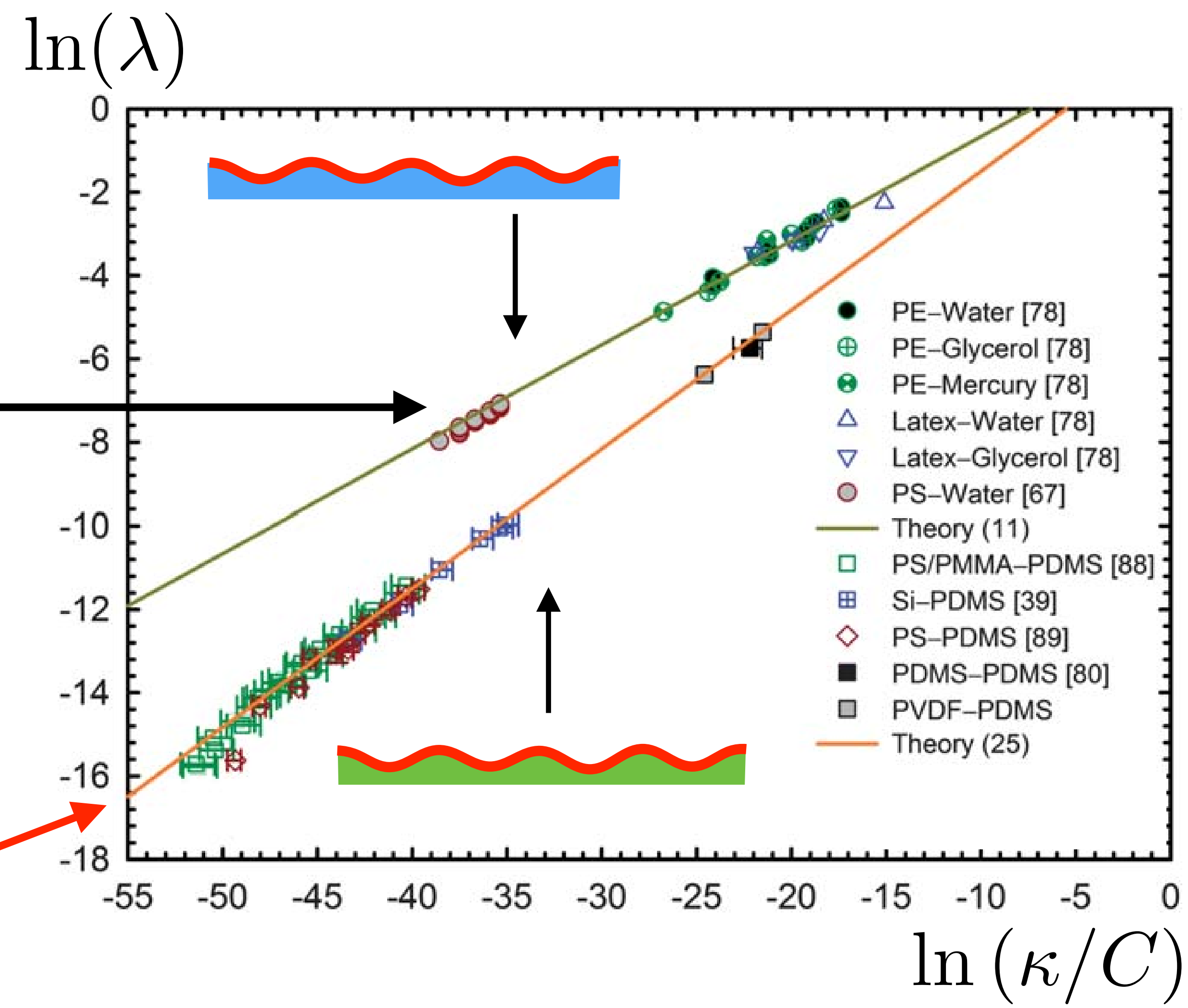


wavelength of wrinkles on liquid substrates

$$\lambda = 2\pi \left(\frac{\kappa}{\rho g}\right)^{1/4}$$

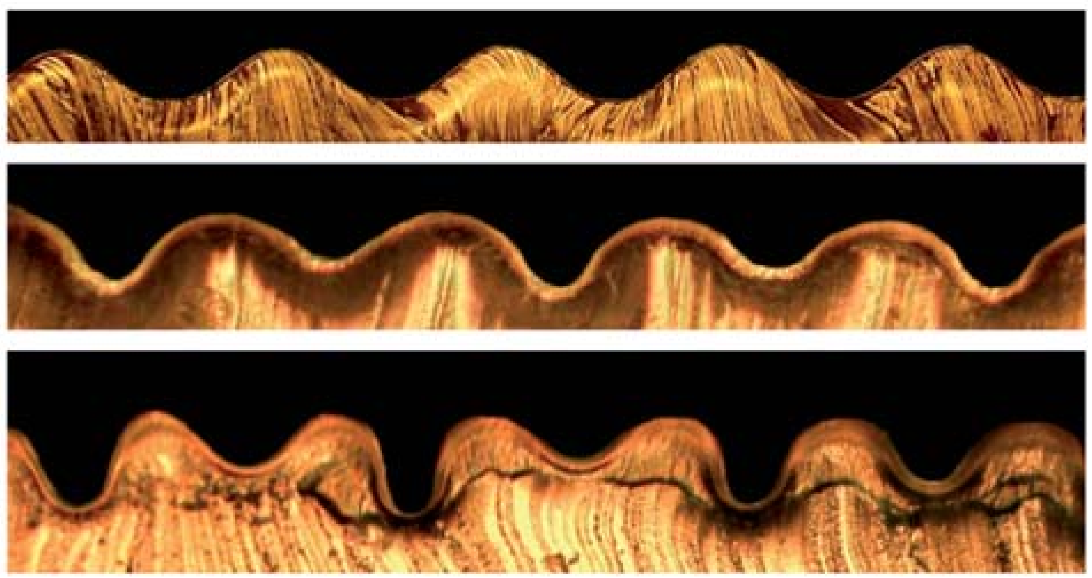
wavelength of wrinkles on soft elastic substrates

$$\lambda = 2\pi \left(\frac{3\kappa}{E_s}\right)^{1/3}$$



F. Brau et al., Soft Matter 9, 8177 (2013)

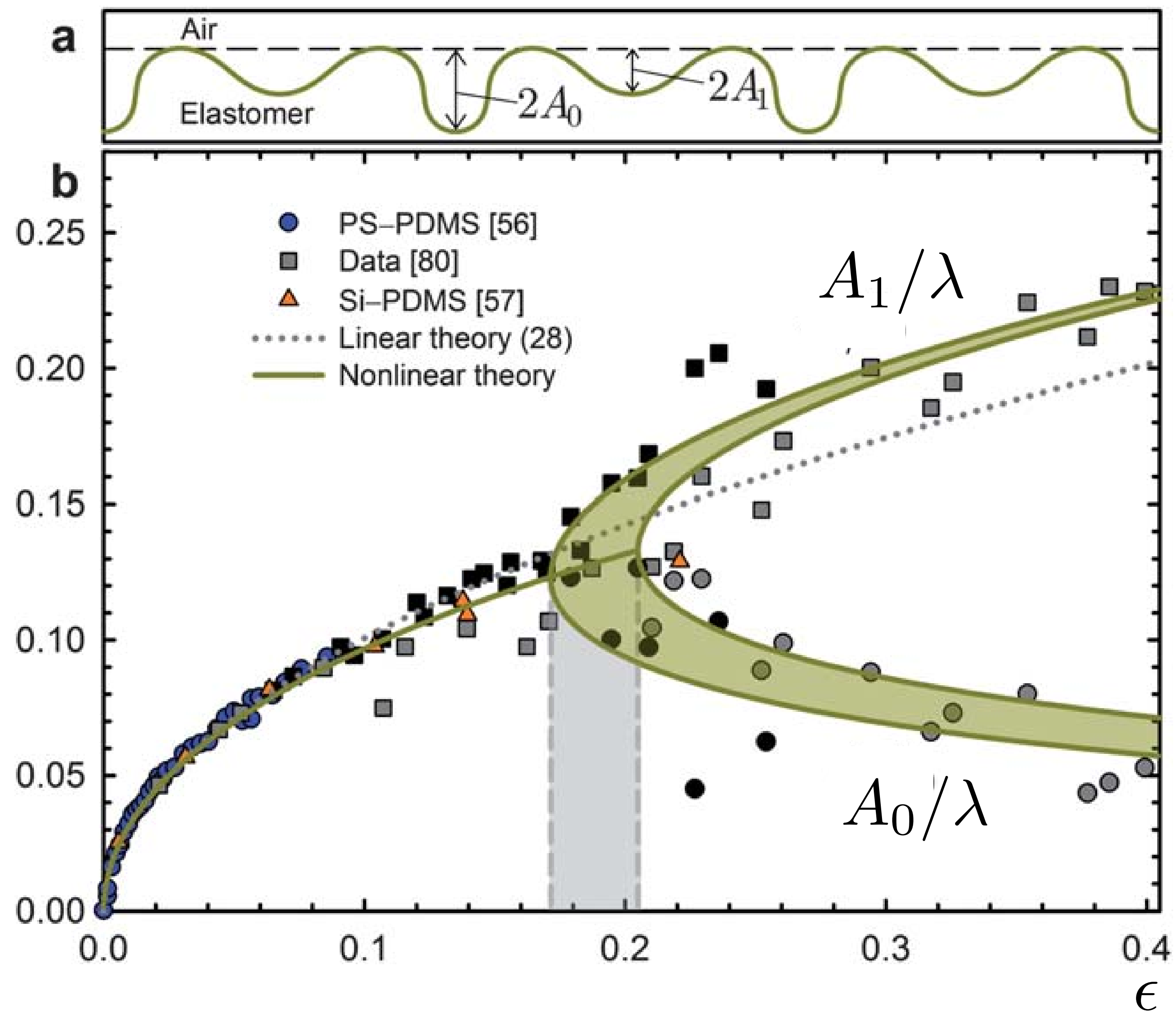
Compression of stiff thin membranes on soft elastic substrates



In order to explain period doubling (quadrupling, ...) one has to take into account the full nonlinear strain tensor of the soft substrate

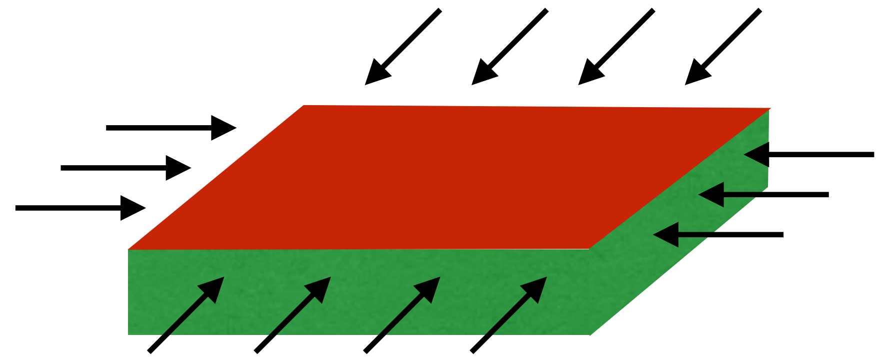
$$2u_{ij}^s = \left(\partial_i u_j^s + \partial_j u_i^s\right) + \sum_k \partial_i u_k^s \partial_j u_k^s$$

$$i, j, k \in x, y, z$$



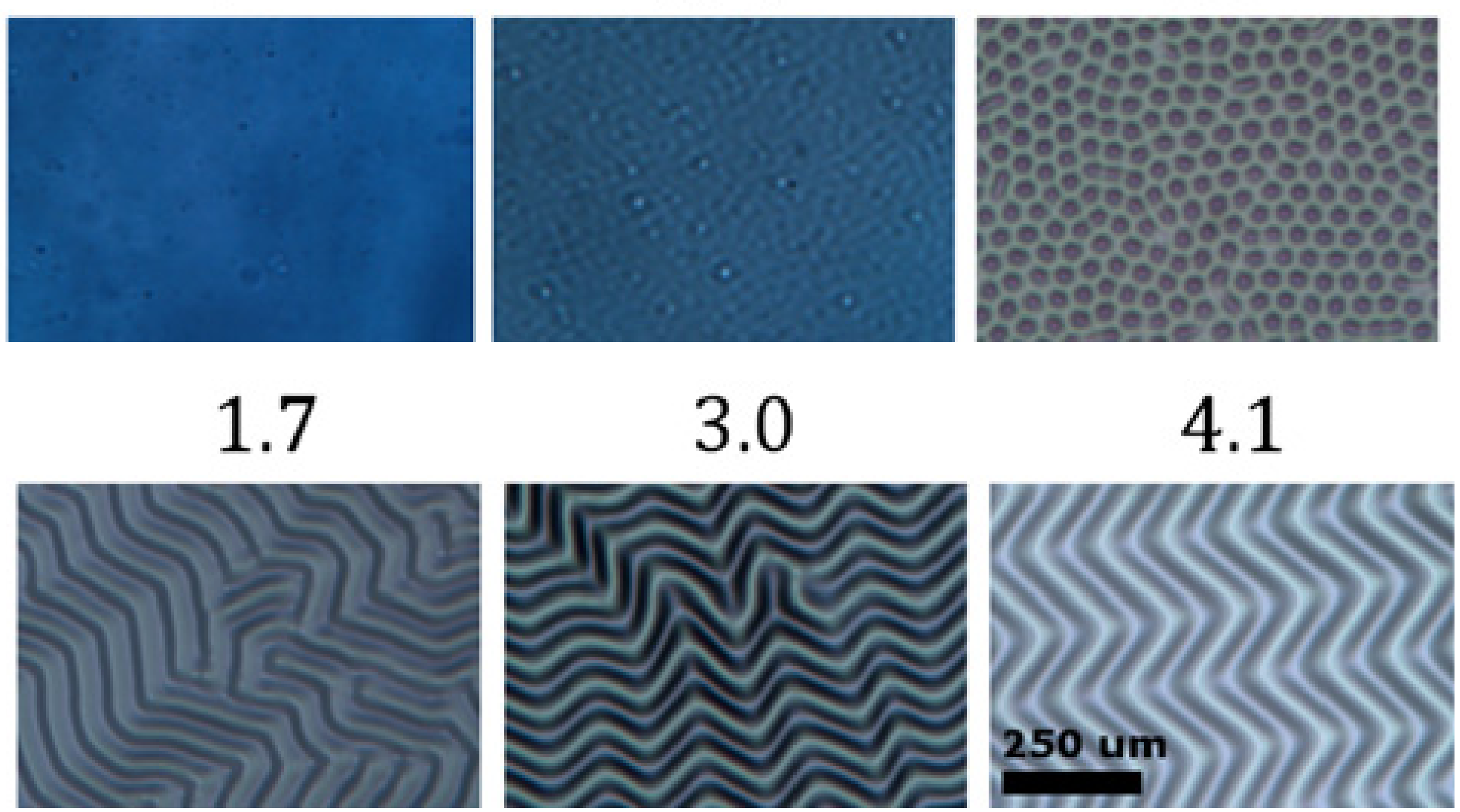
F. Brau et al., Soft Matter 9, 8177 (2013)

