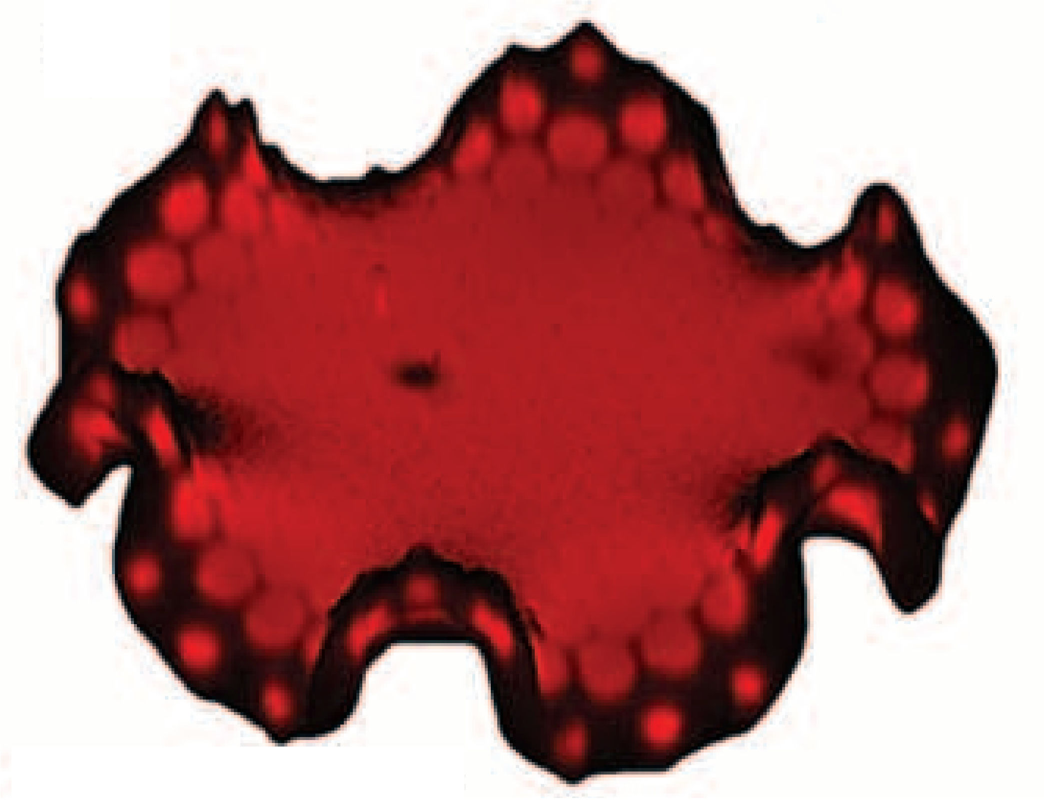
MAE 545: Lecture 20 (12/3)

Shapes of growing/swelling sheets and coiling of rods



Shapes of flowers and leaves

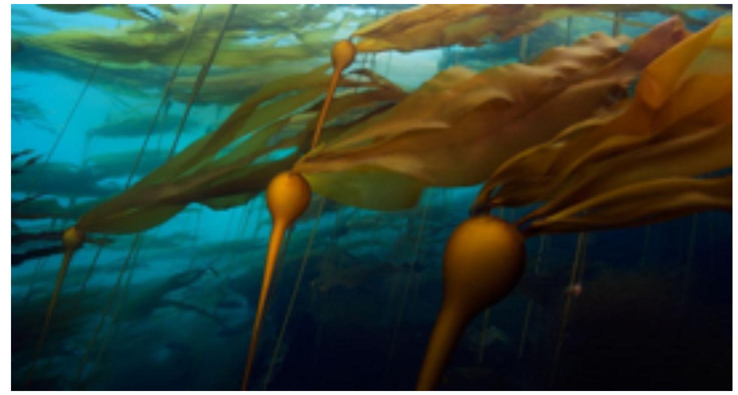
saddles

wrinkled edges

helices

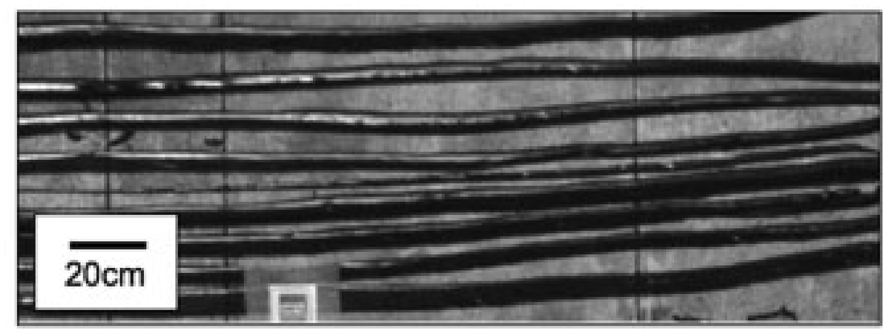
Wrinkled and straight blades in macroalgae

bull kelp (seaweed)



Slow water flow environment (v~0.5 m/s)

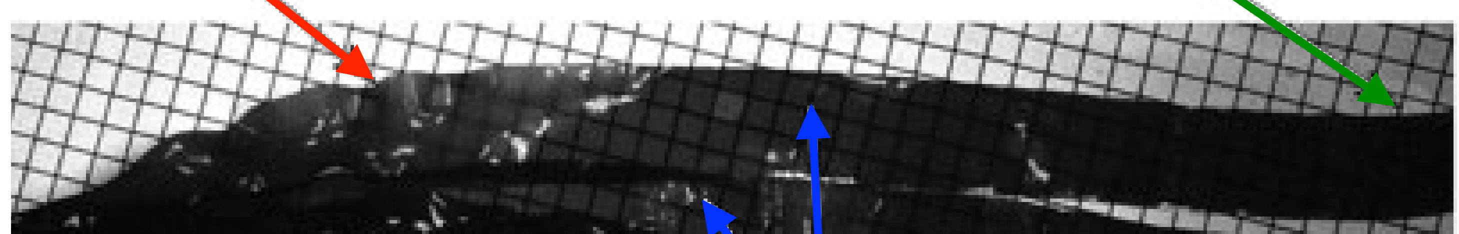
Fast water flow environment (v~1.5 m/s)



new growth after transplantation (wrinkled)

old growth before transplanted (flat)

Transplantation of blade from one environment to the other changes morphology!

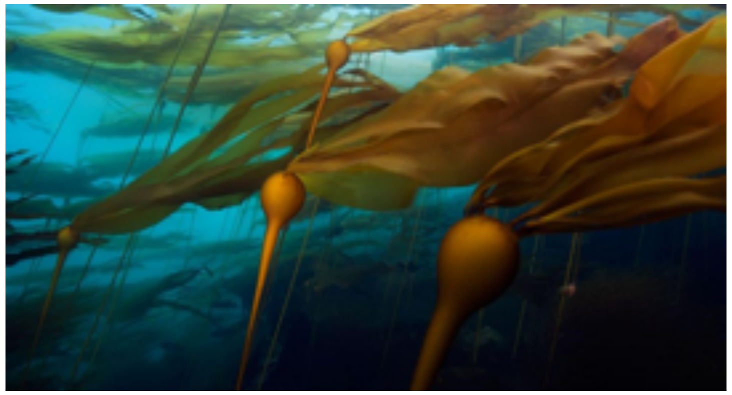


blades

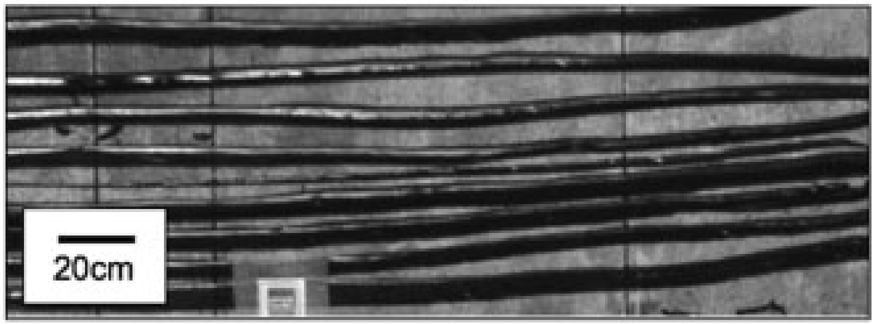
M. Koehl et al., Integ. Comp. Biol. 48, 834 (2008)

Wrinkled and straight blades in macroalgae

bull kelp
(seaweed)



Slow water flow environment (v~0.5 m/s)	Fast water flow environment (v~1.5 m/s)



Slow water flow environment (v~0.5 m/s)

increased drag

blades flap like flags

flapping prevents bundling of blades, which can thus receive more sunlight (photosynthesis)

Fast water flow environment (v~1.5 m/s)

reduced drag to prevent detachment from base (=death)

minimal flapping

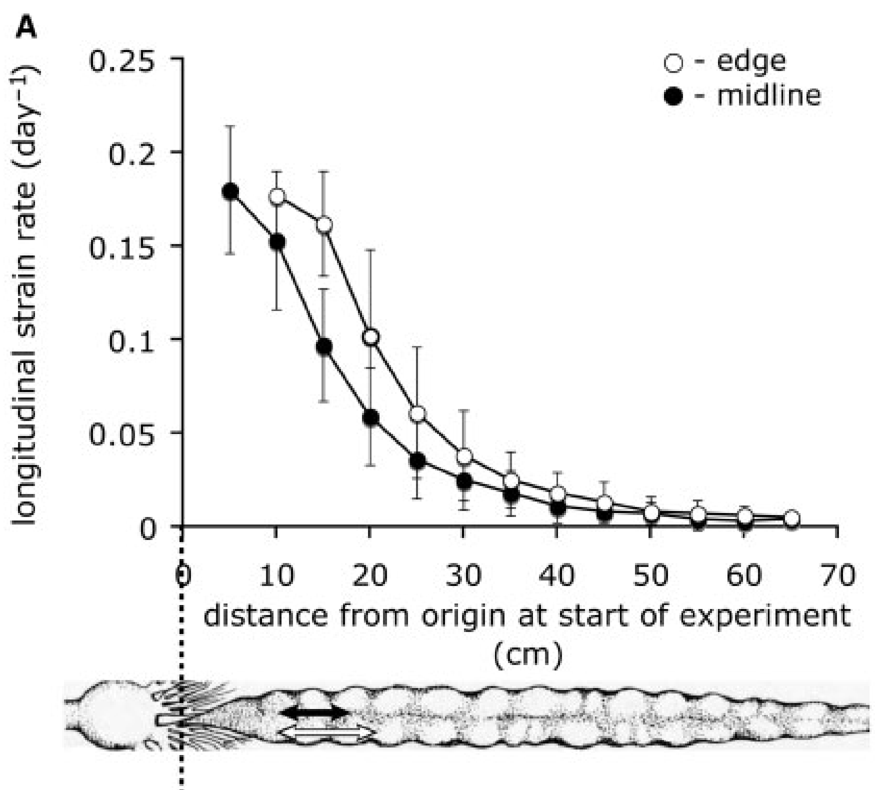
blades bundle together and some blades on the bottom receive less sunlight

M. Koehl et al., Integ. Comp. Biol. 48, 834 (2008)

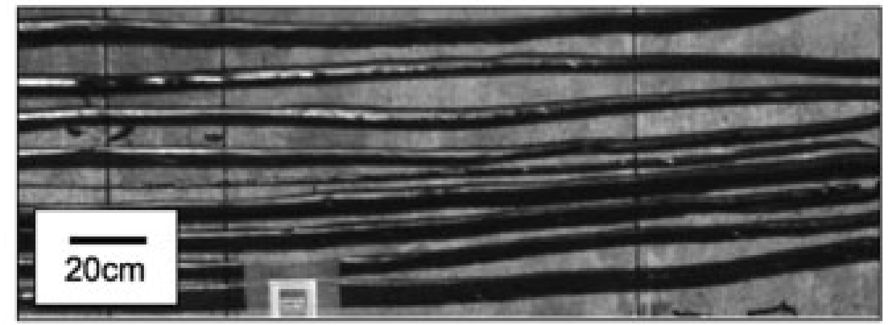
Wrinkled and straight blades in macroalgae

Slow water flow environment (v~0.5 m/s)

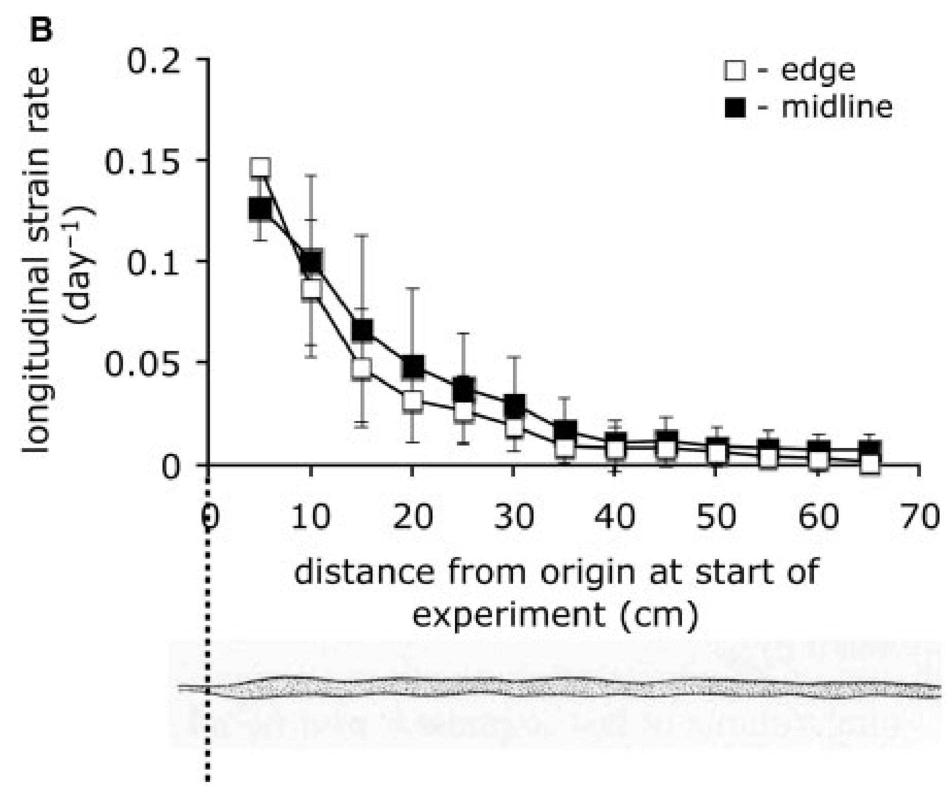
edges of blades grow faster than the midline



Fast water flow environment (v~1.5 m/s)



edges of blades grow at the same speed as the midline

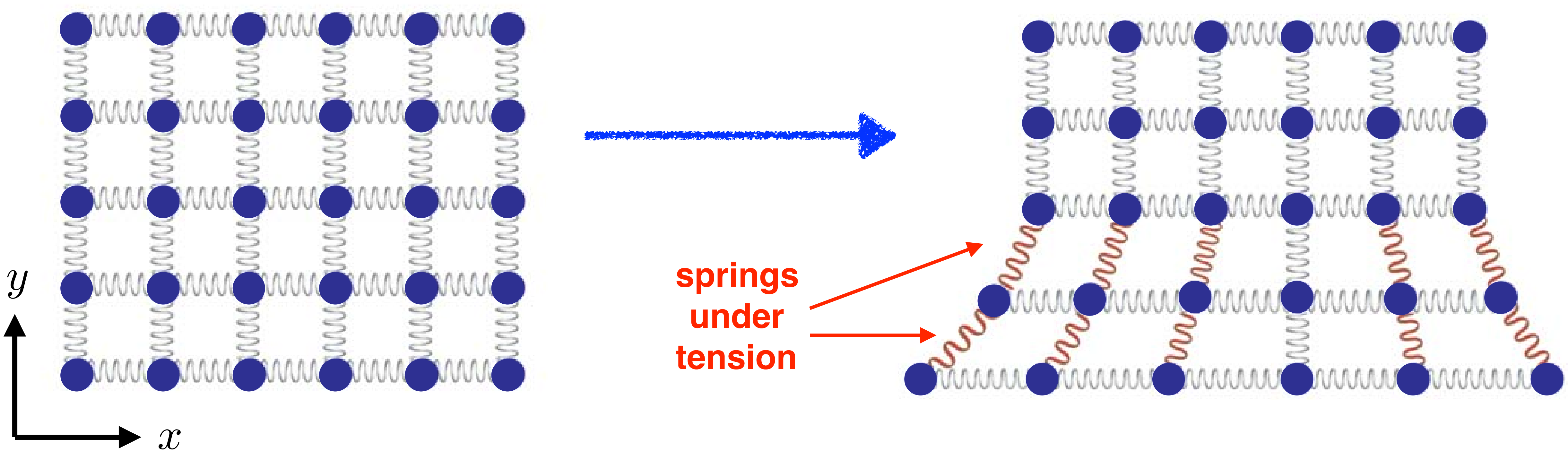


M. Koehl et al., Integ. Comp. Biol. 48, 834 (2008)

Differential growth produces internal stress

before growth

faster growth of the bottom edge in x direction



$$d\ell^2 = \sum_{i,j} g_{ij}\, dx^i dx^j$$

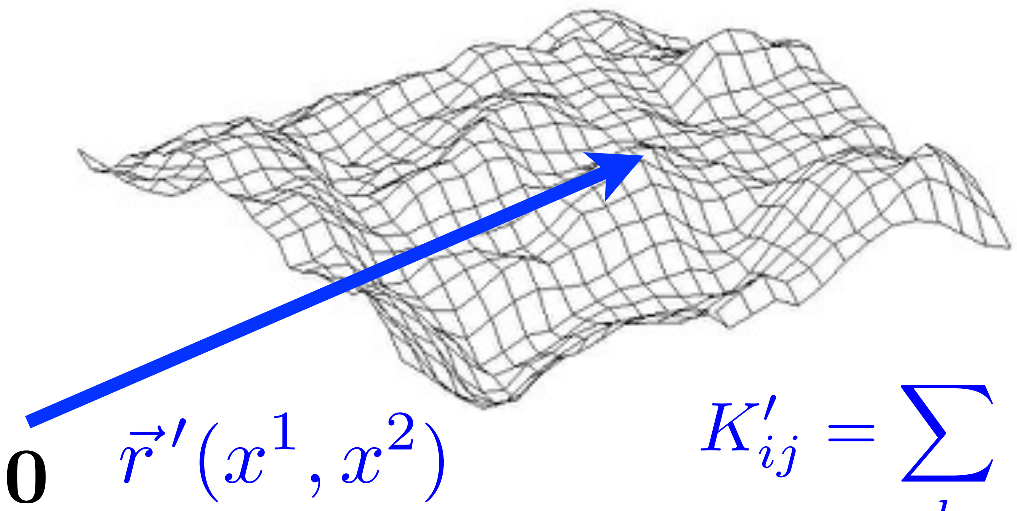
Growth modifies the metric tensor of sheet!

$$g_{ij} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} \longrightarrow g_{ij} = \begin{pmatrix} f(y), & 0 \\ 0, & 1 \end{pmatrix}$$

Note: If growth is different between the top and bottom of the sheet, then the curvature tensor K_{ij} is modified as well!

Mechanics of growing membranes

Growth defines preferred metric tensor g_{ij} and preferred curvature tensor K_{ij}.



$$g'_{ij} = \frac{\partial \vec{r}\,'}{\partial x^i} \cdot \frac{\partial \vec{r}\,'}{\partial x^j}$$

$$K'_{ij} = \sum_k \left(g'^{-1}\right)_{ik} \left(\vec{n}' \cdot \frac{\partial^2 \vec{r}\,'}{\partial x^k \partial x^j}\right)$$

strain tensors

$$u_{ij} = \frac{1}{2} \sum_k \left(g^{-1}\right)_{ik} \left(g'_{kj} - g_{kj}\right)$$

$$b_{ij} = K'_{ij} - K_{ij}$$

The equilibrium membrane shape $\vec{r}\,'(x^1, x^2)$ corresponds to the minimum of elastic energy:

$$U = \int \sqrt{g}\, dx^1 dx^2 \frac{1}{2} \left[(B - \mu)\mathrm{tr}(u_{ij})^2 + 2\mu \sum_{i,j} u_{ij}^2 + \kappa \mathrm{tr}(b_{ij})^2 + 2\kappa_G \det(b_{ij}) \right]$$

Growth can independently tune the metric tensor g_{ij} and the curvature tensor K_{ij}, which may not be compatible with any surface shape that would produce zero energy cost!

Zero energy shape exists only when preferred metric tensor g_{ij} and preferred curvature tensor K_{ij} satisfy Gauss-Codazzi-Mainardi relations!

Mechanics of growing membranes

One of the Gauss-Codazzi-Mainardi equations (Gauss's Theorema Egregium) relates the Gauss curvature to metric tensor

$$\det(K'_{ij}) = \mathcal{F}(g'_{ij})$$

The equilibrium membrane shape $\vec{r}\,'(x^1, x^2)$ corresponds to the minimum of elastic energy:

$$U = \int \sqrt{g}dx^1 dx^2 \frac{1}{2}\left[(B-\mu)\mathrm{tr}(u_{ij})^2 + 2\mu\sum_{i,j} u_{ij}^2 + \kappa \mathrm{tr}(b_{ij})^2 + 2\kappa_G \det(b_{ij})\right]$$

scaling with membrane thickness d

$$B, \mu \sim Ed$$

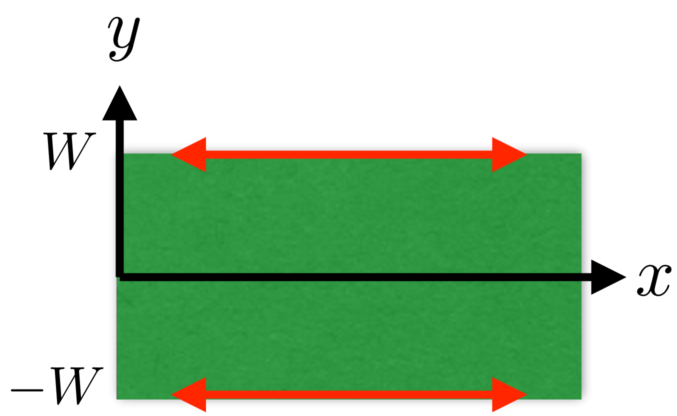
$$\kappa, \kappa_G \sim Ed^3$$

For very thin membranes the equilibrium shape matches the preferred metric tensor to avoid stretching, compressing and shearing. This also specifies the Gauss curvature!

$$g'_{ij} = g_{ij}$$

$$\det(K'_{ij}) = \mathcal{F}(g_{ij})$$

Example



Assume that differential growth in x direction produces metric tensor of the form

$$g_{ij} = \begin{pmatrix} f(y), & 0 \\ 0, & 1 \end{pmatrix} \qquad f(y) = 1 + ce^{(|y|-W)/\lambda}$$

For thin membranes the metric tensor wants to be matched

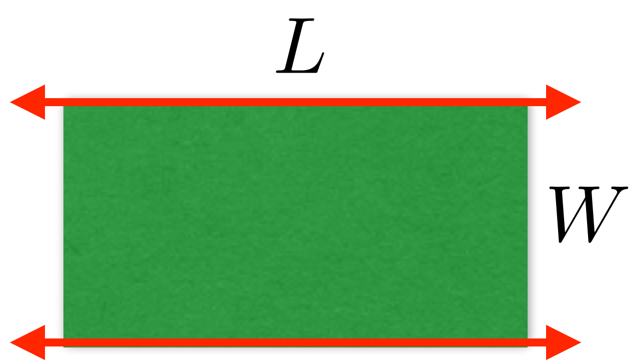
$$g'_{ij} = g_{ij}$$

Gauss's Theorema Egregium provides Gauss curvature

$$\det(K'_{ij}(y)) = \mathcal{F}(g_{ij}) = -\frac{1}{f}\frac{d^2 f(y)}{dy^2} = -\frac{1}{\lambda^2} \times \frac{ce^{(|y|-W)/\lambda}}{\left(1 + ce^{(|y|-W)/\lambda}\right)} < 0$$

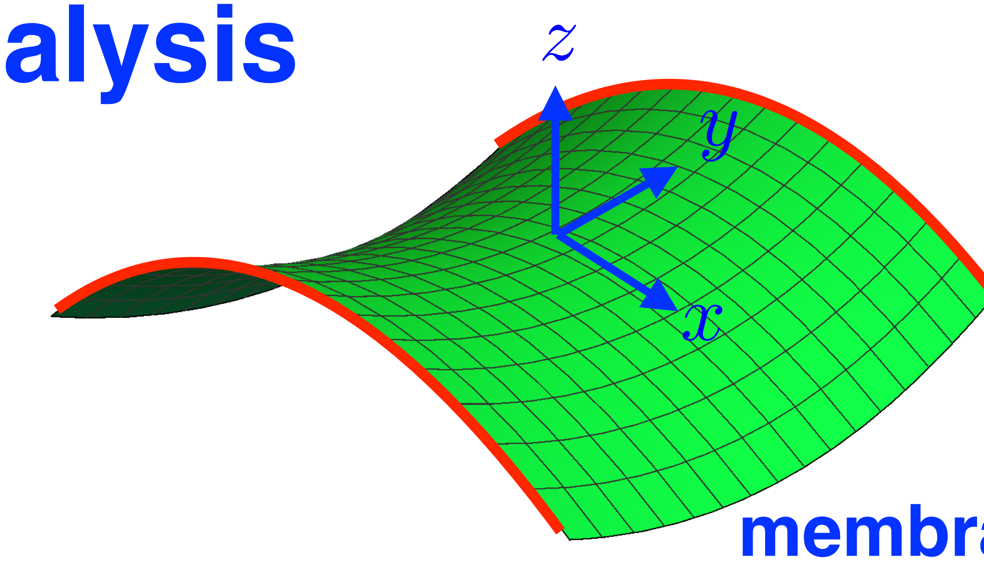
For thin membranes faster growth on edges produces shapes that locally look like saddles!