Uniform compression of stiff thin membranes on soft elastic substrates



hexagonal crystal of bumps

ϵ/ϵ_c	$\lesssim 1$	$\gtrsim 1$	1.3
ϵ/ϵ_c	1.7	3.0	4.1

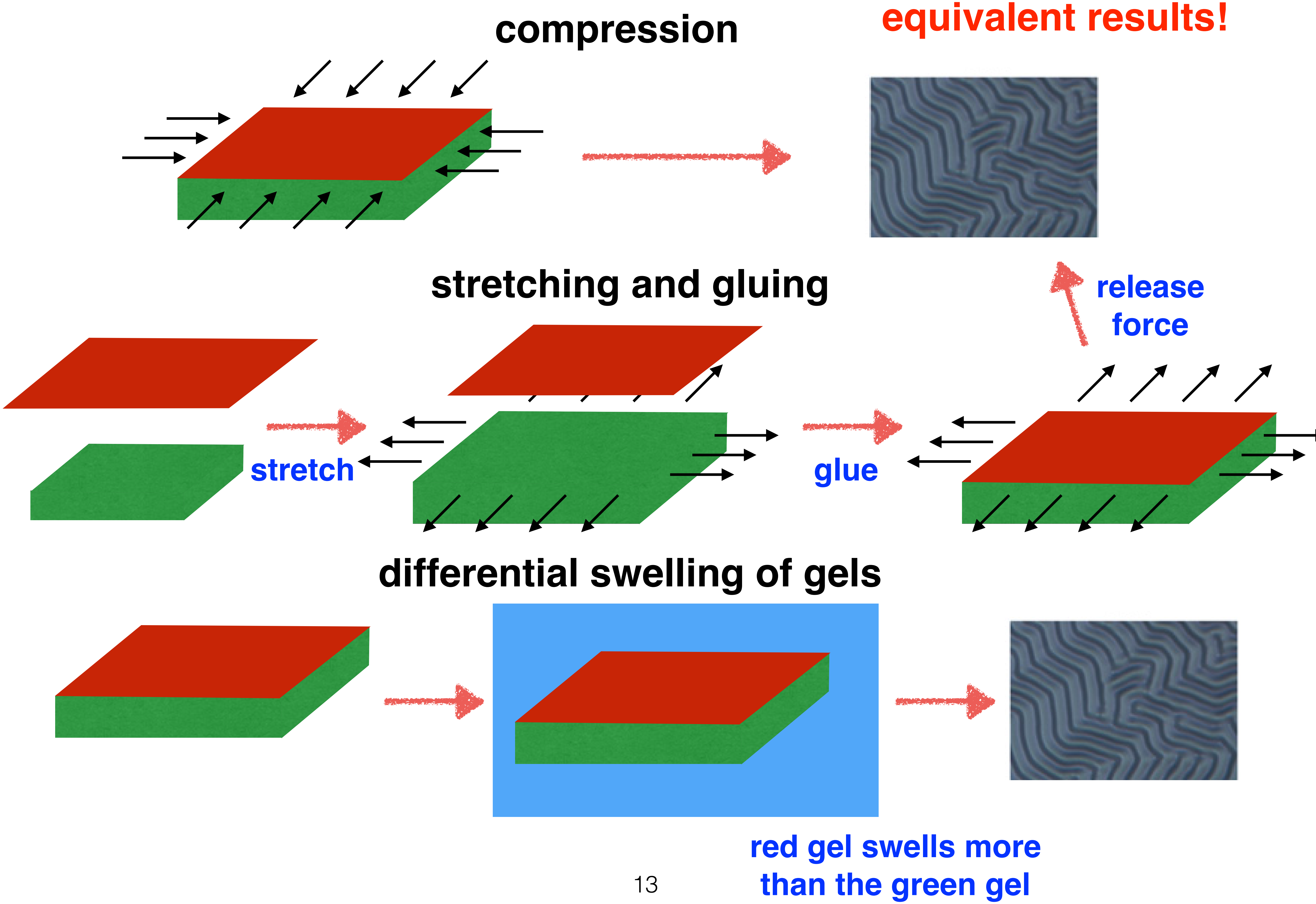


herringbone pattern

S. Cai et al., J. Mech. Phys. Solids 59, 1094 (2011)

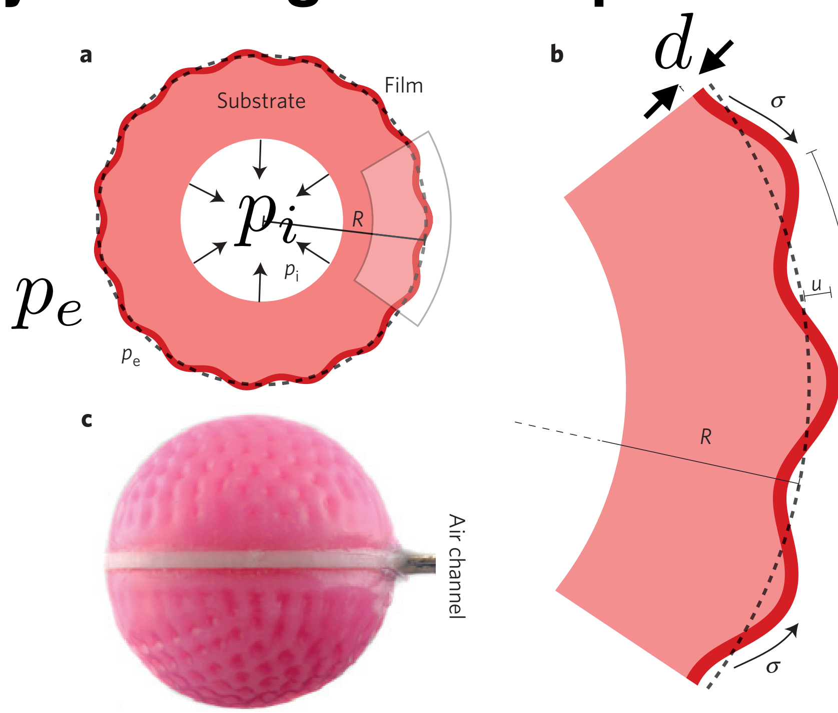
Experimental protocols

All protocols produce equivalent results!

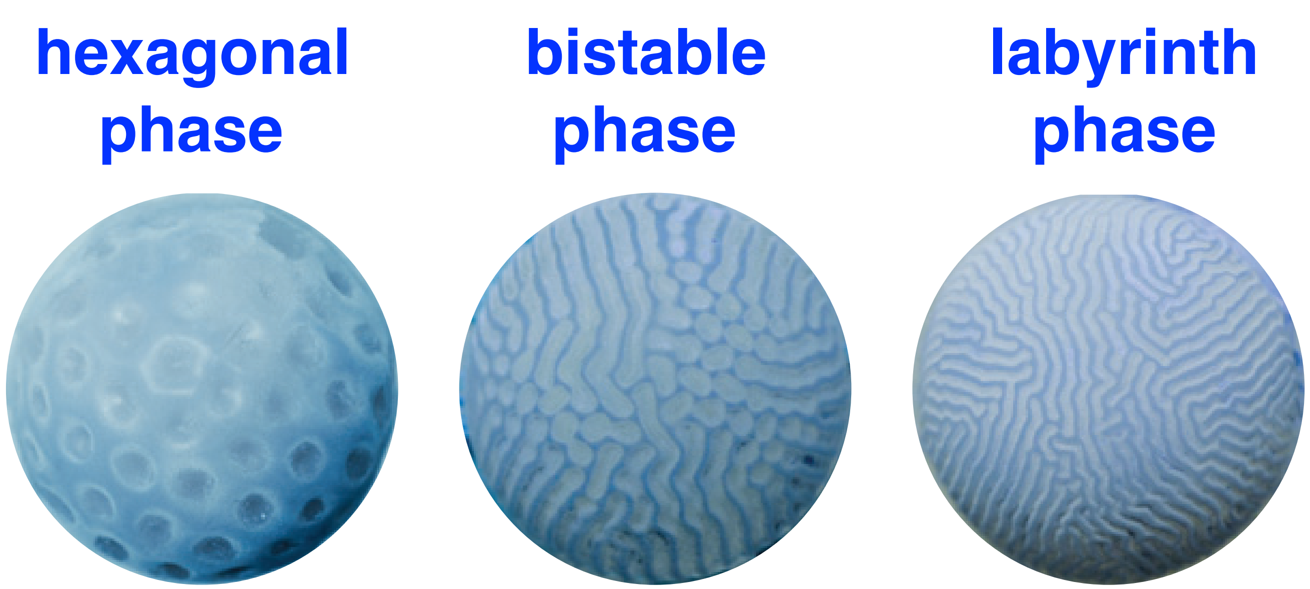


Compression of stiff thin membranes on a spherical soft substrates

Spherical shells are compressed by reducing internal pressure

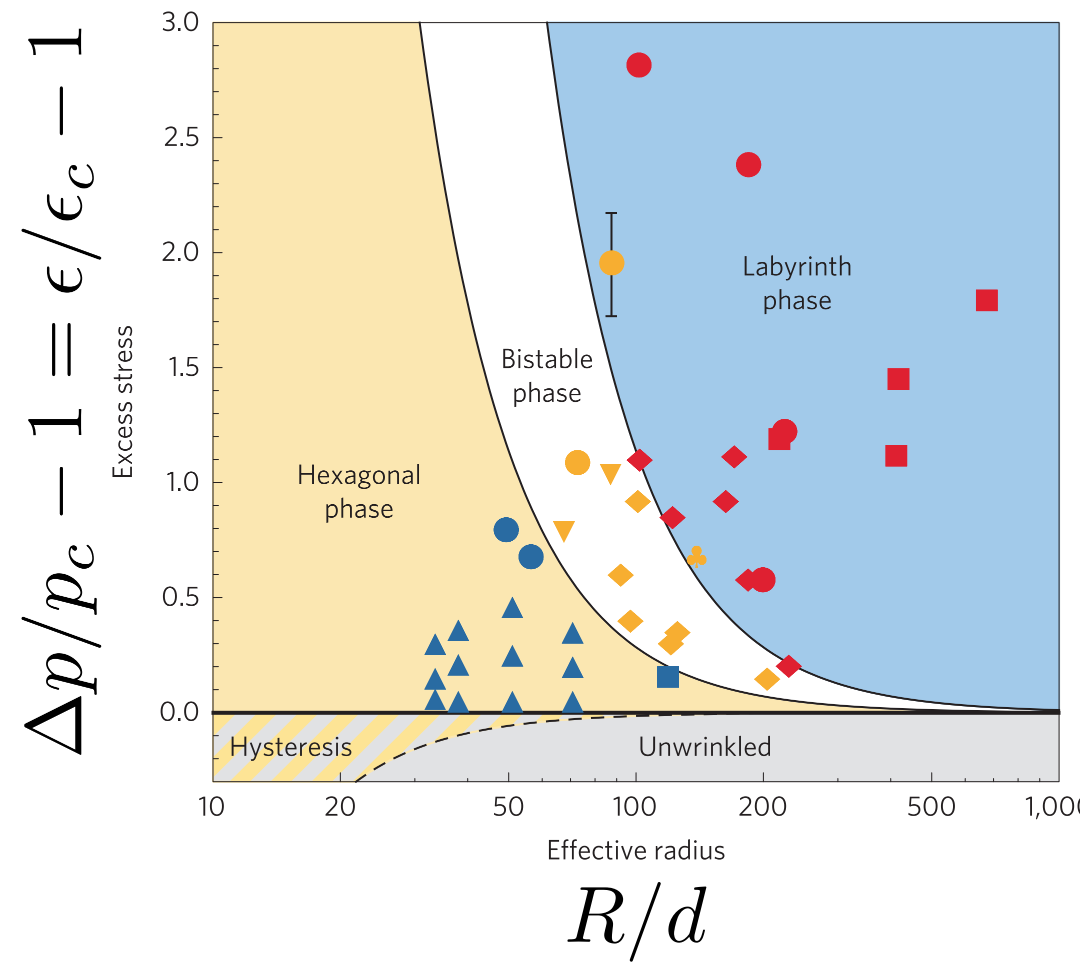


$R = 20\text{mm}$



R/d

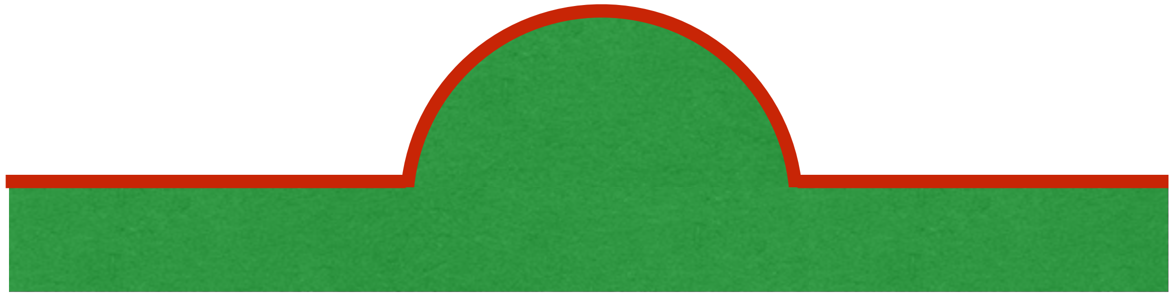
Phase diagram



N. Stoop et al., Nat. Materials 14, 337 (2015)

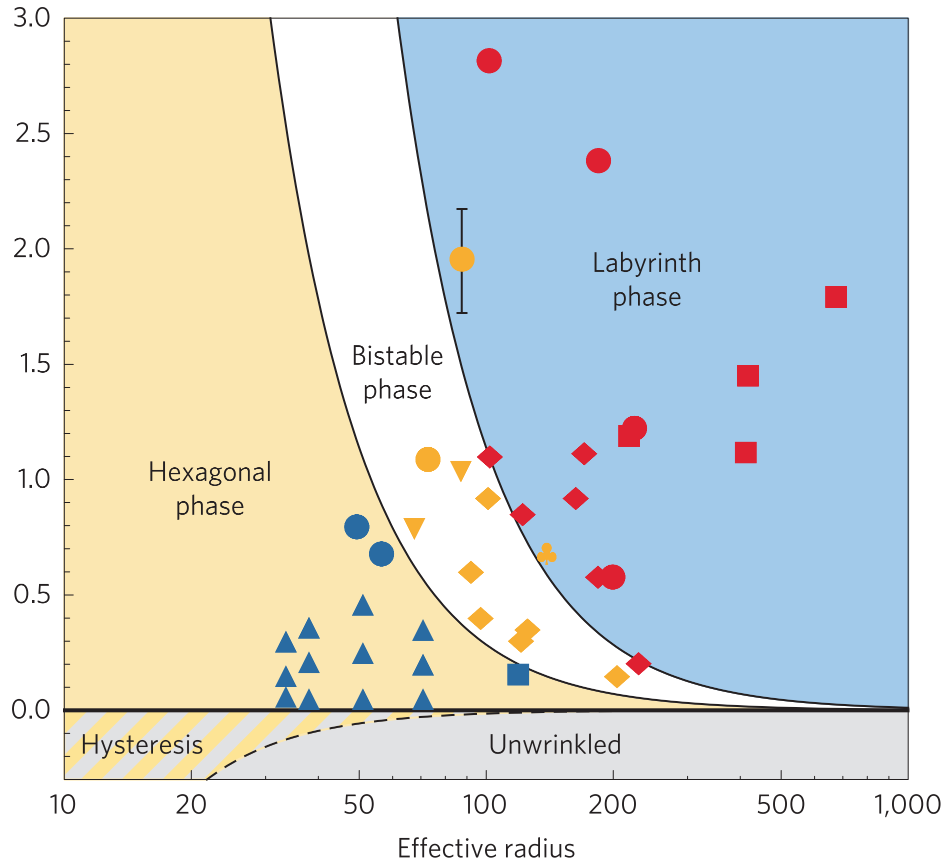
Compression of stiff thin membranes on a spherical soft substrates