Scaling analysis



faster growth increases the edge length by factor ϵ

stress released by bending

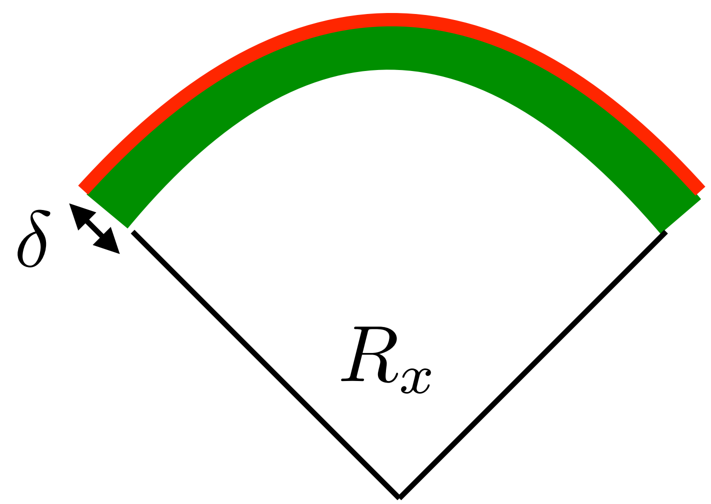


membrane thickness d

y-z cross-section

$$\frac{1}{R_y} \sim \frac{\delta}{W^2}$$

projection to x-z plane



$$\frac{L(1+\epsilon)}{L} \sim \frac{R_x+\delta}{R_x}$$

$$\frac{1}{R_x} \sim \frac{\epsilon}{\delta}$$

Membrane bending energy

$$U_b \sim A \times \kappa \times \left(\frac{1}{R_x^2} + \frac{1}{R_y^2} + \frac{1}{R_x R_y}\right) \sim A \times E_m d^3 \times \left(\frac{\epsilon^2}{\delta^2} + \frac{\delta^2}{W^4} + \frac{\epsilon}{W^2}\right)$$

Minimize U_b with respect to δ :

$$\delta \sim W\sqrt{\epsilon}$$

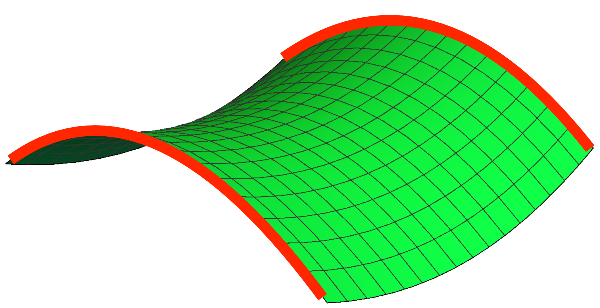
$$U_b \sim \frac{A E_m d^3 \epsilon}{W^2}$$

Scaling analysis

membrane compression

$$U_c \sim A E_m d \epsilon^2$$

membrane bending



$$U_b \sim \frac{A E_m d^3 \epsilon}{W^2}$$

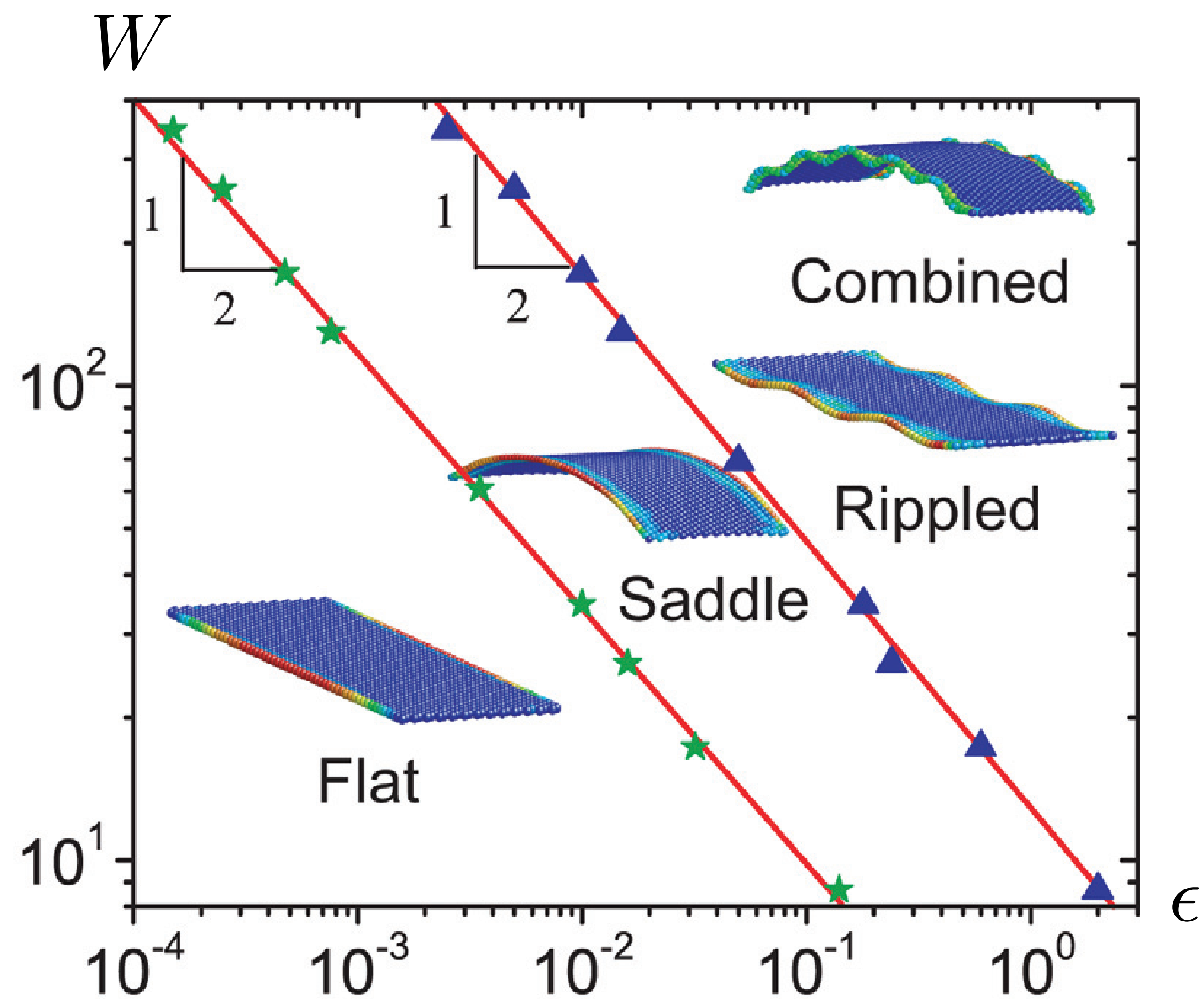
membrane bends above the critical strain

$$\epsilon > \epsilon_c \sim \frac{d^2}{W^2}$$

amplitude of bending at the critical strain

$$\delta^* \sim W \sqrt{\epsilon_c} \sim d$$

numerical simulations



H. Liang and L. Mahadevan, PNAS 106, 22049 (2009)

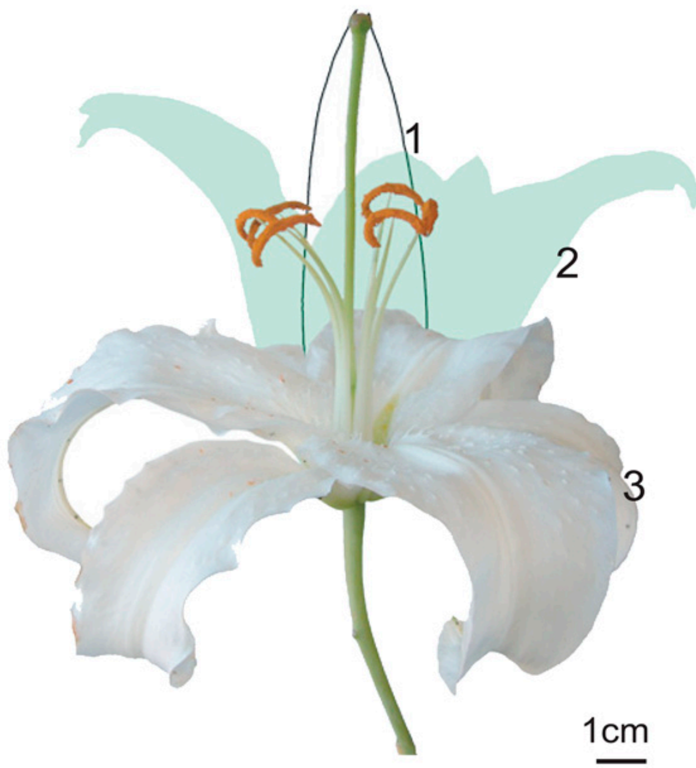
Shapes of flowers and leaves

Faster growth of the edge is consistent with observed saddles and edge wrinkles, which indeed correspond to the negative Gauss curvature!

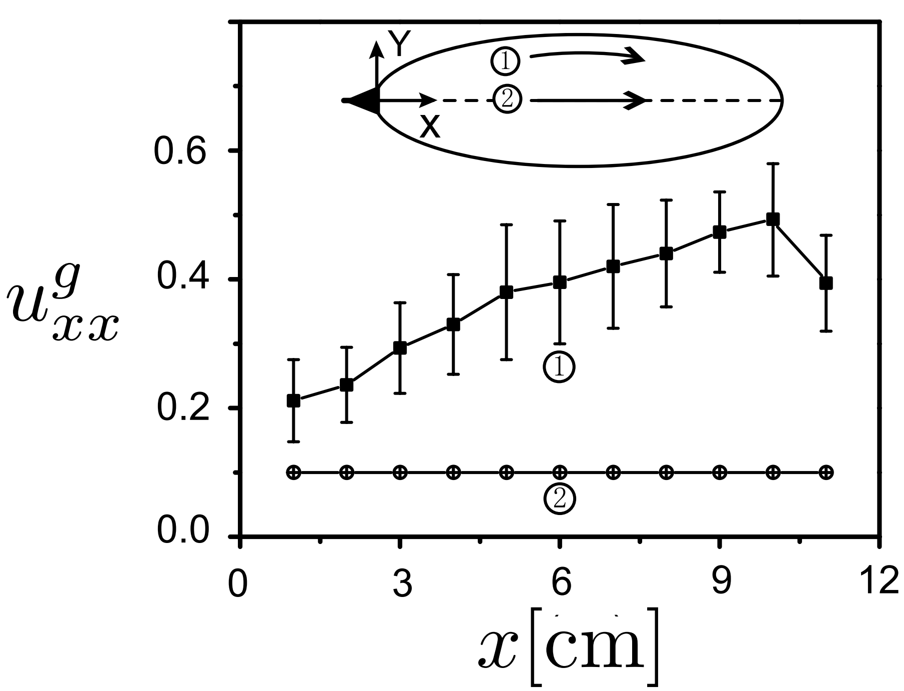
saddles

wrinkled edges (+saddles)