Swelling of gels



Phase diagram

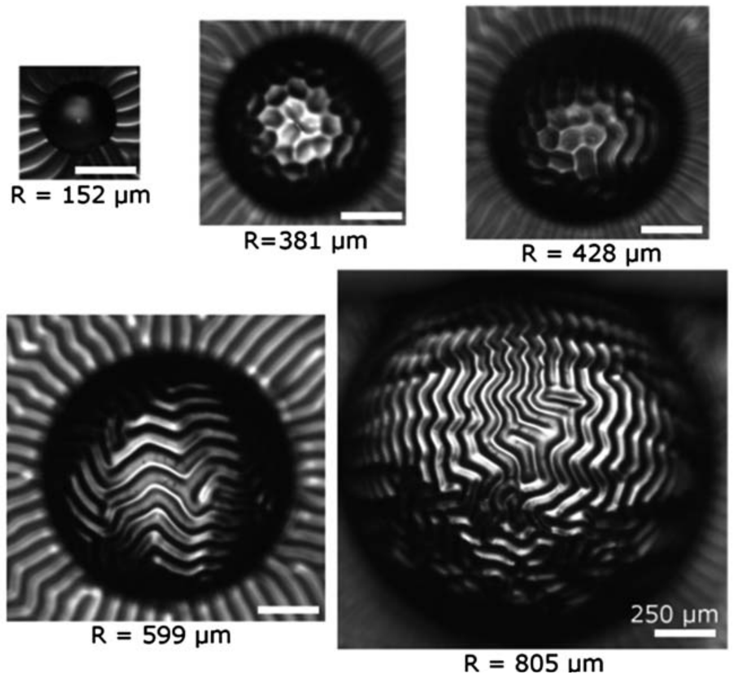
$\Delta p / p_c - 1 = \epsilon / \epsilon_c - 1$



R/d

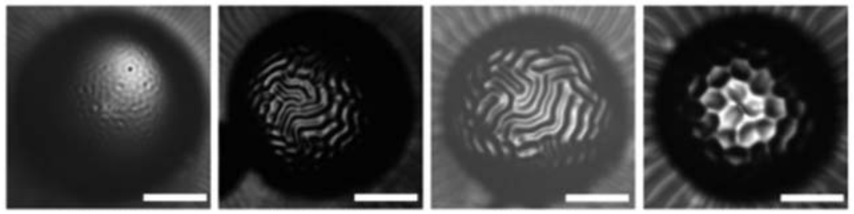
D. Breid and A.J. Crosby, Soft Matter 9, 3624 (2013)

Modifying radius R



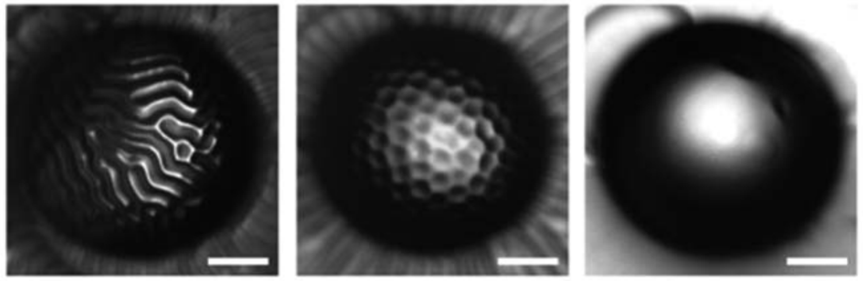
Modifying membrane thickness d

$R = 381 \mu m$

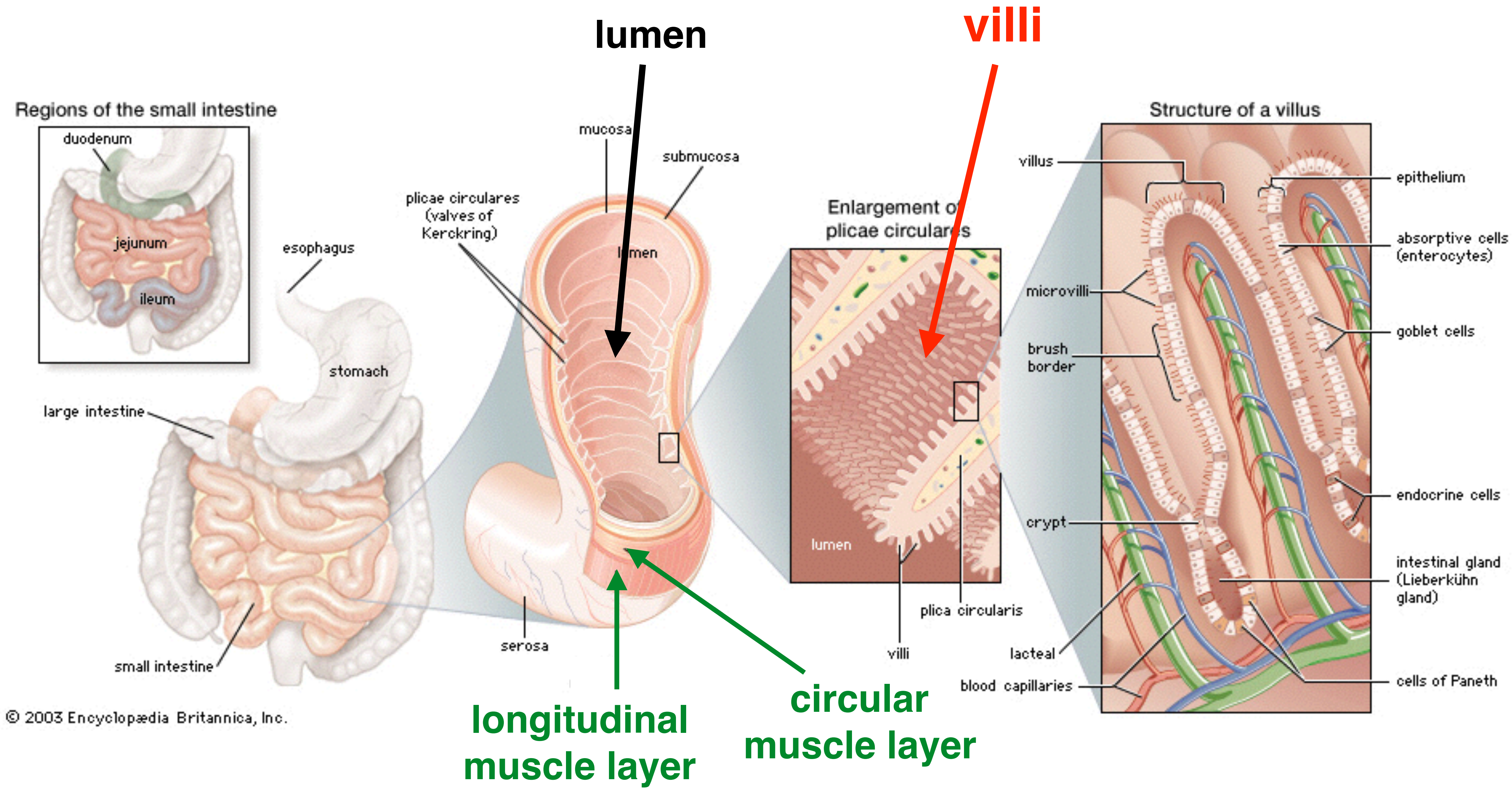


Modifying swelling strain ϵ

$R = 522 \mu m$



How are villi formed in guts?



Villi increase internal surface area of intestine for faster absorption of digested nutrients.

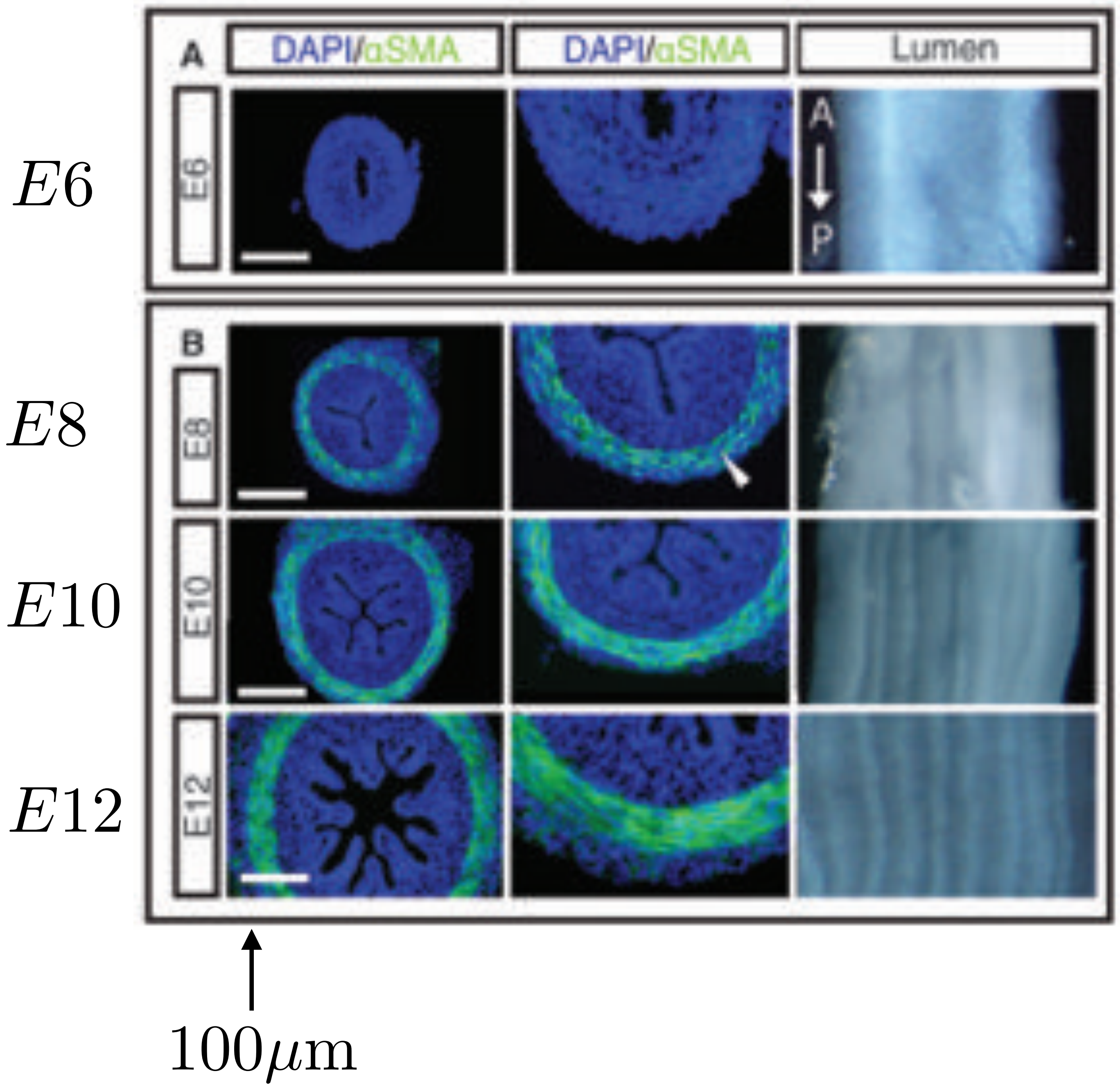
Lumen patterns in chick embryo

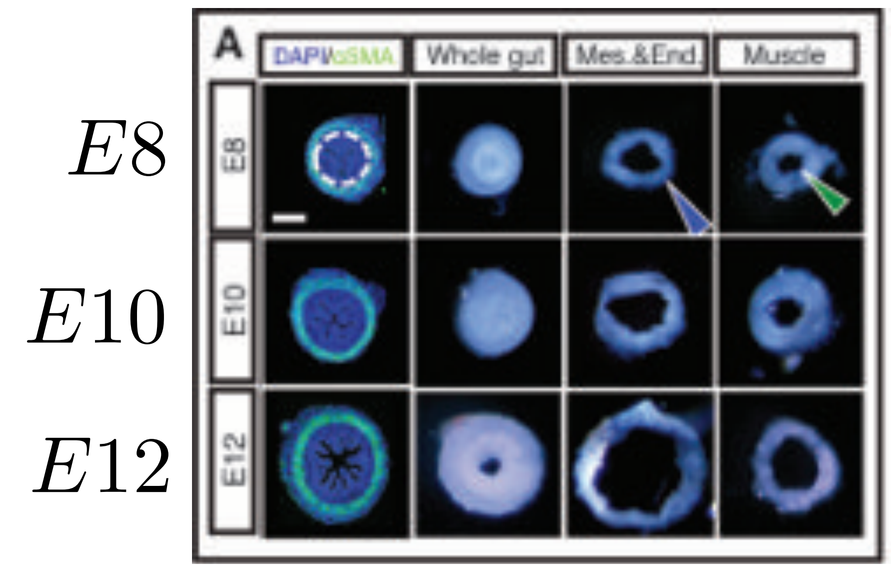
DAPI marks cell nuclei

aSMA marks smooth muscle actin

EX: age of chick embryo in days

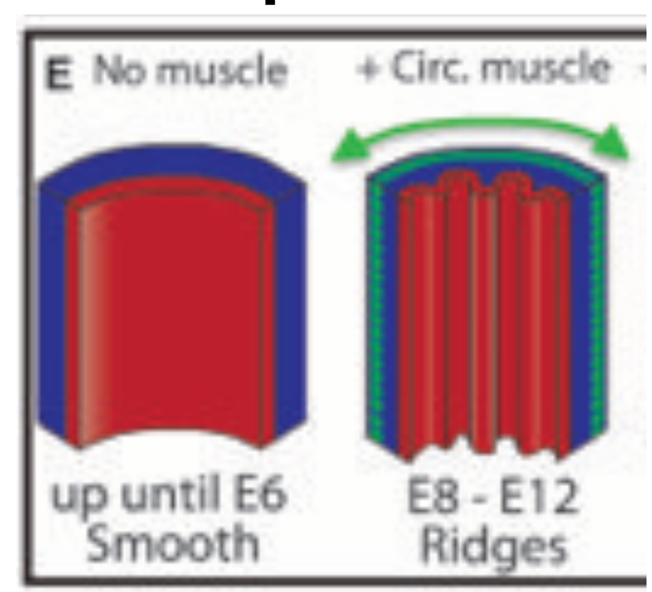
Stiff muscles grow slower than softer mesenchyme and endoderm layers





radial compression due to differential growth produces striped wrinkles

endoderm
mesenchyme
muscle



A. Shyer et al., Science 342, 212 (2013)

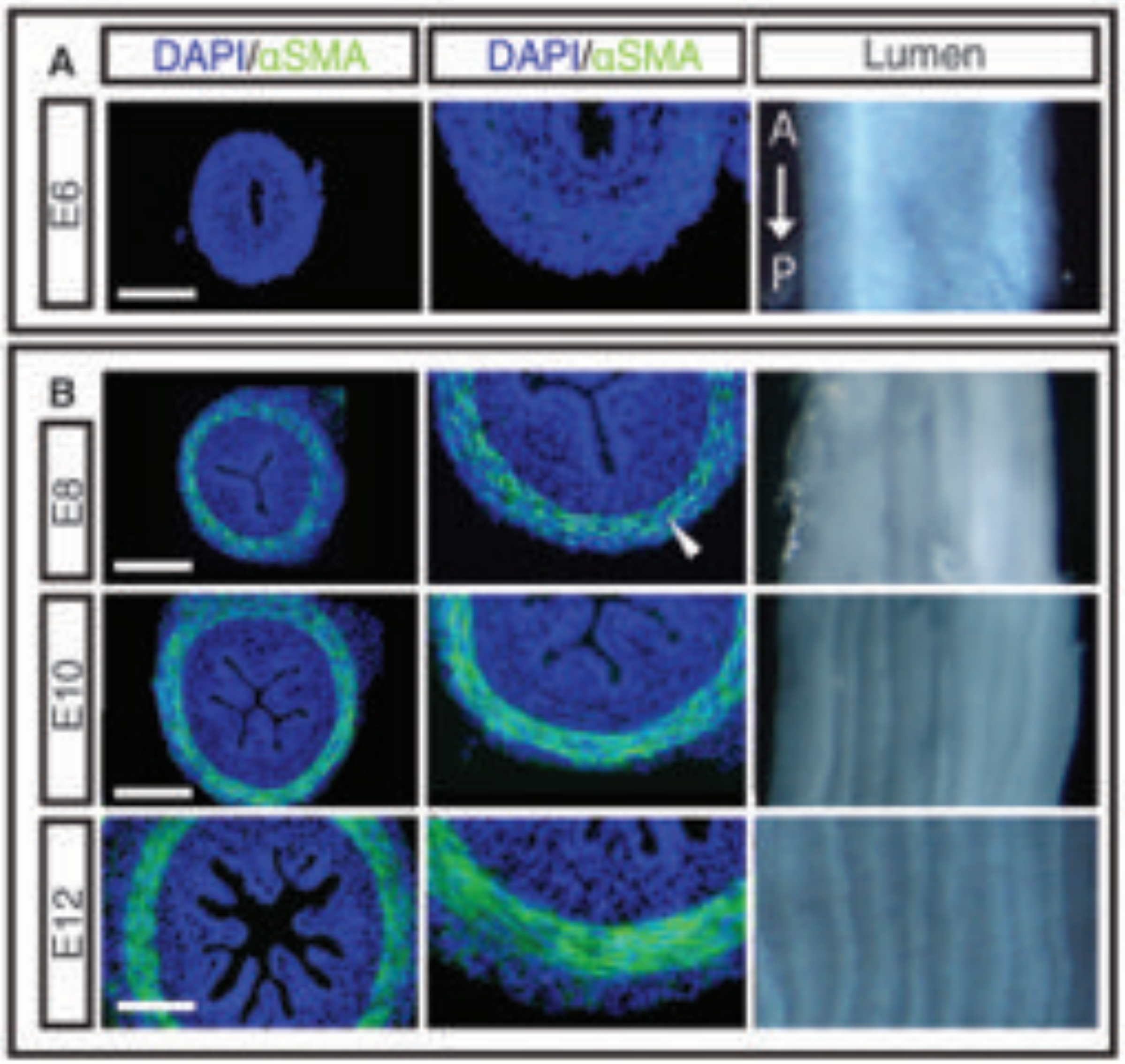
Lumen patterns in chick embryo

E6

E8

E10

E12



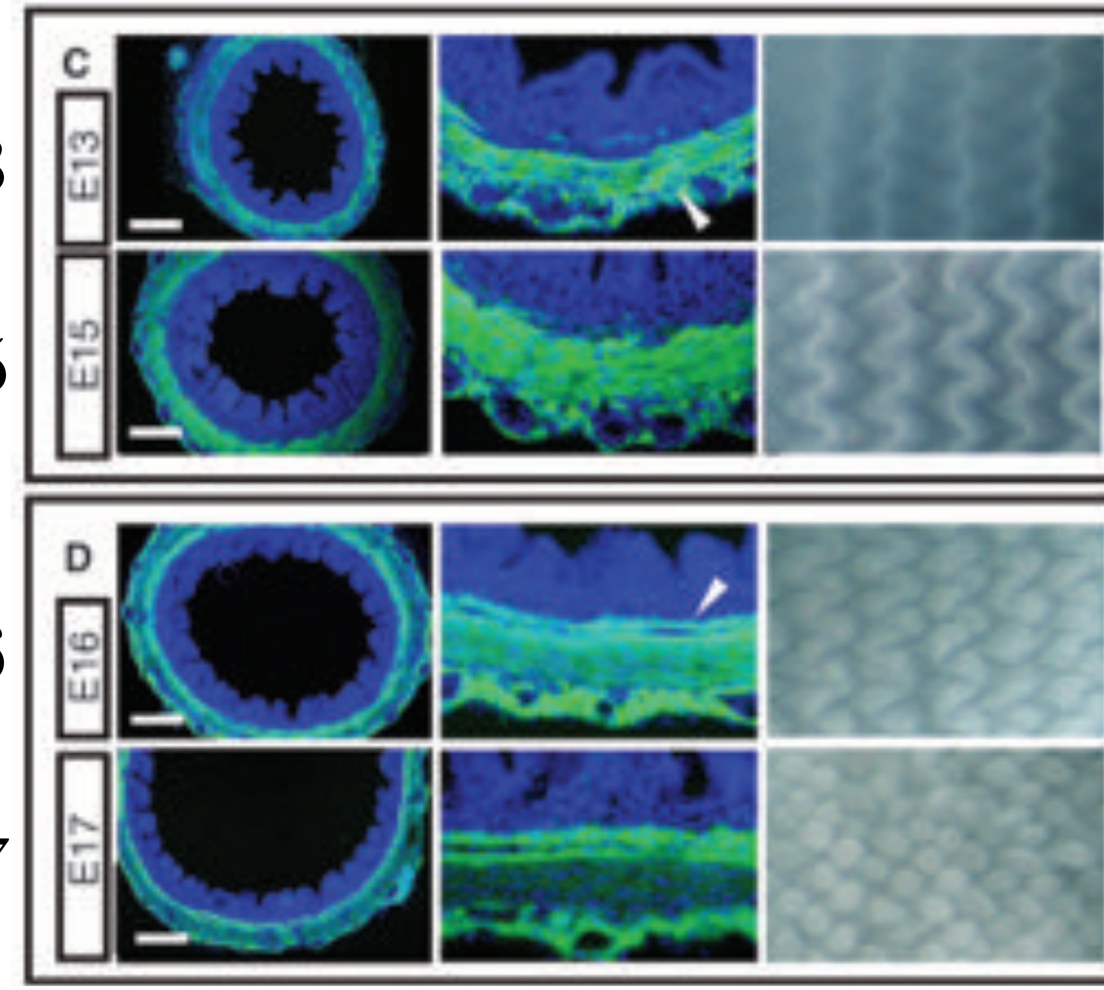
Formation of longitudinal muscles at E13 produces longitudinal compression