Growth of a blooming lily



faster growth of the edge

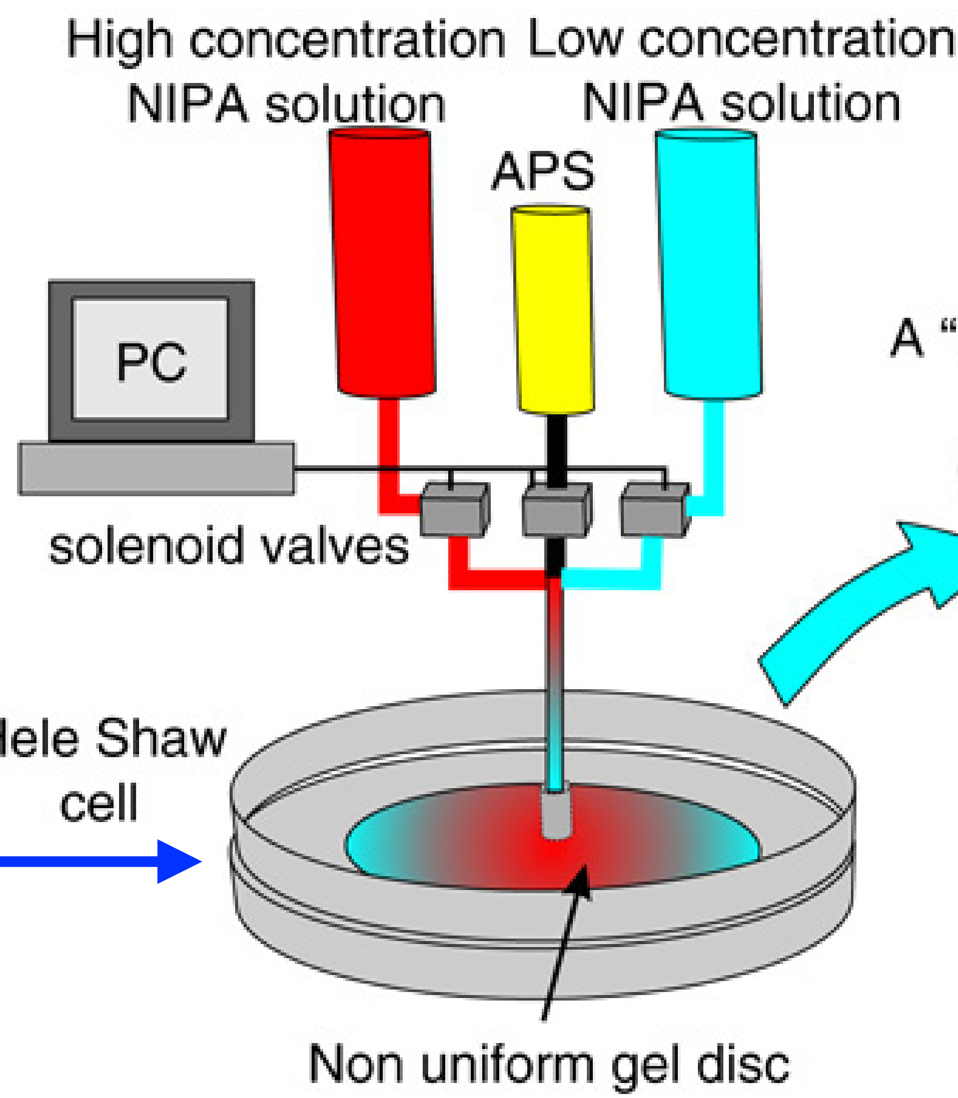


in lab blooming takes 4.5 days under constant fluorescent light (1 frame/min)

H. Liang and L. Mahadevan, PNAS 108, 5516 (2011)

Shaping of gel membranes by differential shrinking

Computer software controls valves to inject a predefined time depend concentration of NIPA polymers in water solution.

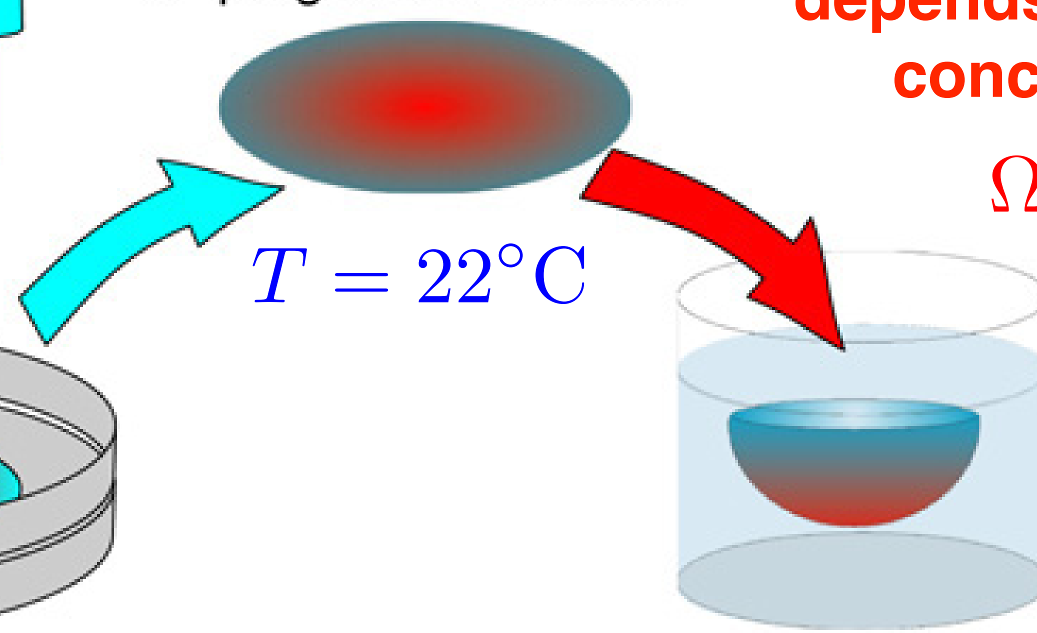


thickness 0.25 or 0.5 mm

Frozen NIPA concentration profile

$C(r)$

A “programed” flat disc



At higher temperatures gel shrinks because some of the water gets expelled. Shrinking depends on the NIPA concentration.

$\Omega(C(r))$

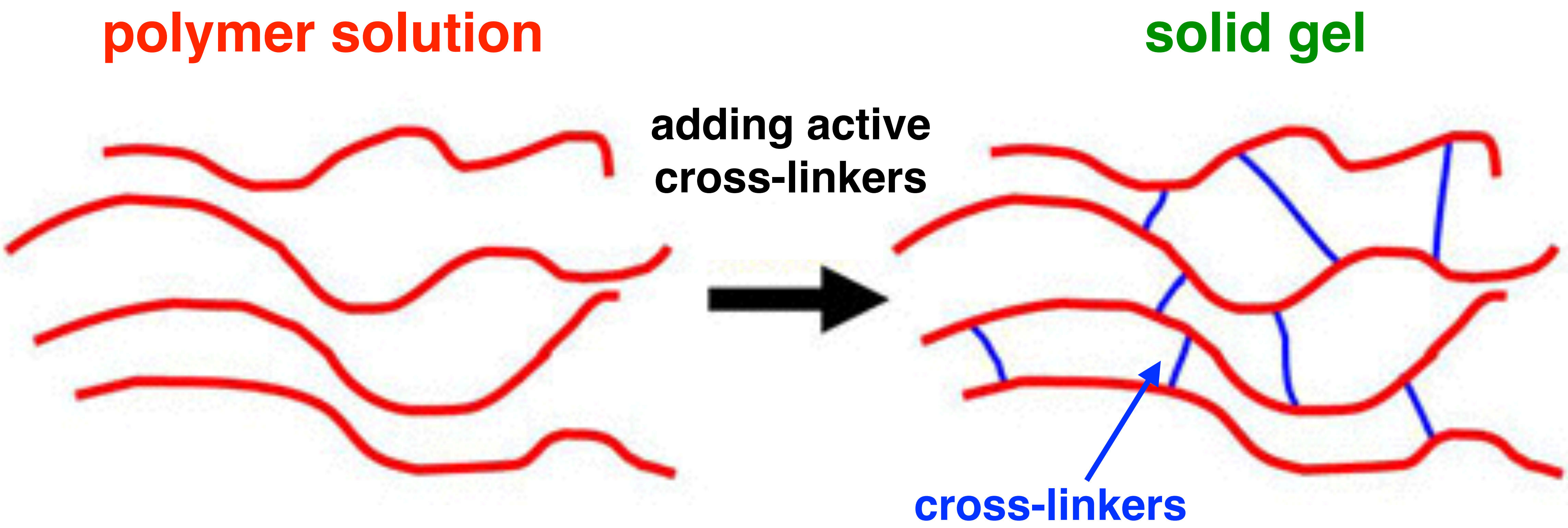
“Activation” of the metric” in hot water

$T = 45°\mathrm{C}$

Active cross-linkers (APS) polymerize the polymer solution within one minute, before polymers get a chance to diffuse around.

E. Efrati et al., Physica D 235, 29 (2007)

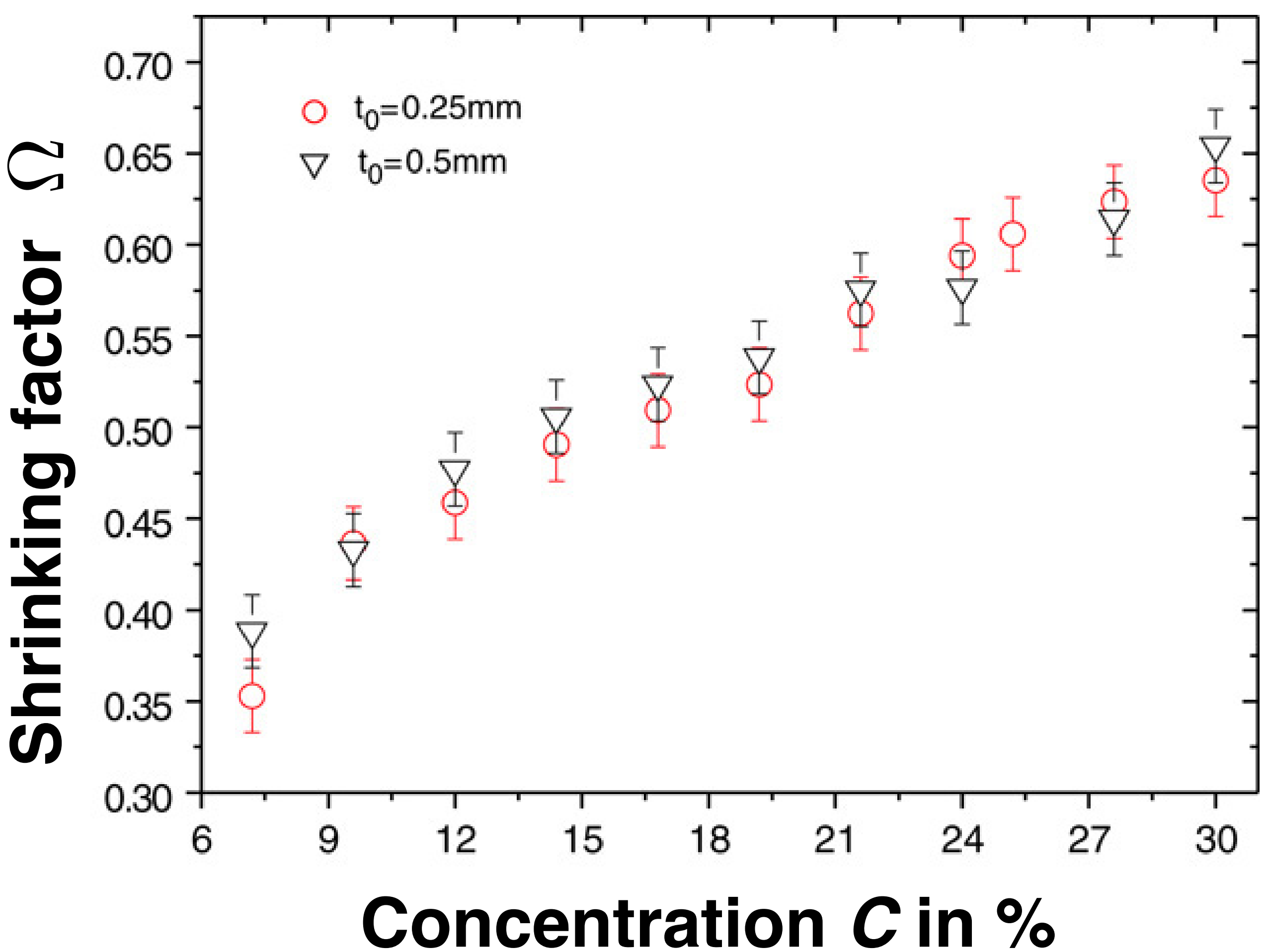
Cross-linking of polymers result in a solid gel



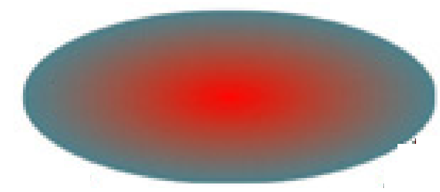
Note: some cross-linkers can be chemically activated by UV light exposure. Duration of UV light exposure controls the degree of cross-linking and therefore the Young's modulus *E* for gels.