E13

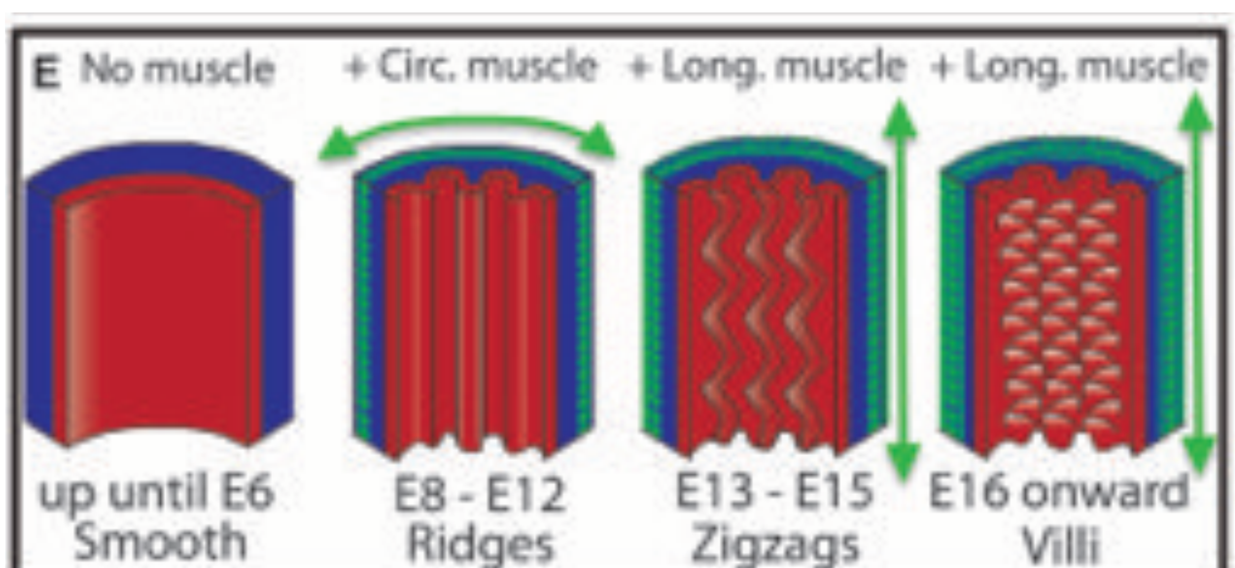
E15

E16

E17



endoderm
mesenchyme
muscle



A. Shyer et al., Science 342, 212 (2013)

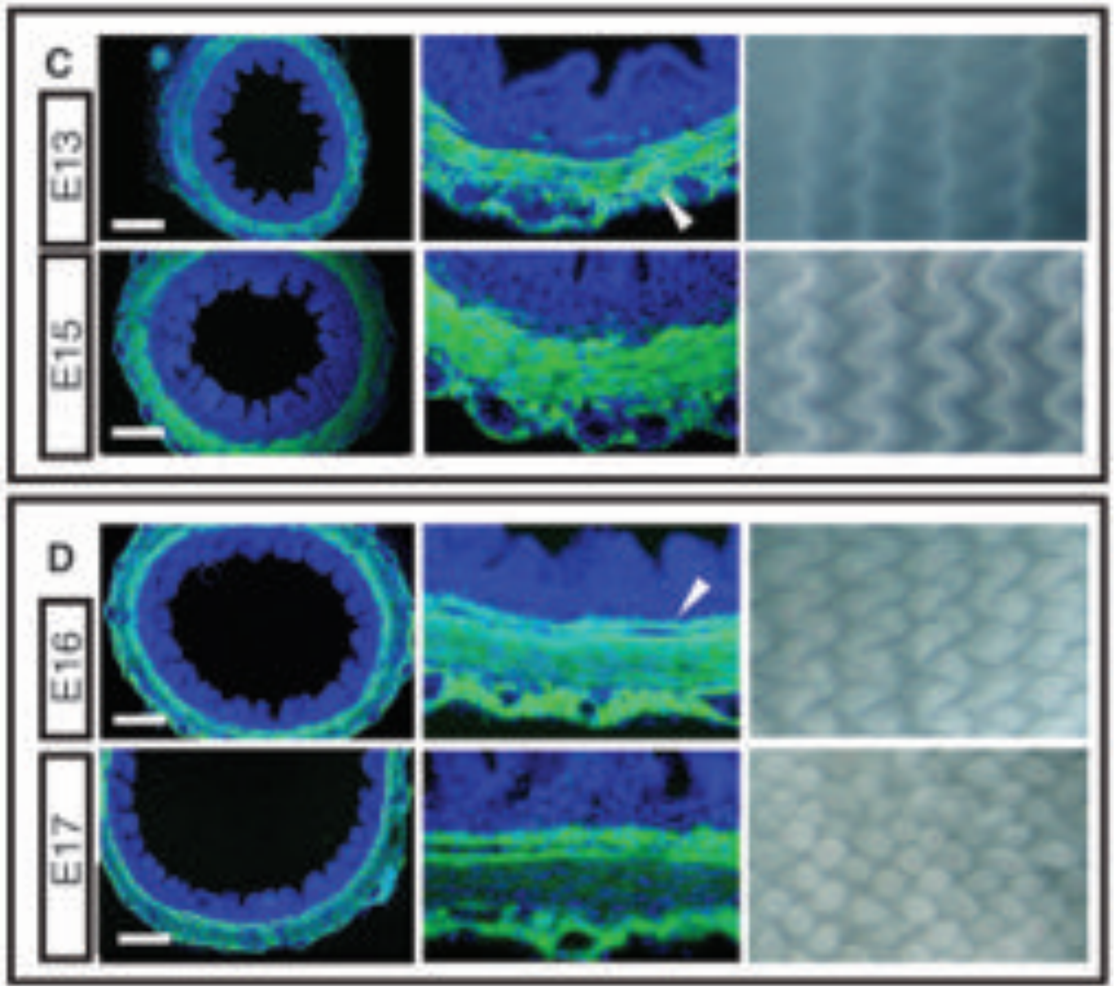
Lumen patterns in chick embryo

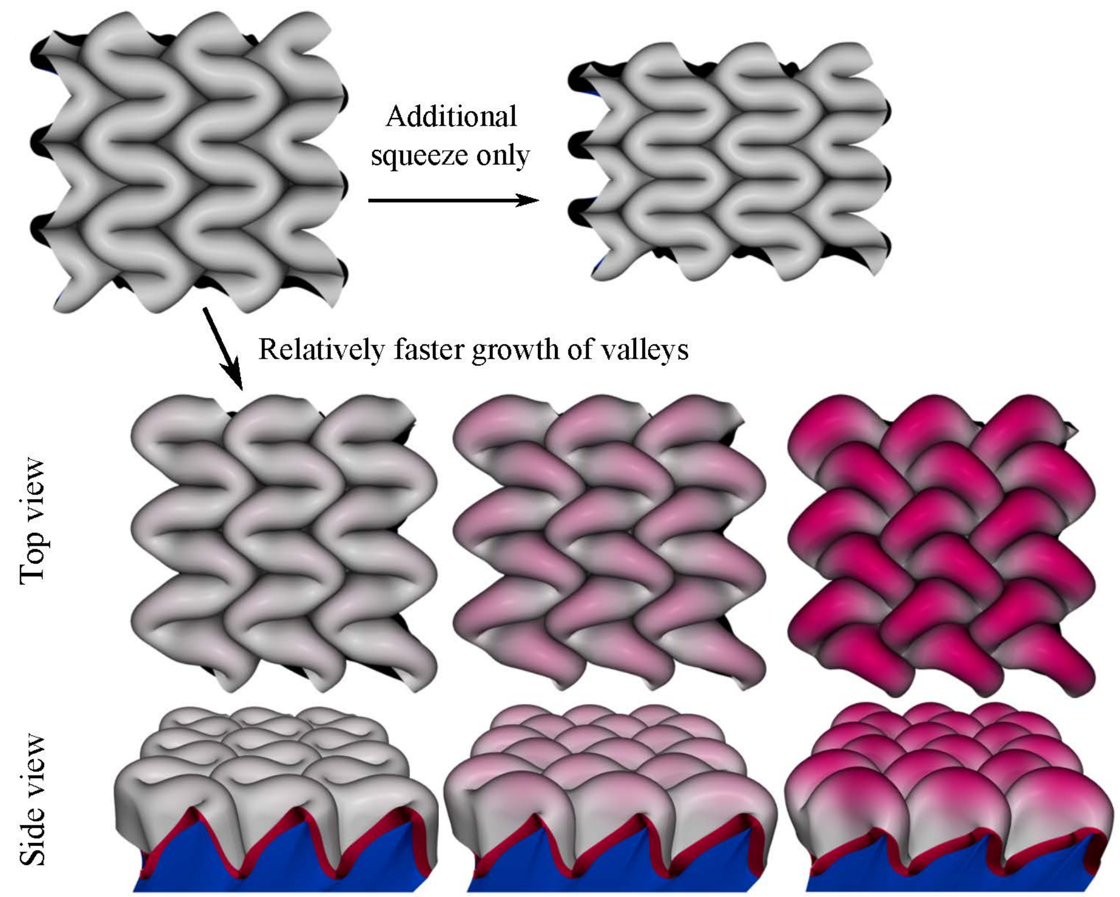
$E13$

$E15$

$E16$

$E17$

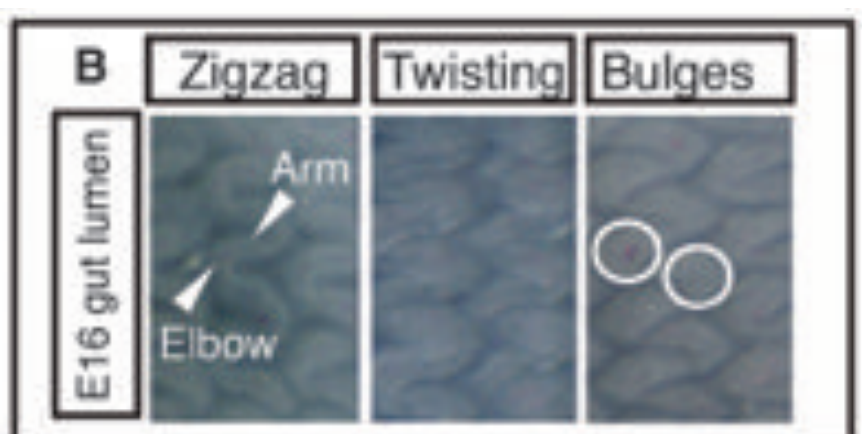


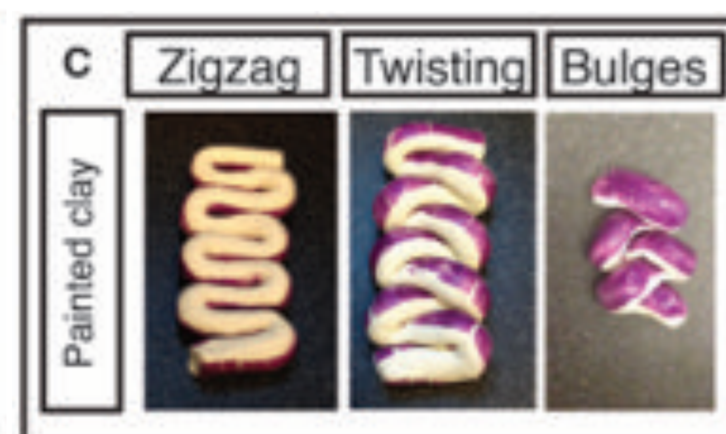


Zigzag **Twisting** **Bulges**

Villi start forming at E16 because of the faster growth in valleys

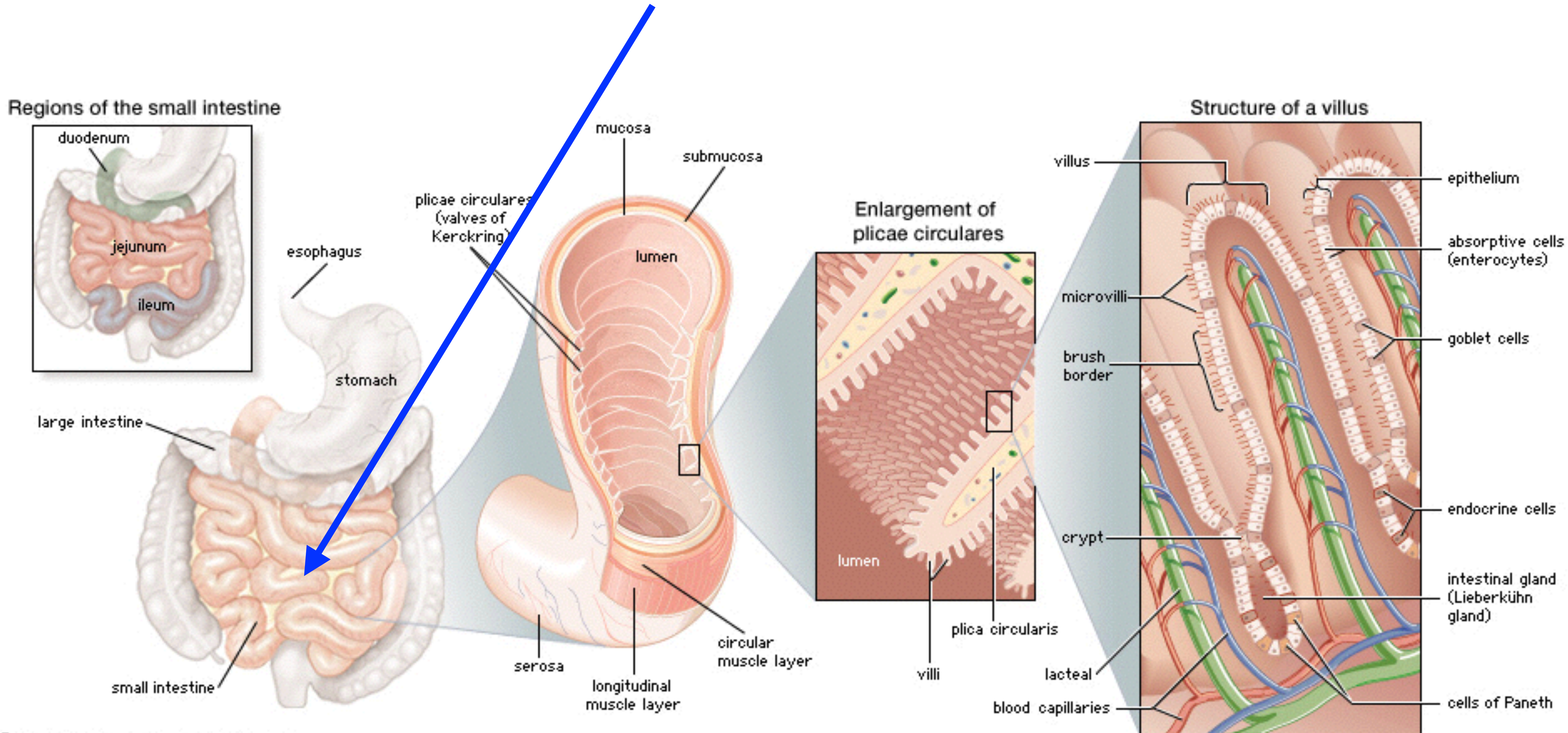
The same mechanism for villi formation also works in other organisms!





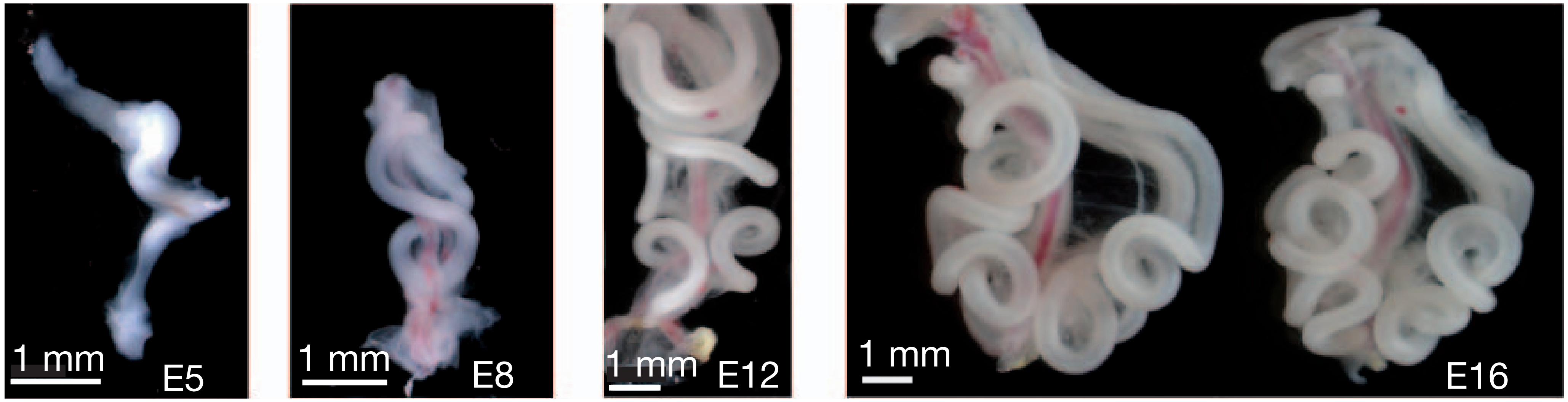
A. Shyer et al., Science 342, 212 (2013)

Why are guts shaped like that?



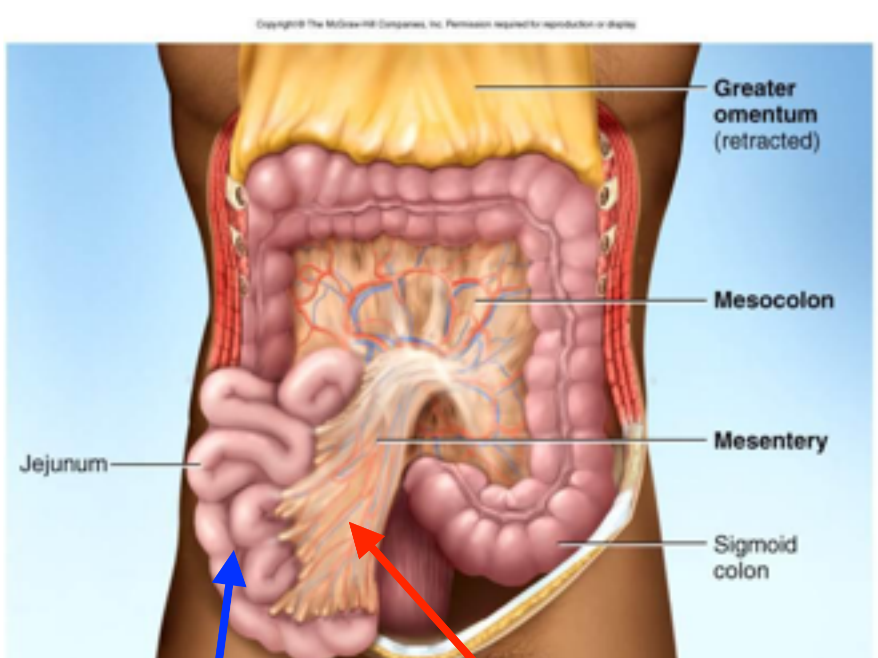
Guts in chick embryo

Surgically removed guts from chick embryo

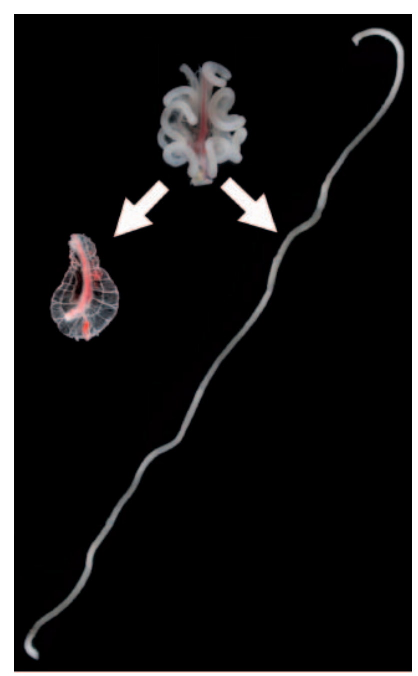


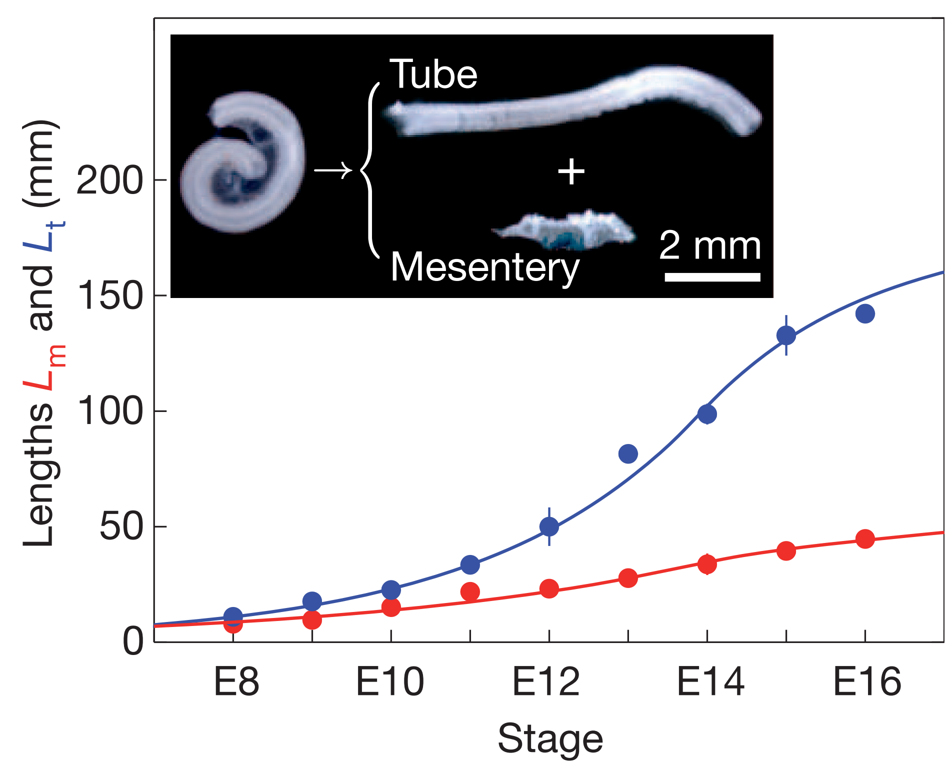
Tube straightens after separation from mesentery

Tube grows faster than mesentery sheet!



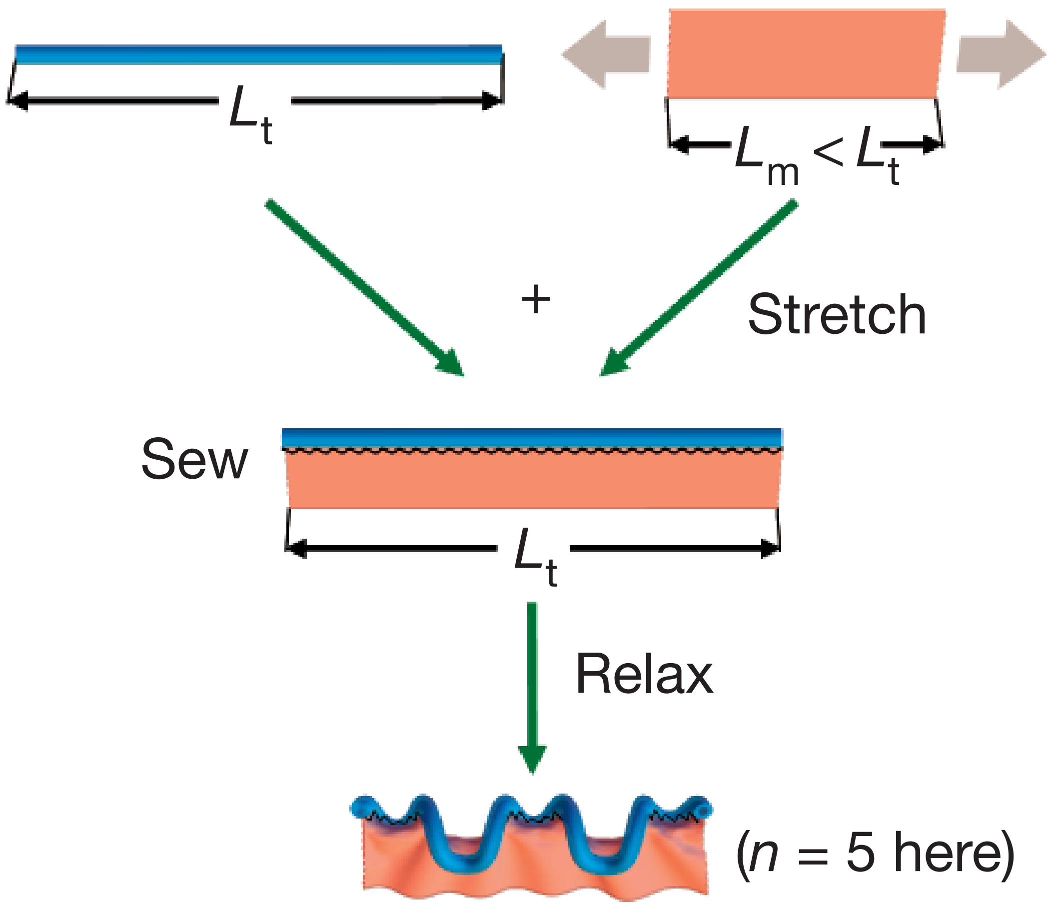
tube **mesentery**



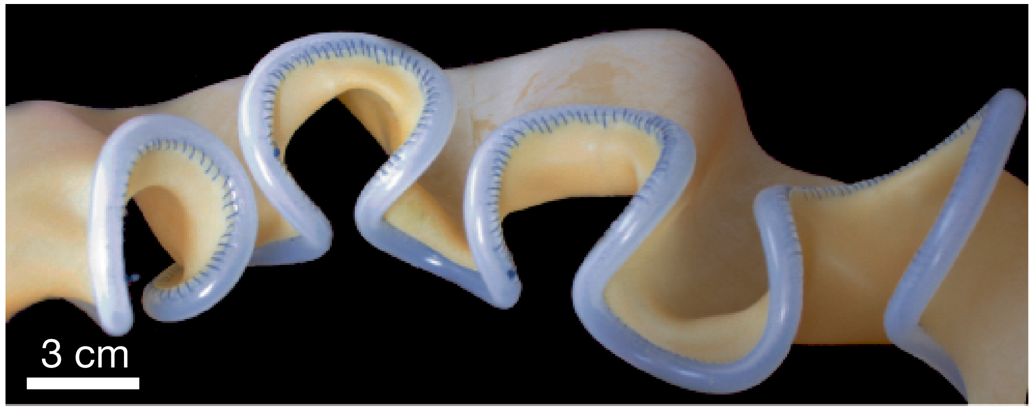


T. Savin et al., Nature 476, 57 (2011)

Synthetic analog of guts



Rubber model of guts

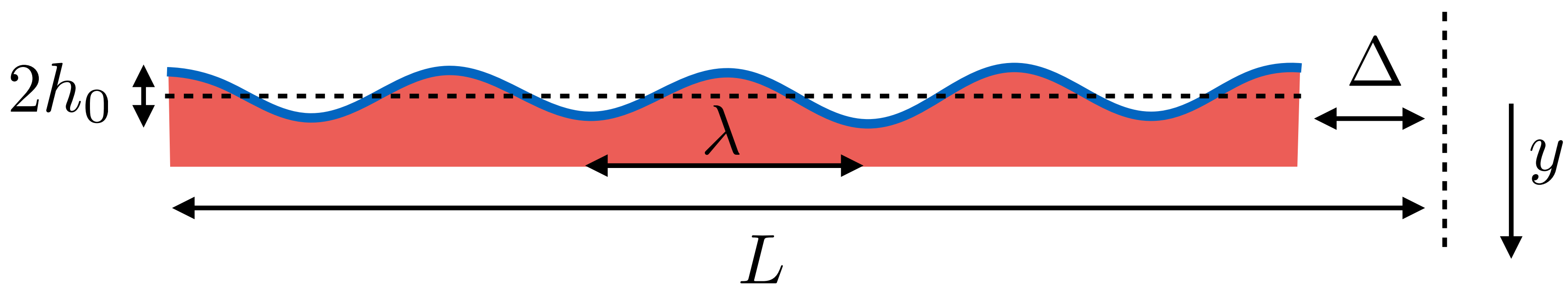


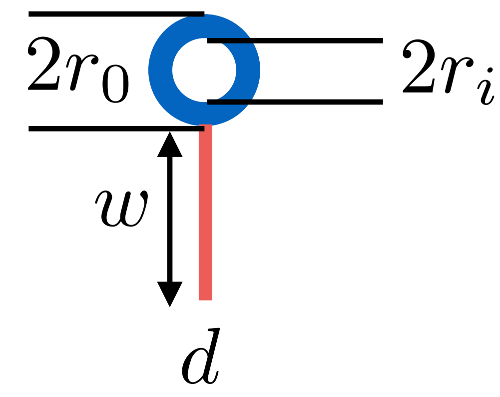
Chick guts at E12

What is the wavelength of this oscillations?

T. Savin et al., Nature 476, 57 (2011)

Compression of stiff tube on soft elastic mesentery sheet





assumed profile $h(s) = h_0 \cos(2\pi s/\lambda)$

amplitude of wrinkles

$$h_0 = \frac{\lambda}{\pi}\sqrt{\frac{\Delta}{L}} = \frac{\lambda\sqrt{\epsilon}}{\pi}$$

deformation of the soft mesentery decays exponentially away from the surface

$$h(s, y) \approx h_0 \cos(2\pi s/\lambda) e^{-y/\lambda}$$

bending energy of stiff tube

$$U_b \sim L \times \kappa_t \times \frac{1}{R^2} \sim L \times E_t I_t \times \frac{h_0^2}{\lambda^4} \sim \frac{L E_t I_t \epsilon}{\lambda^2}$$

deformation energy of soft mesentery

$$U_m \sim A \times E_m d \times \epsilon_m^2 \sim L\lambda \times E_m d \times \frac{h_0^2}{\lambda^2} \sim L E_m d \lambda \epsilon$$

minimize total energy (U_b+U_m) with respect to λ

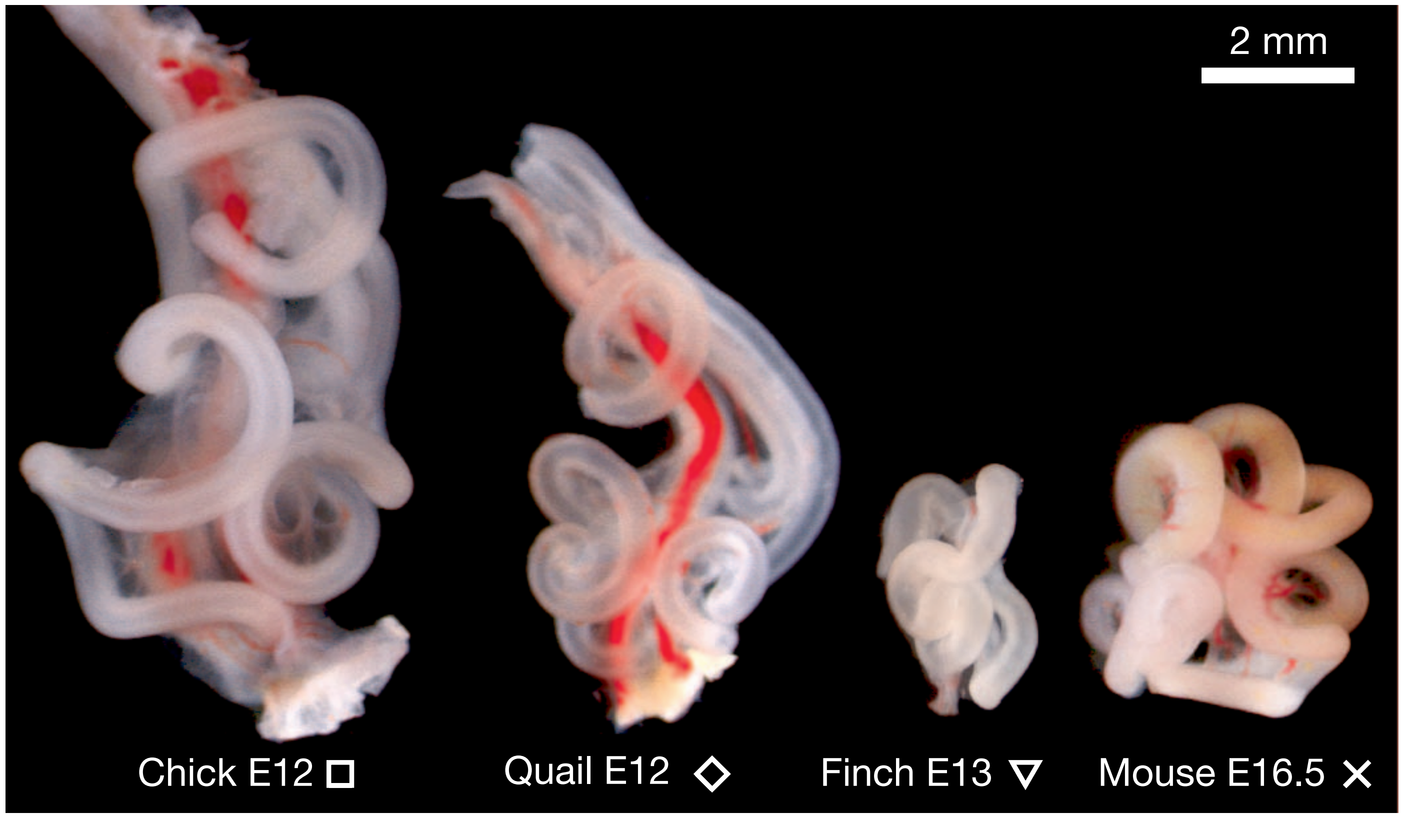
$$\lambda \sim \left(\frac{E_t I_t}{E_m d}\right)^{1/3}$$