Shaping of gel membranes by differential shrinking

Shrinking of gels at T=45°C



$T = 22°\mathrm{C}$

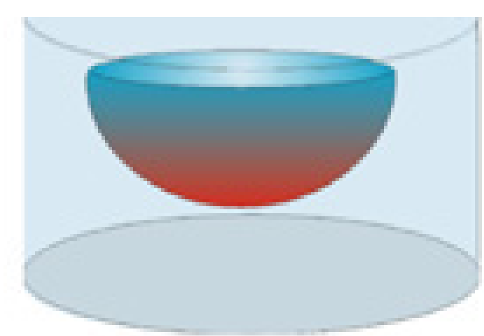


$C(r)$

$$g_{ij} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

Shrinking modifies metric

$T = 45°\mathrm{C}$



$$g_{ij} = \begin{pmatrix} \Omega(r), & 0 \\ 0, & \Omega(r) \end{pmatrix}$$

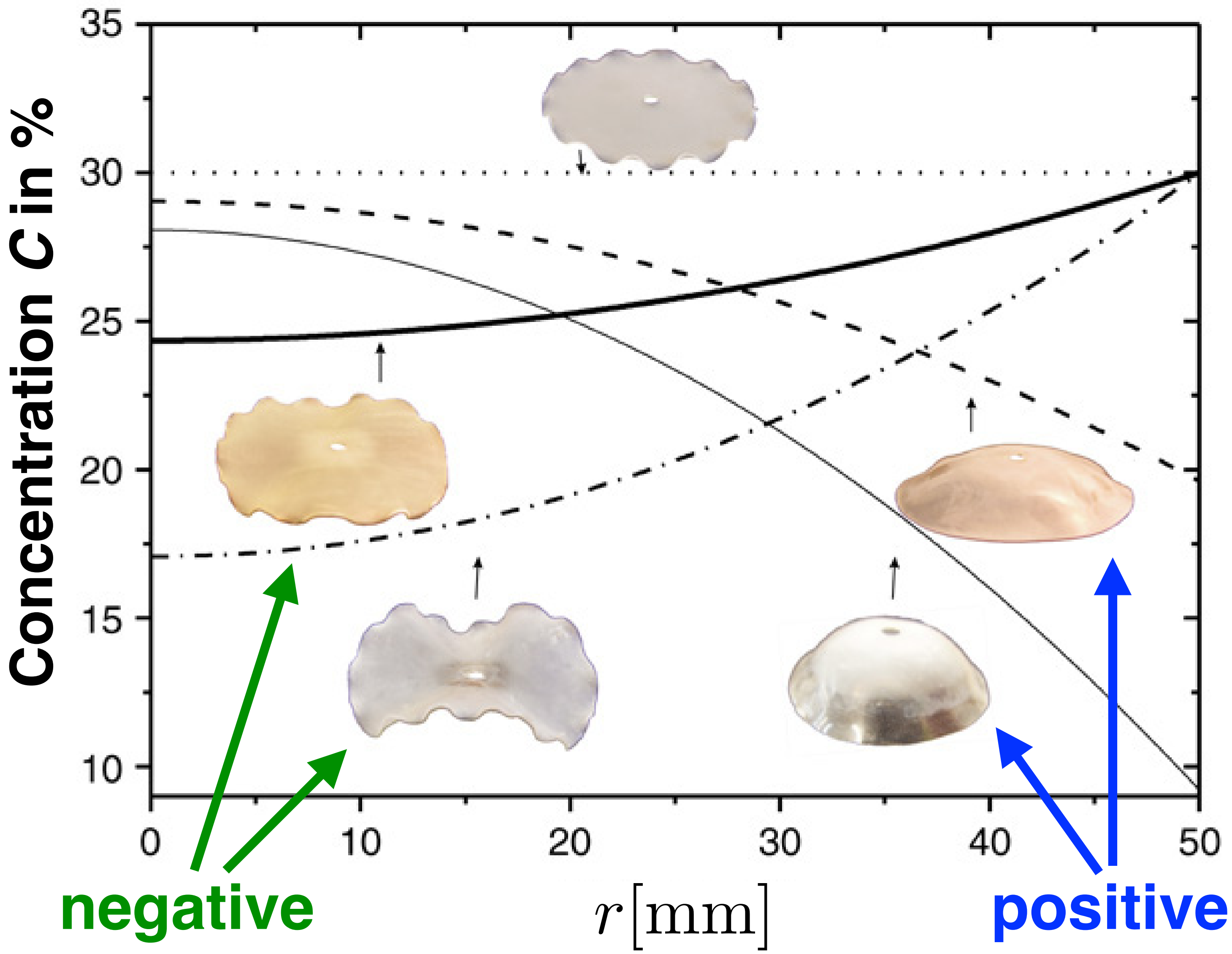
locally isotropic shrinking

For thin membranes the target Gauss curvature is

$$\det(K'_{ij}(r)) = -\frac{\nabla^2(\ln\Omega(r))}{2\Omega(r)}$$

E. Efrati et al., Physica D 235, 29 (2007)

Shaping of gel membranes by differential shrinking

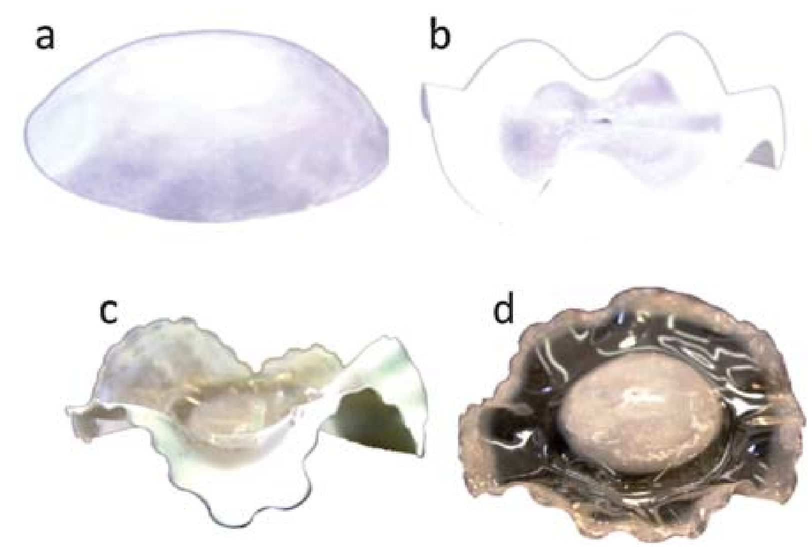


negative Gauss curvature

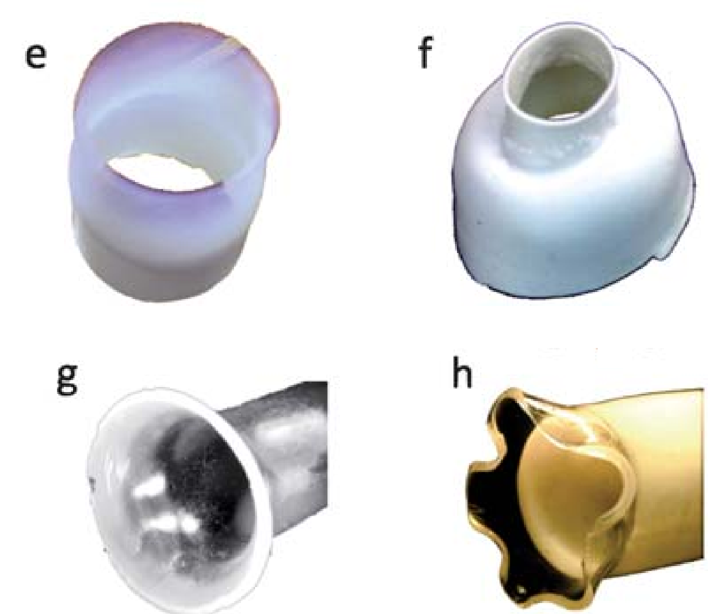
positive Gauss curvature

E. Efrati et al., Physica D 235, 29 (2007)

Shrinking of sheets



Shrinking of tubes

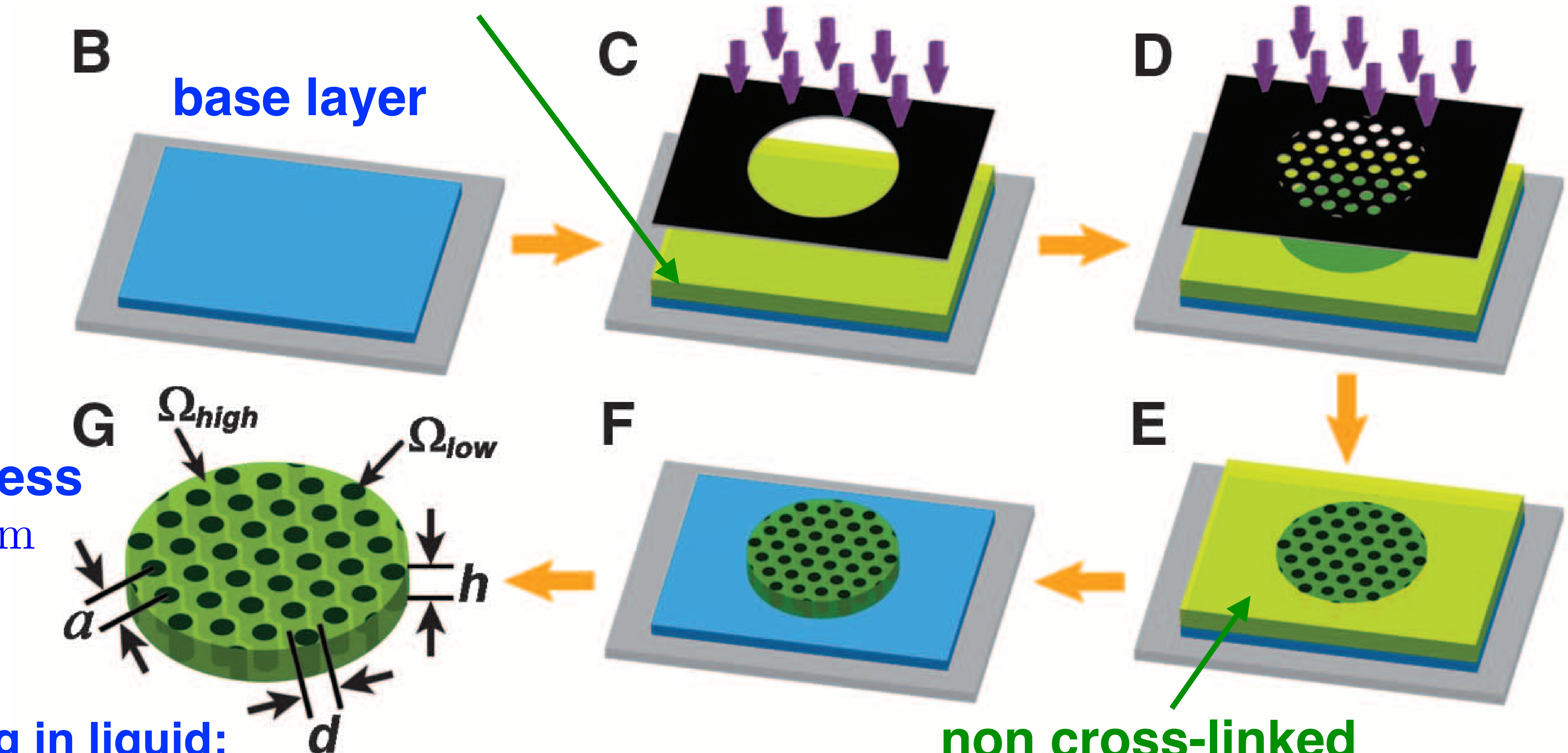


E. Sharon and E. Efrati, Soft Matter 6, 5693 (2010)

Shaping of gel membrane properties by lithography

thin film of polymer solution with premixed inactive cross-linkers

UV light activates cross-linkers. Time of UV light exposure determines the degree of polymer cross-linking.



thickness $7\text{-}17\mu m$

swelling in liquid:

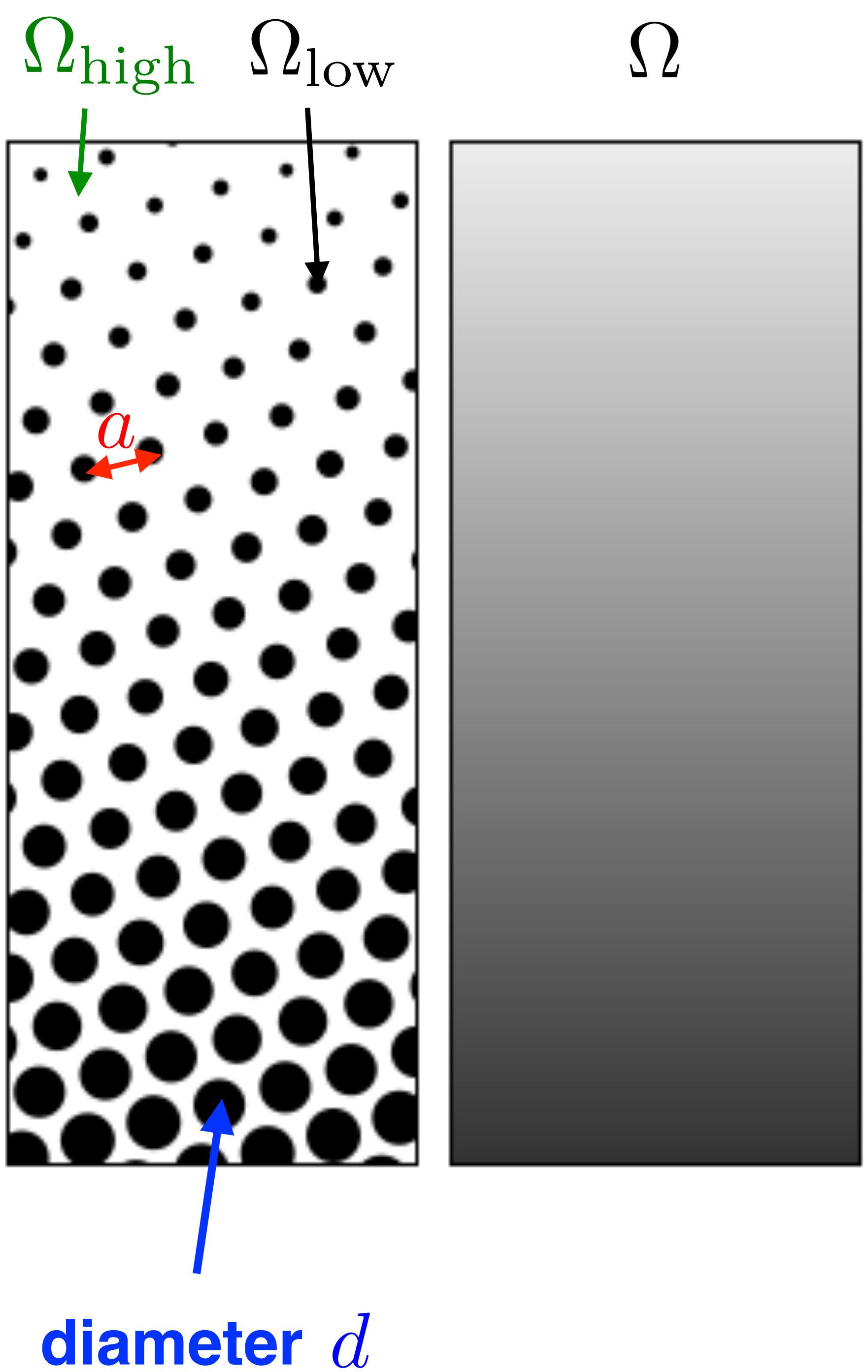
more cross-linked regions swell less Ω_{low}

less cross-linked regions swell more Ω_{high}

non cross-linked polymer solution is washed away

 J. Kim et al., Science 335, 1201 (2012)

Halftoning

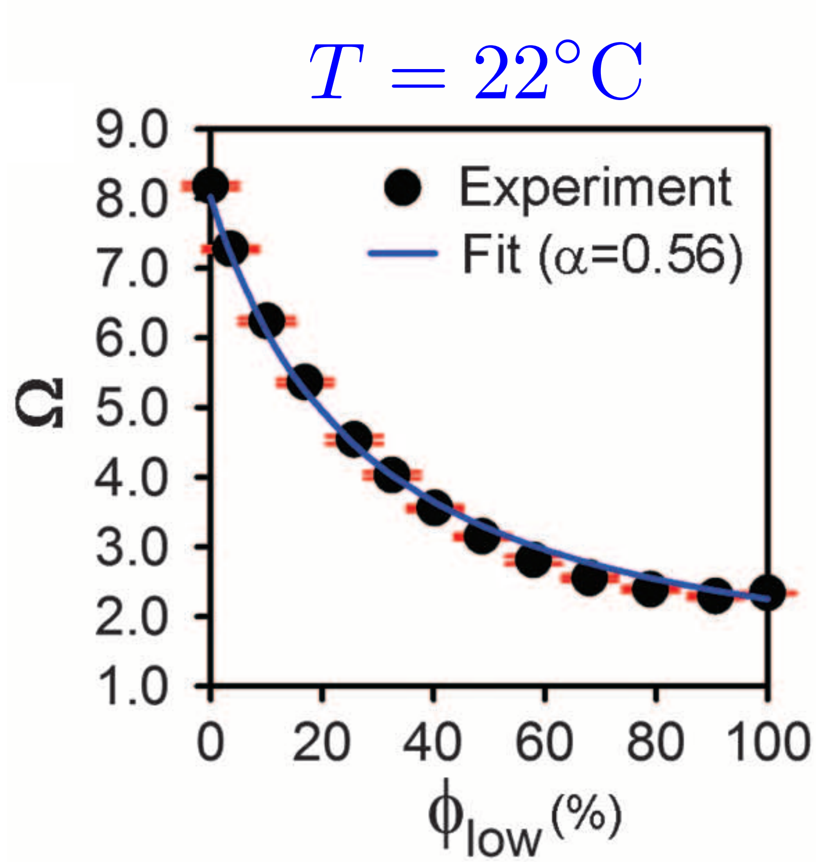


local area fraction of the low swelling regions

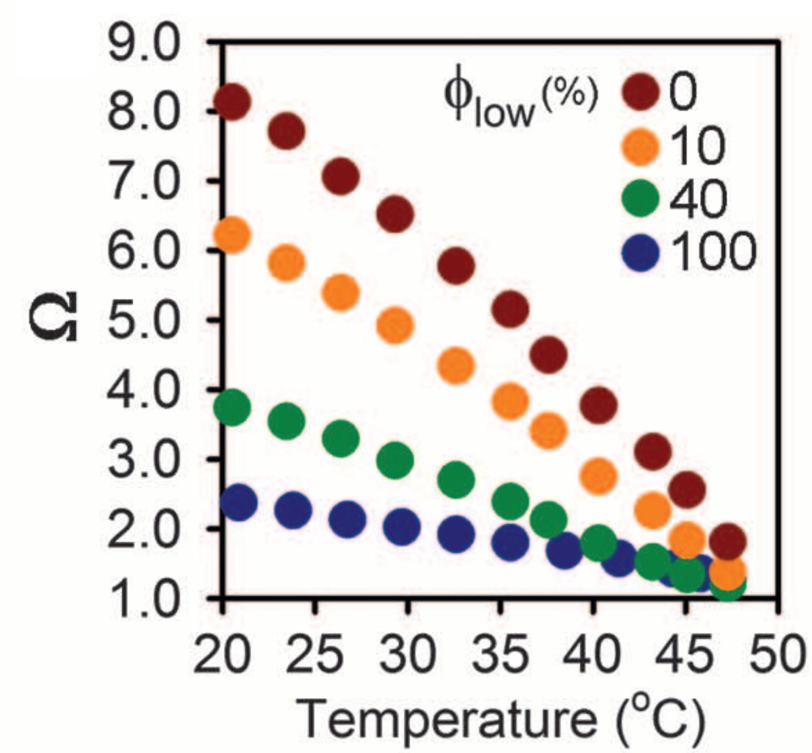
$$\phi_{\text{low}} = \frac{\Delta A_{\text{low}}}{\Delta A_{\text{low}} + \Delta A_{\text{high}}} = \frac{\pi}{2\sqrt{3}}\left(\frac{d}{a}\right)^2$$

Effective swelling Ω can be estimated from local force balance as

$$\frac{\phi_{\text{low}} + \alpha(1-\phi_{\text{low}})}{\Omega^{1/2}} = \frac{\phi_{\text{low}}}{\Omega_{\text{low}}^{1/2}} + \frac{\alpha(1-\phi_{\text{low}})}{\Omega_{\text{high}}^{1/2}}$$

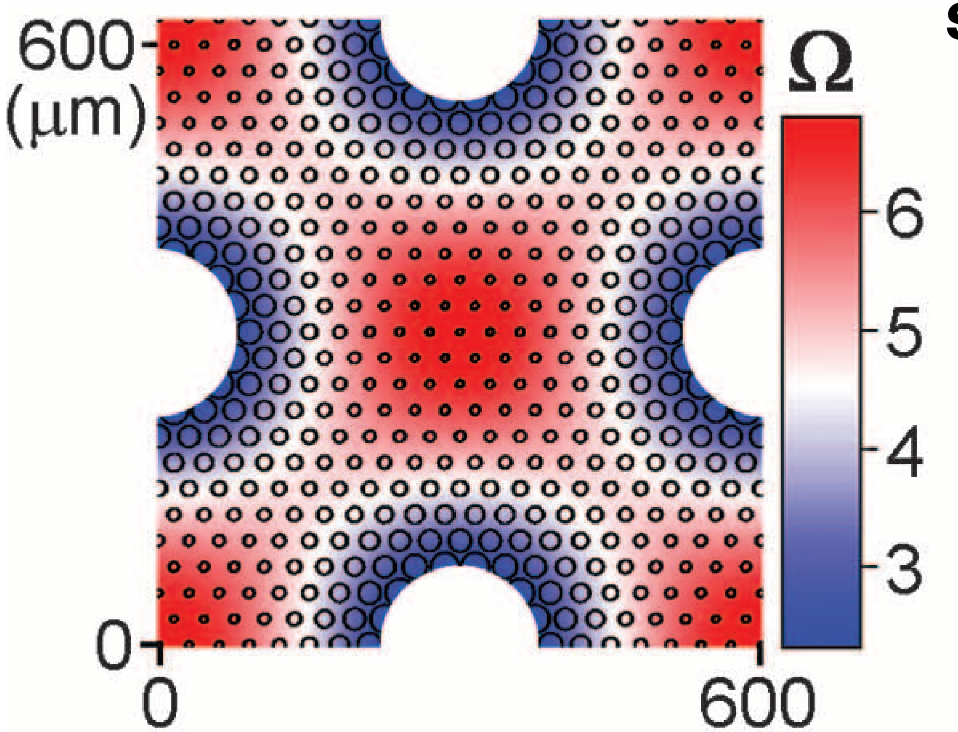


swelling depends on *T*

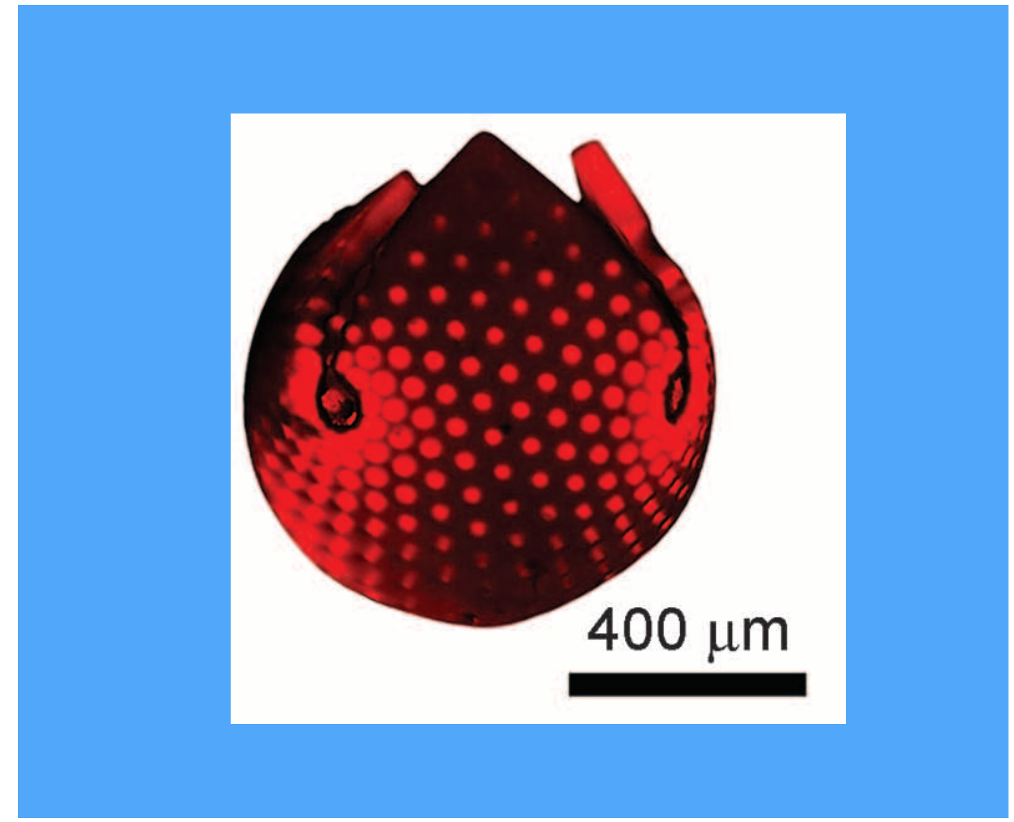


 J. Kim et al., Science 335, 1201 (2012)

Shaping of gel membrane properties by halftone lithography



Differential swelling in liquid deforms square membrane to a closed sphere



metric tensor

$$g_{ij} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

locally isotropic swelling

$$g_{ij} = \begin{pmatrix} \Omega(x,y), & 0 \\ 0, & \Omega(x,y) \end{pmatrix}$$

For thin membranes the target Gauss curvature is

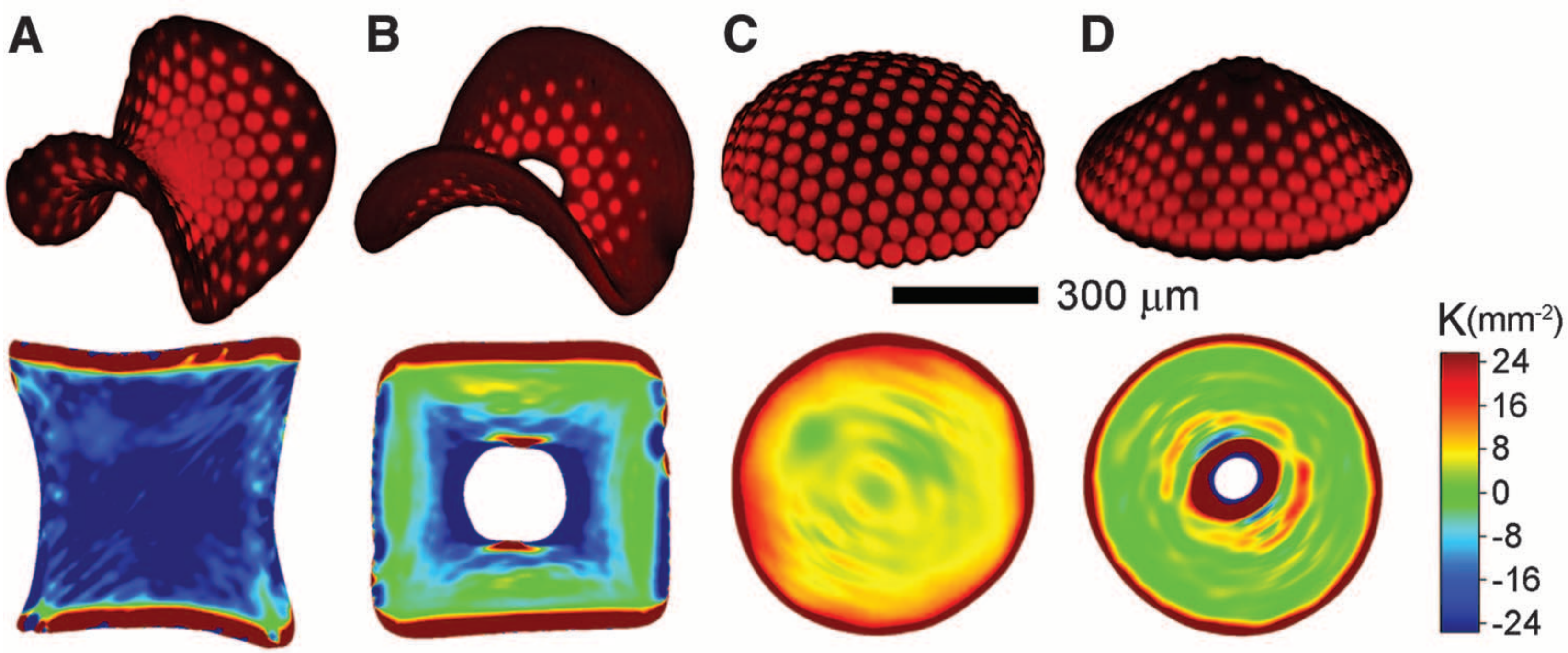
$$\det(K'_{ij}(x,y)) = -\frac{\nabla^2(\ln\Omega(x,y))}{2\Omega(x,y)}$$

Inverse problem can be solved with conformal maps.

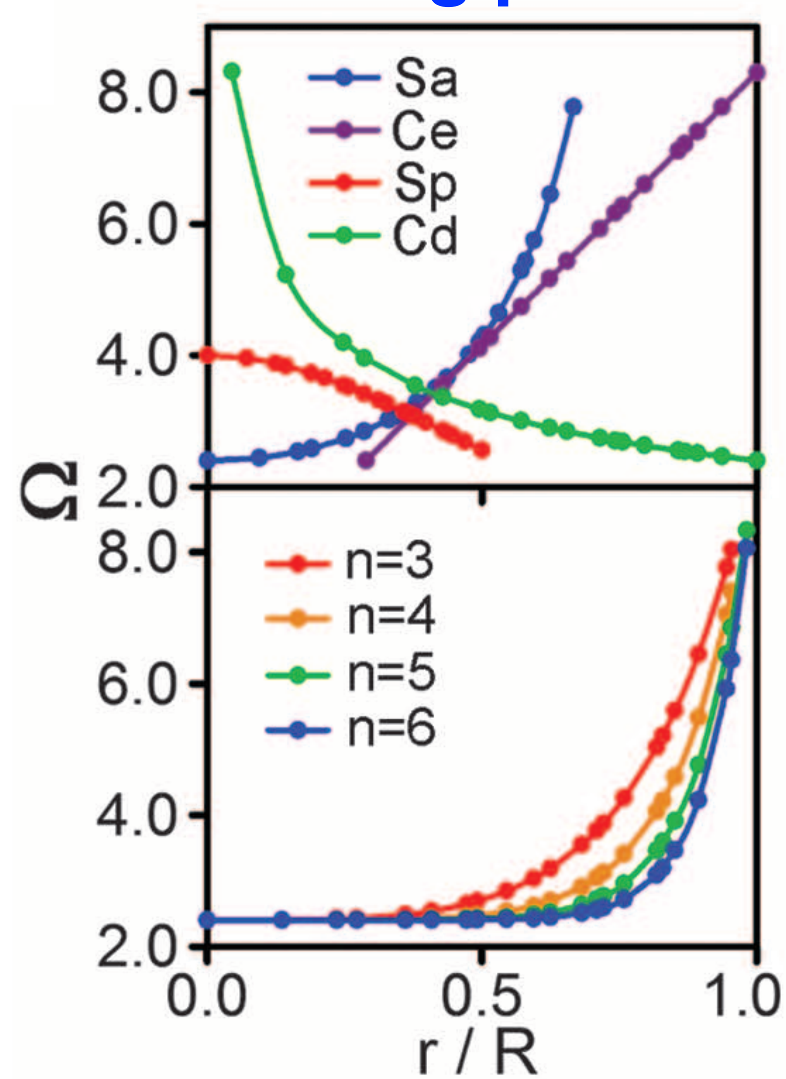
J. Kim et al., Science 335, 1201 (2012)

Shaping of gel membrane properties by halftone lithography

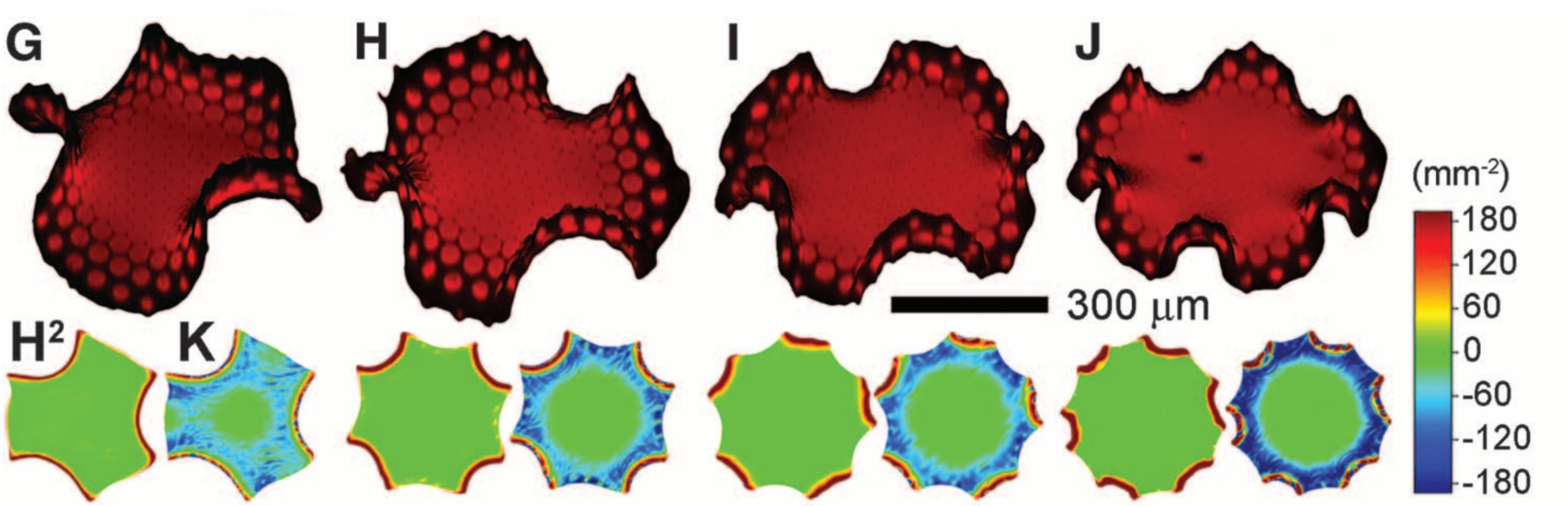
saddle (Sa) | cone with excess angle (Ce) | spherical cap (Sp) | cone with deficit angle (Cd)



swelling profiles



Enneper's minimal surfaces (H=0)