bending stiffness of tube

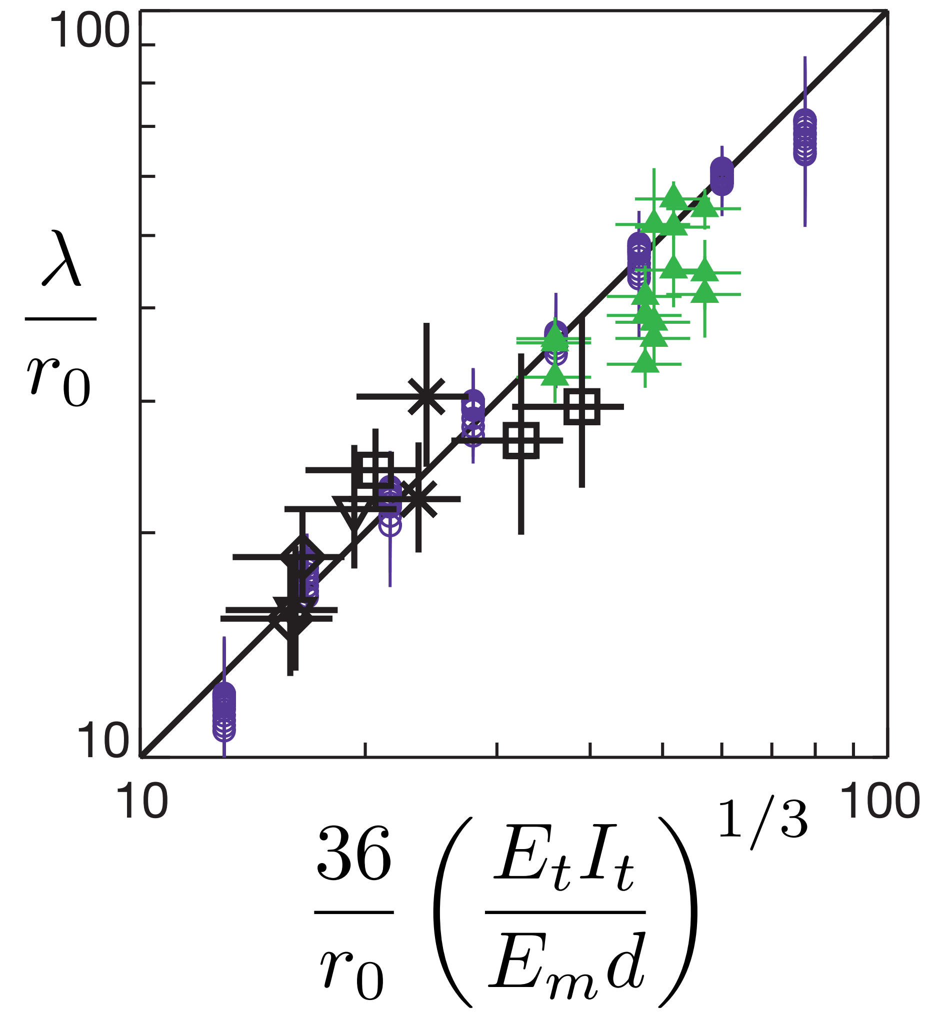
$$\kappa_t = E_t I_t$$

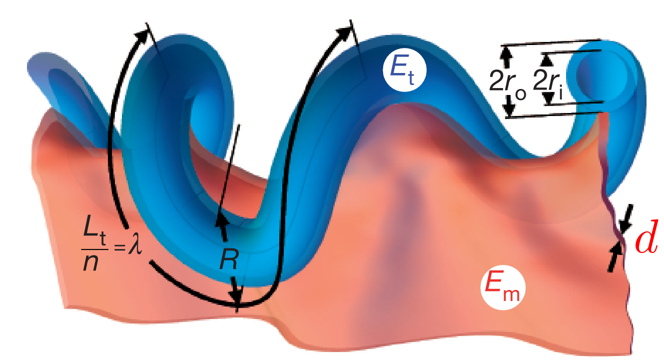
$$\kappa_t \propto E_t(r_0^4 - r_i^4)$$

Wavelength of oscillations in guts



animal data, rubber model, computer simulations



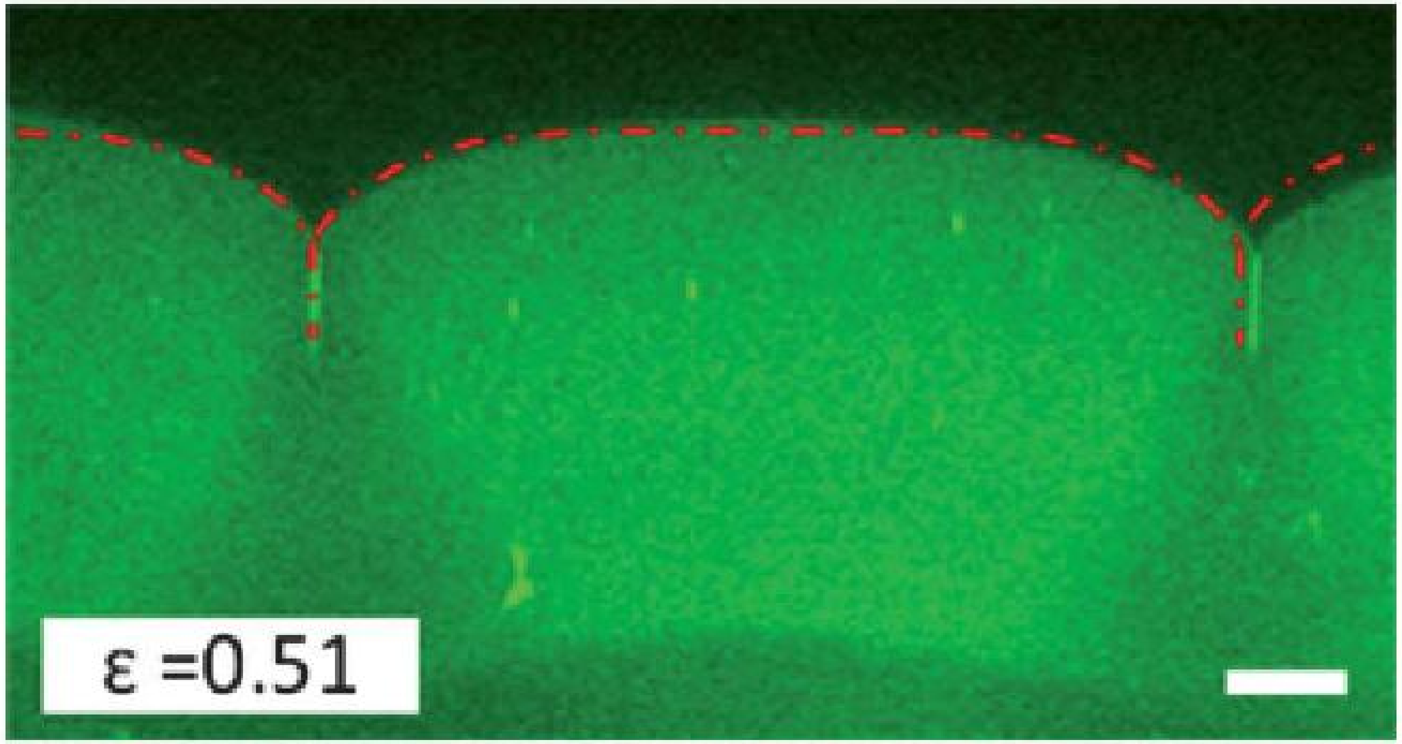


T. Savin et al., Nature 476, 57 (2011)

Compression of soft elastic material

When soft elastic material is compressed by more than 35% surface forms sharp creases. This is effect of nonlinear elasticity!

swollen gel
on a stiff substrate

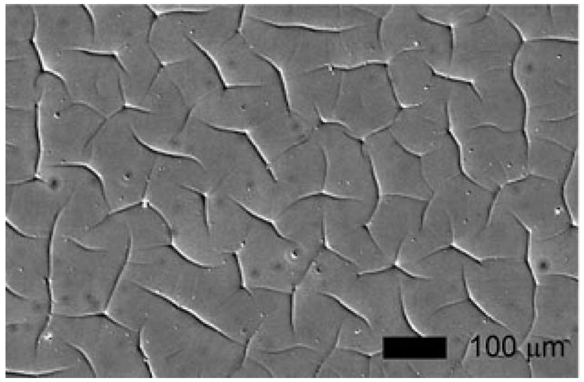


arm of
an infant

rising
dough

Liangfen
(starch jelly)

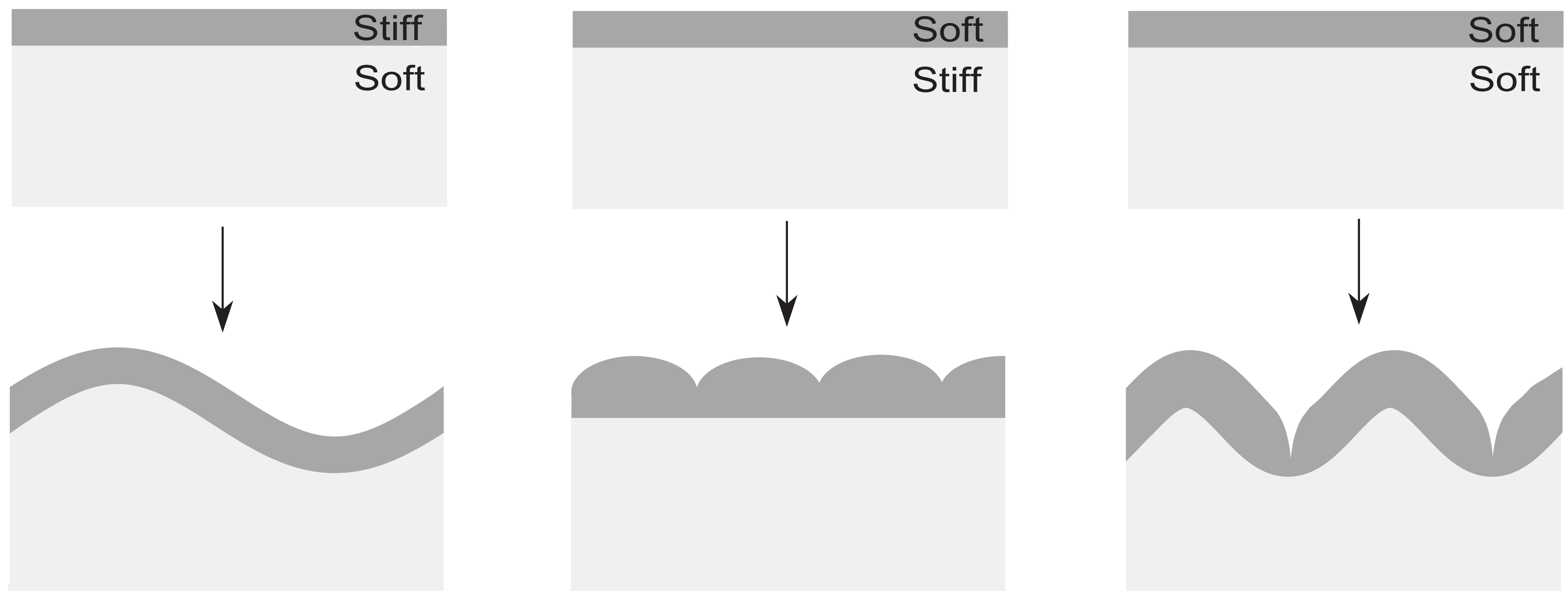
swollen gel
on a stiff substrate



Swelling of thin membranes on elastic substrates

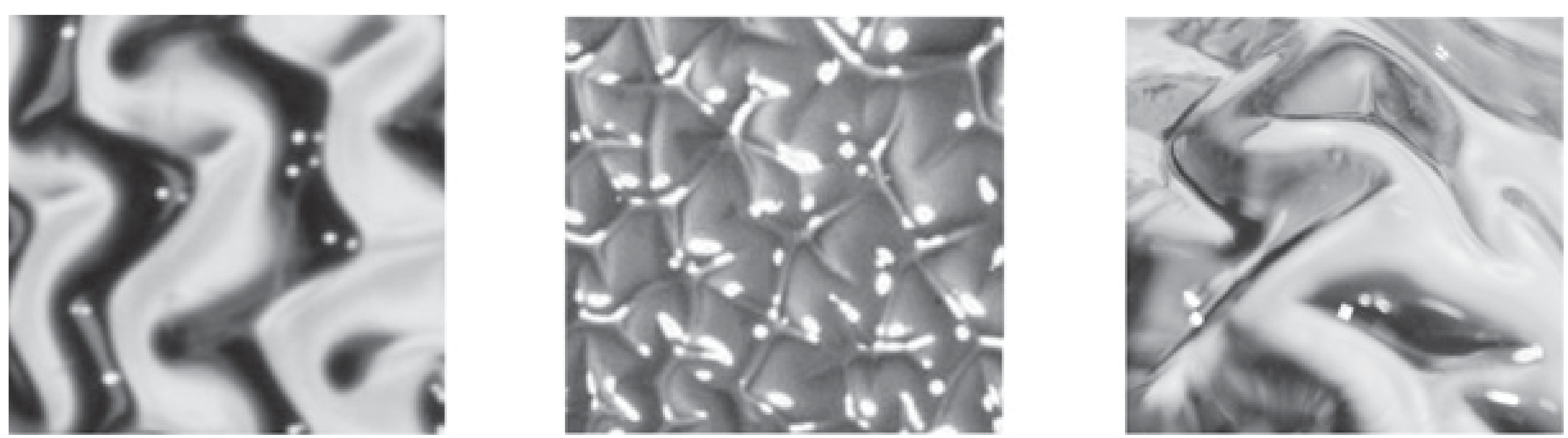
stiff membrane soft substrate	soft membrane stiff substrate	soft membrane soft substrate

swelling



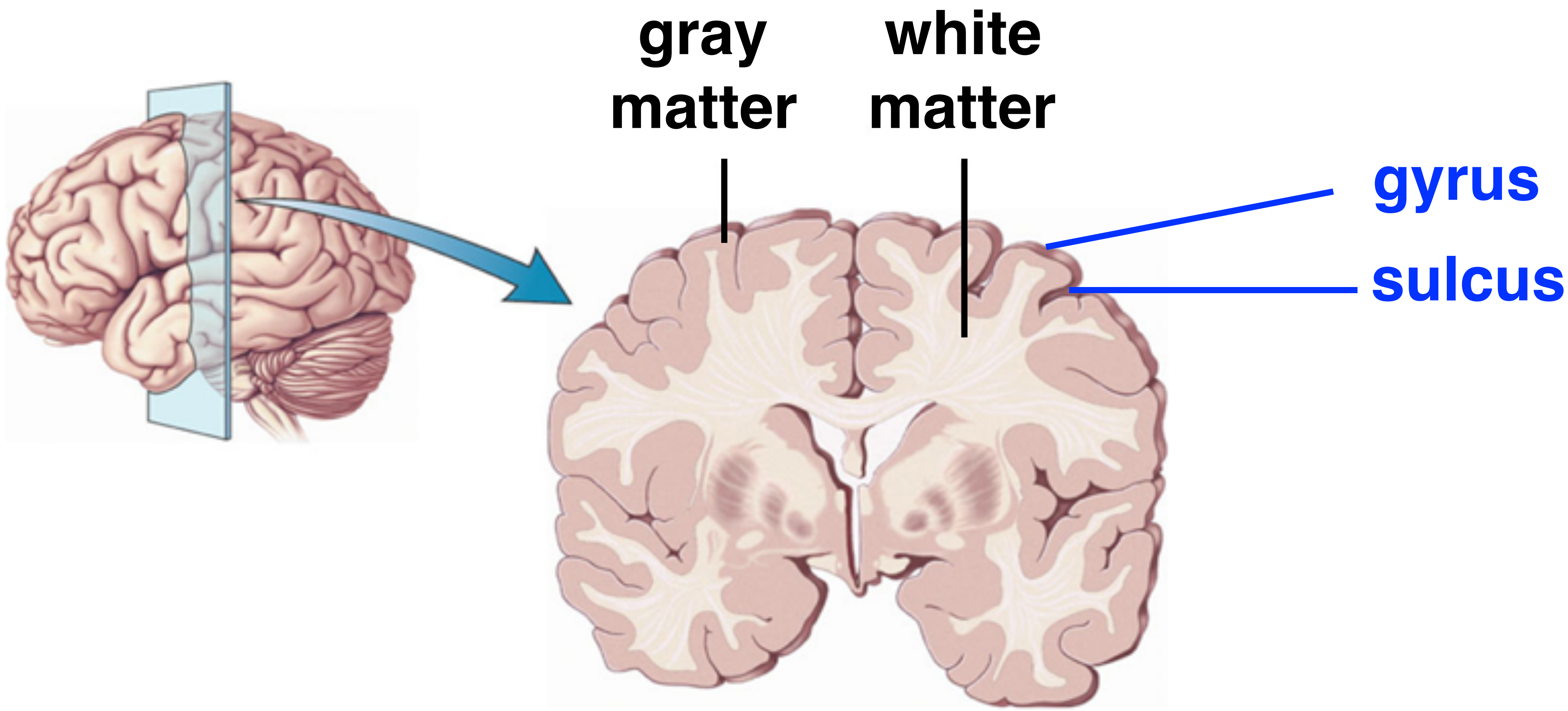
wrinkles	creases	wrinkles +creases

swelling of gels

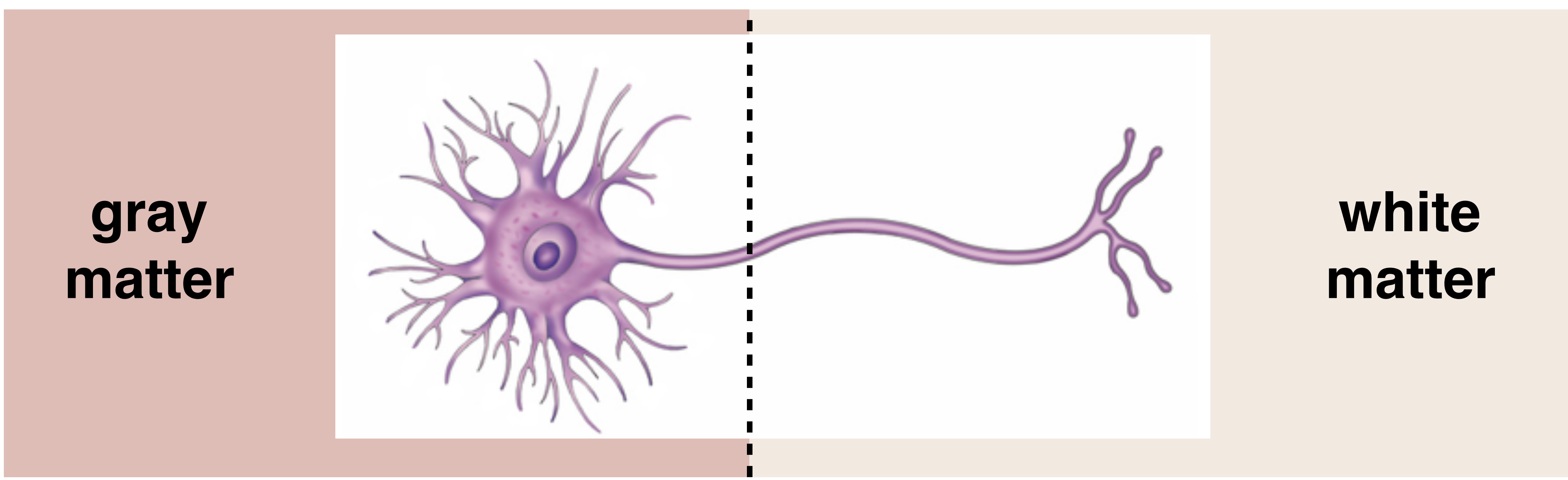


T. Tallinen et al., PNAS 111, 12667 (2014)

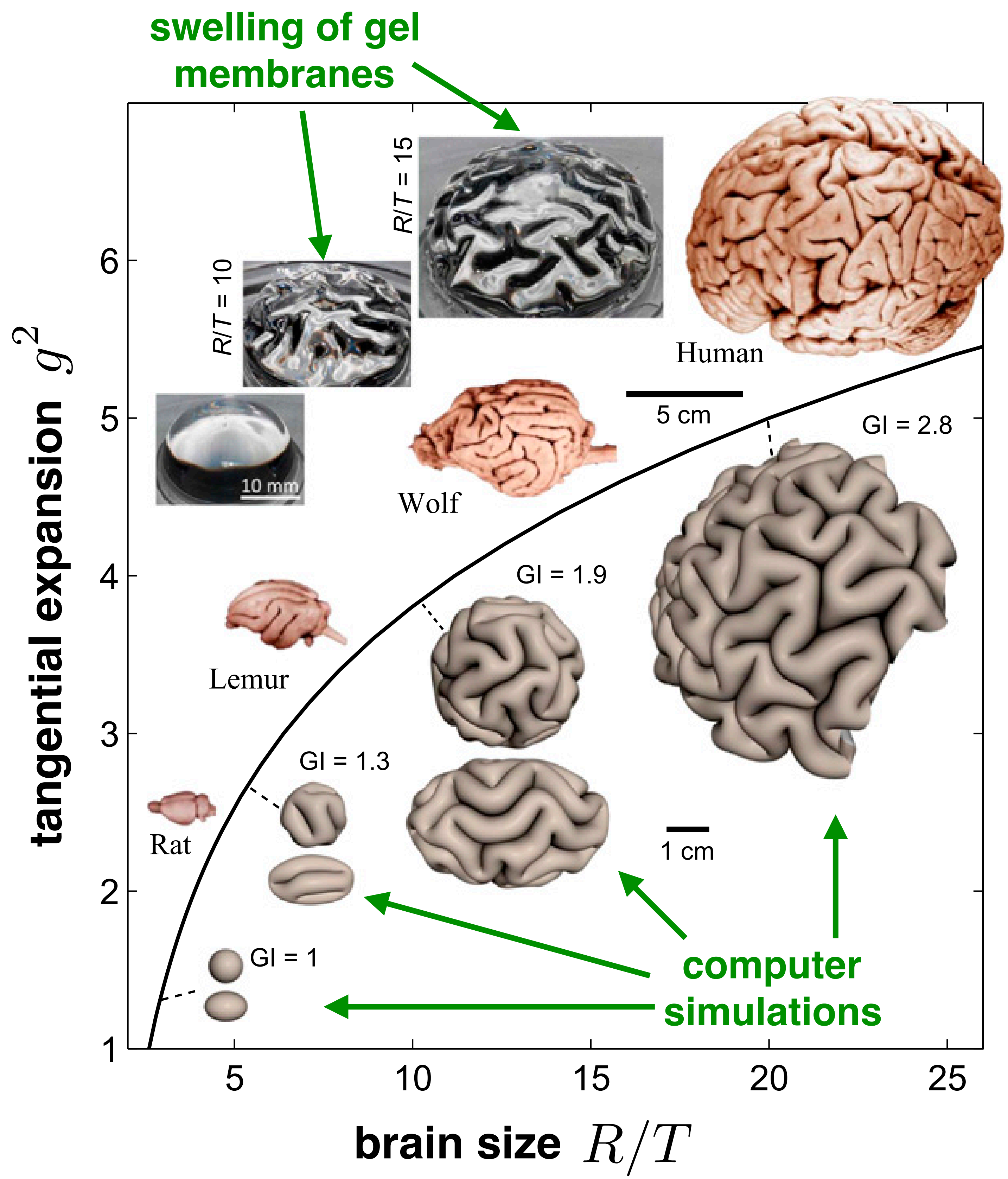
Gyri and Sulci of Brains



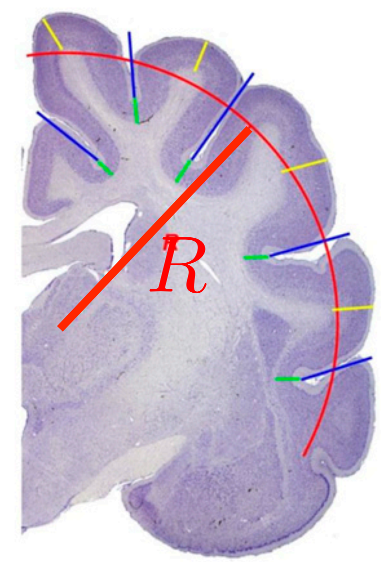
Migration of neurons to the cortex leads to swelling of gray matter!



Brains for various organisms



measurements of brain parameters



***R*: brain size**

***T*: thickness of gray matter**

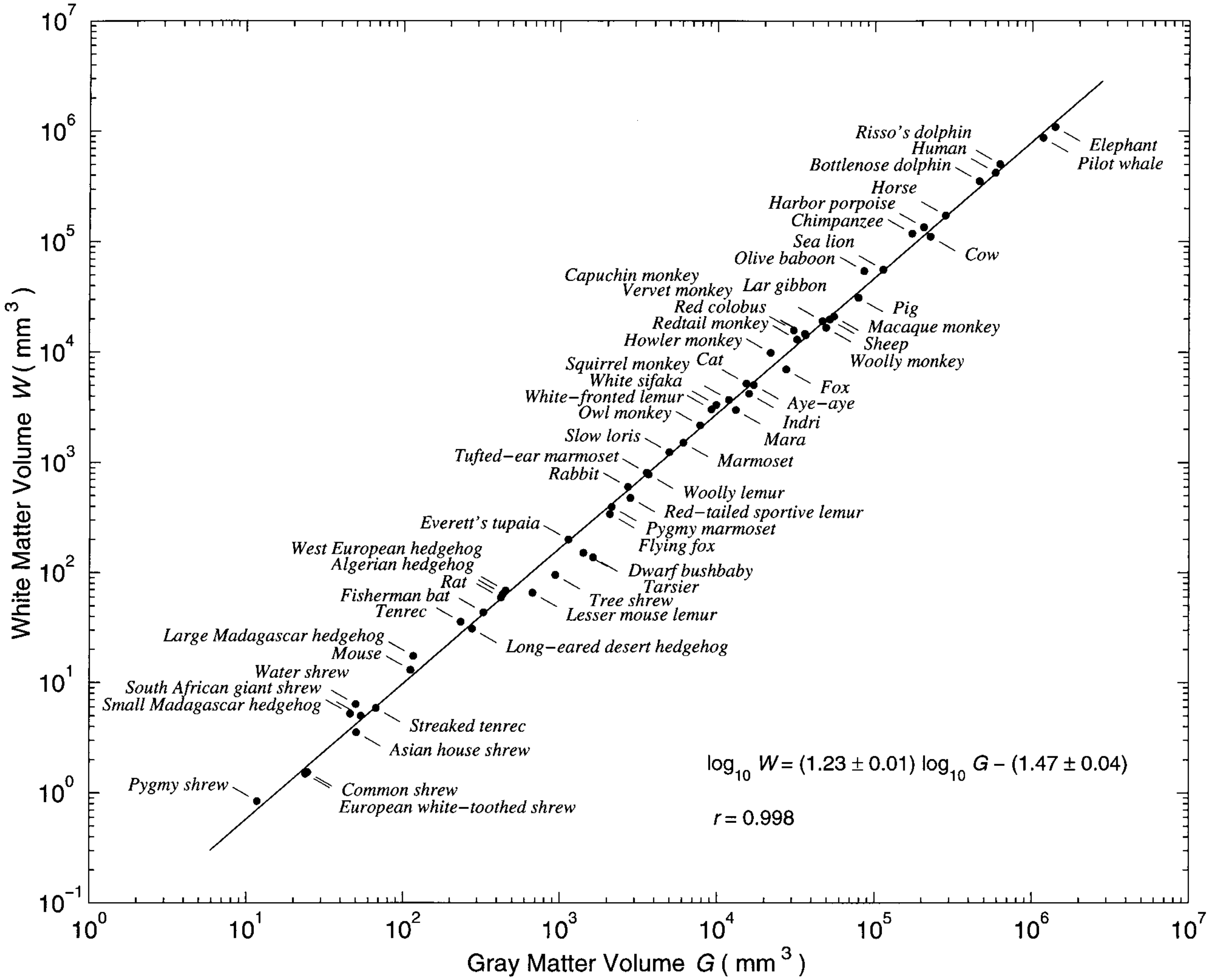
tangential expansion $g = \dfrac{\text{contour length of gray matter}}{\text{length of circular section}}$

gyrification index $\mathrm{GI} = \dfrac{\text{area of brain surface}}{\text{area of convex hull}}$

T. Tallinen et al., PNAS 111, 12667 (2014)

Power law scaling for the brain size of various organisms

$$\text{white matter volume} \propto \left(\text{gray matter volume}\right)^{1.23}$$



another scaling relation

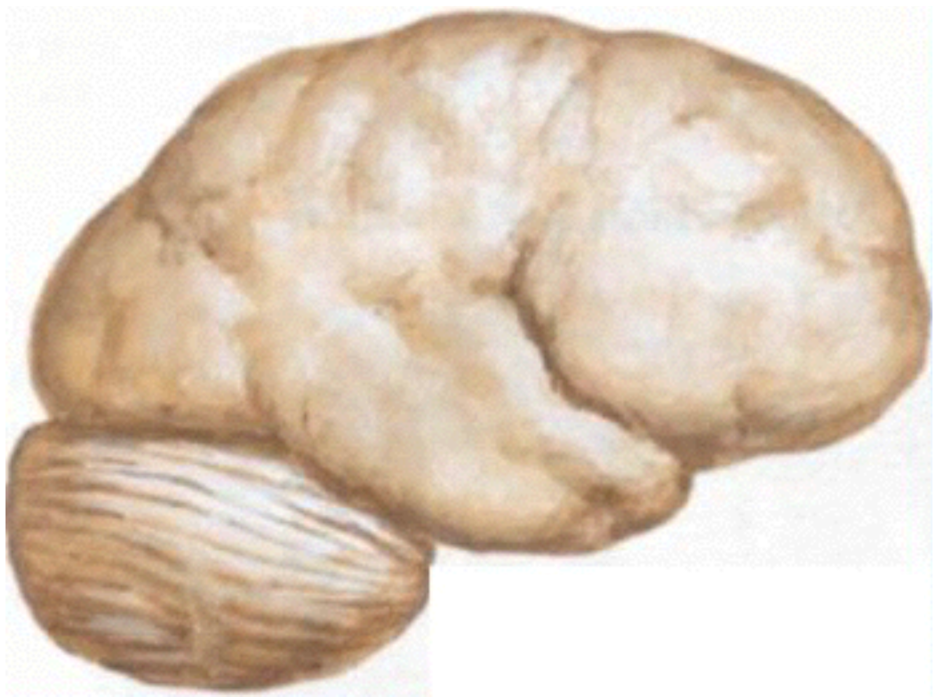
$$\text{gray matter thickness} \propto \left(\text{gray matter volume}\right)^{0.10}$$

Note: power law scalings of various quantities among organisms are very common!

K. Zhang and T.J. Sejnowski, PNAS 97, 5621 (2000)

Brain malformations

lissencephaly
pachygyria
(small number of larger gyri)

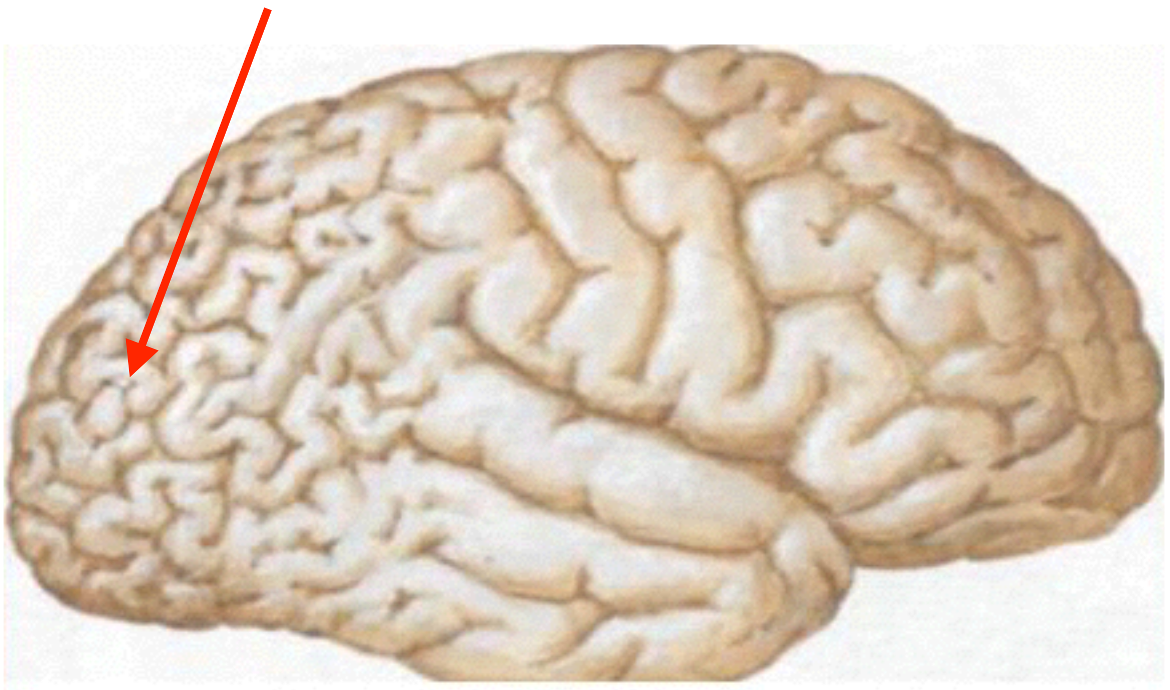


Reduced neuronal migration to cortex

↓

Gray matter is thicker and it swells less!

polymicrogyria
(large number of smaller gyri)



Typically gray matter has only four rather than six layers in the affected areas.