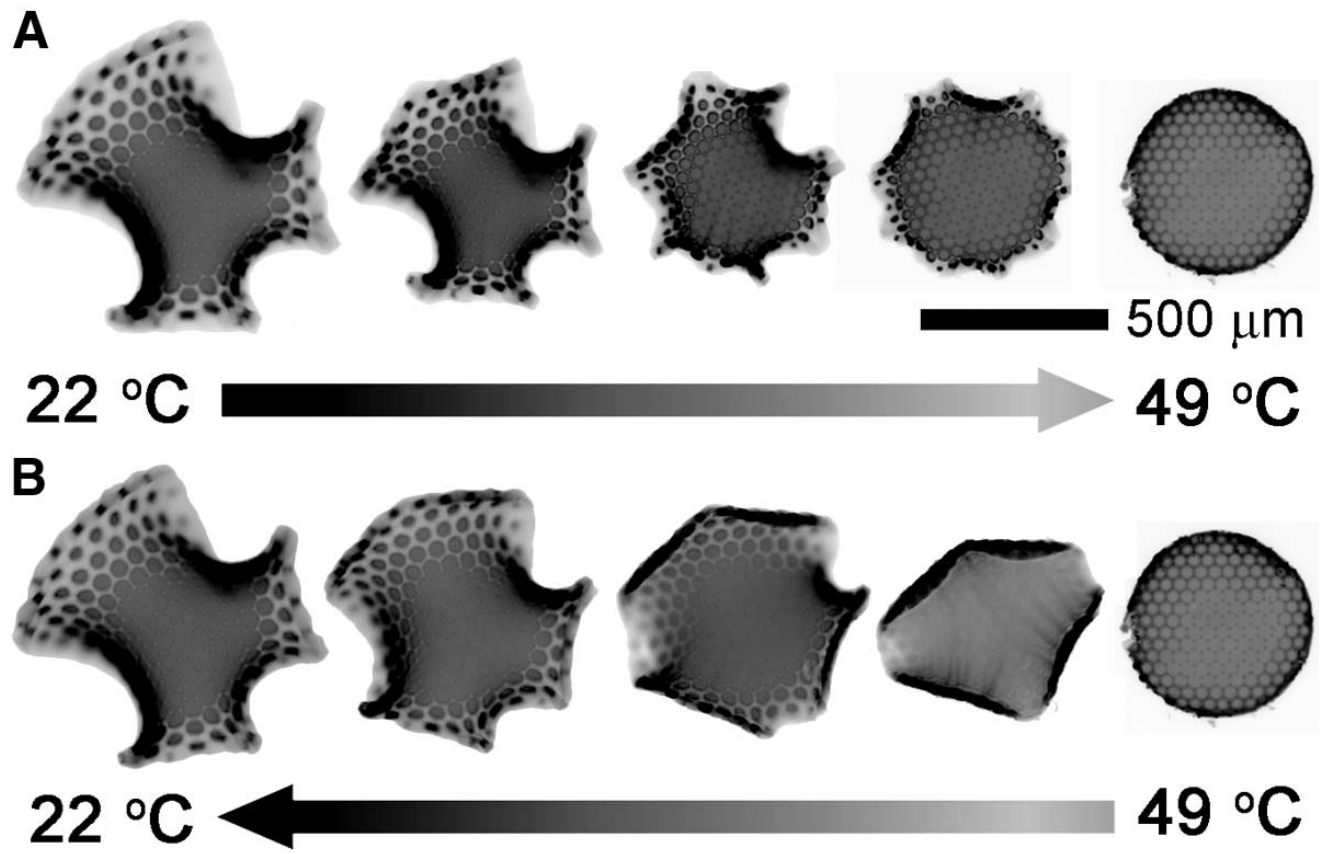


H - mean curvature

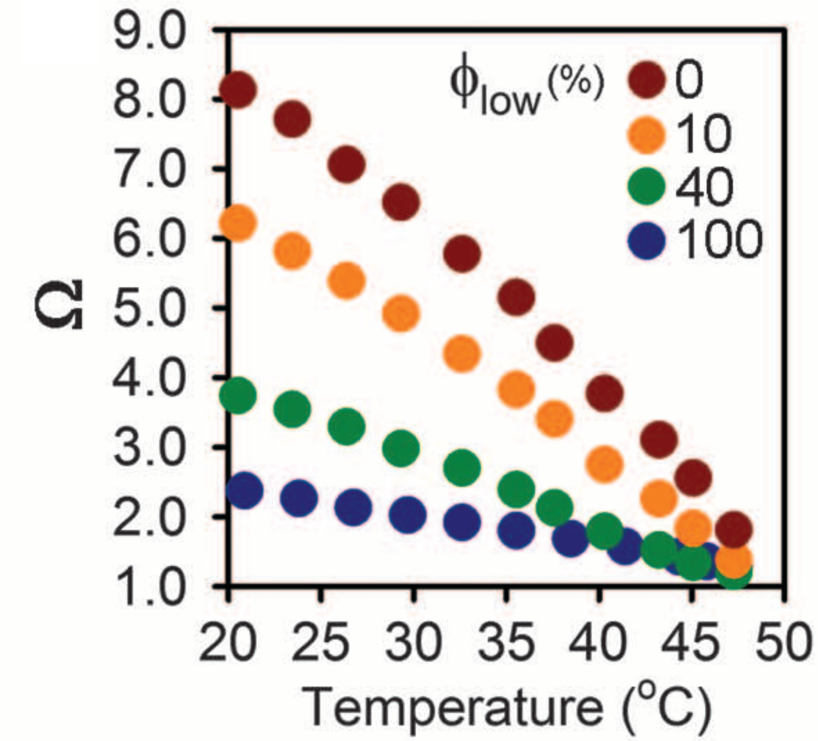
K - Gauss curvature

J. Kim et al., Science 335, 1201 (2012)

Temperature controls swelling and thus the deformed shape



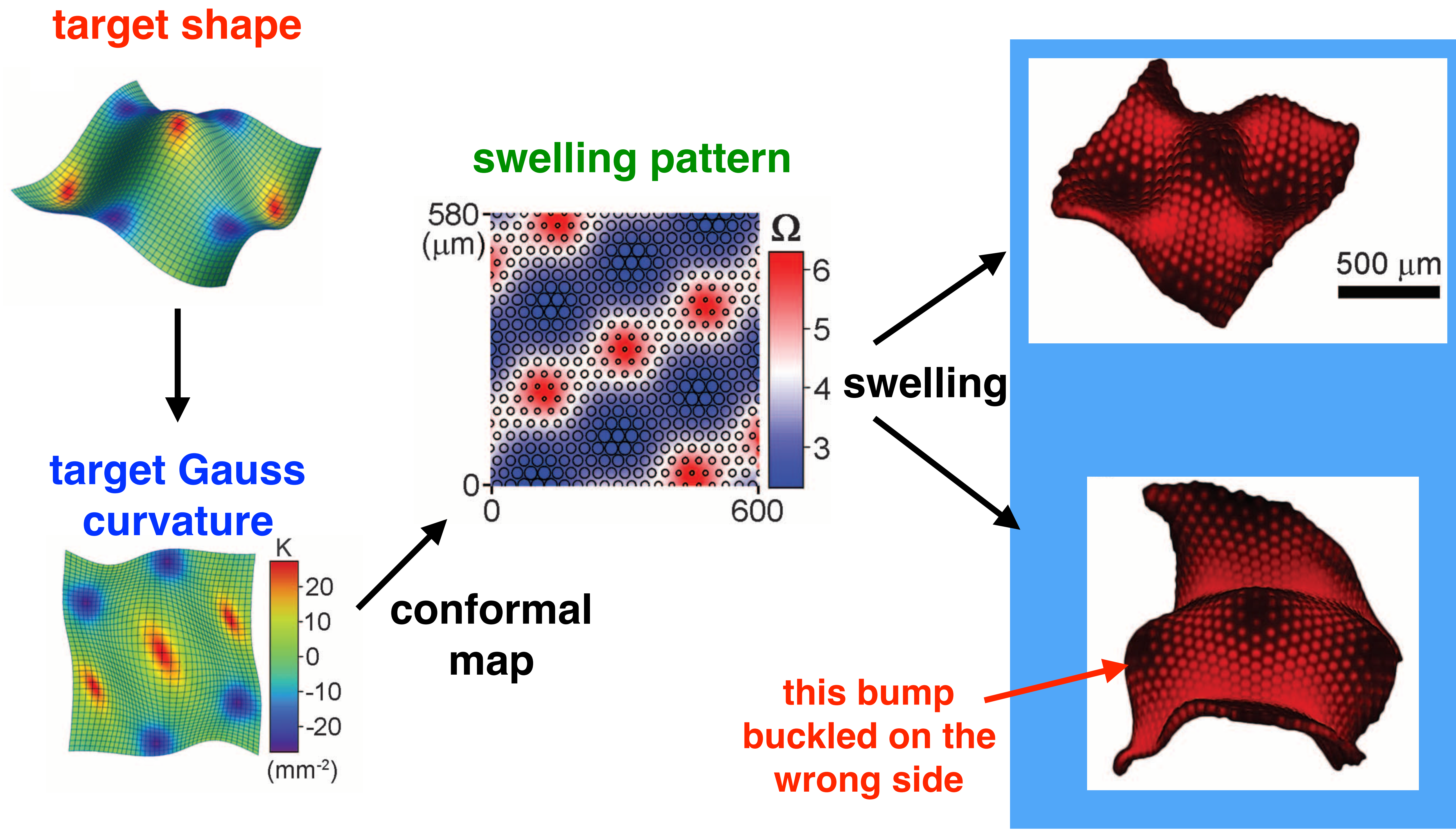
swelling depends on *T*



Note different intermediate shapes!
By slowly varying the temperature
we stay in a local minimum!

 J. Kim et al., Science 335, 1201 (2012)

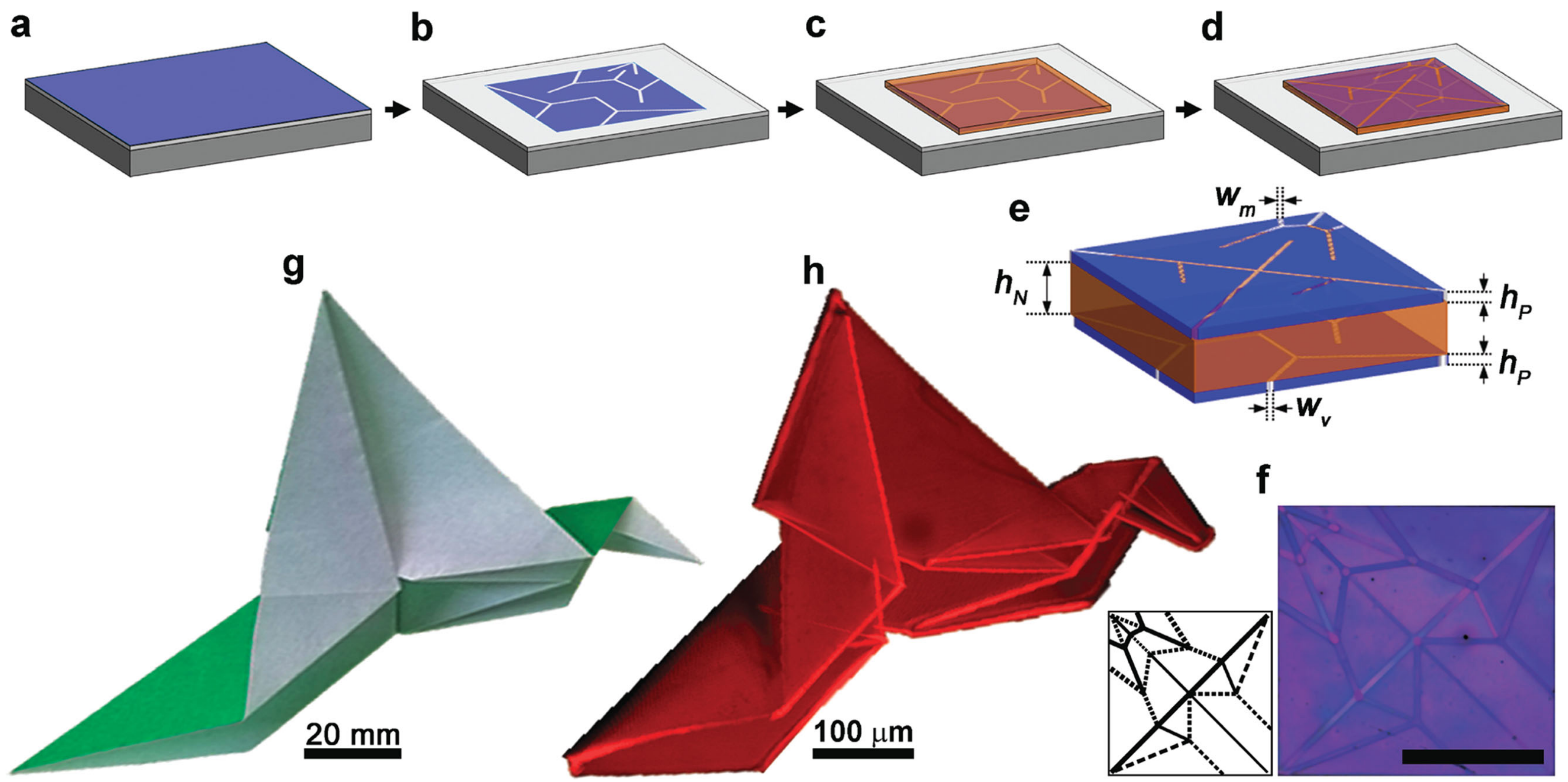
Gaussian curvature does not uniquely specify the shape!



J. Kim et al., Science 335, 1201 (2012)

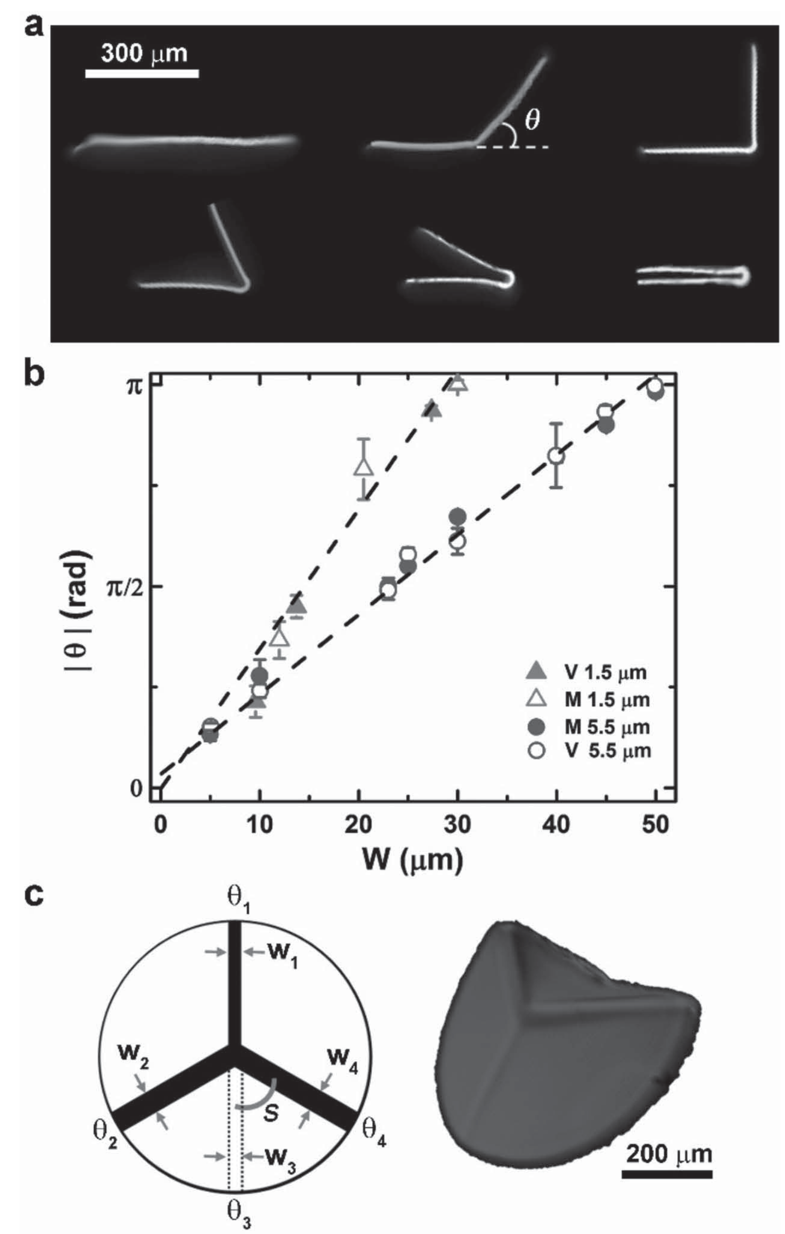
Self folding origami with gel swelling

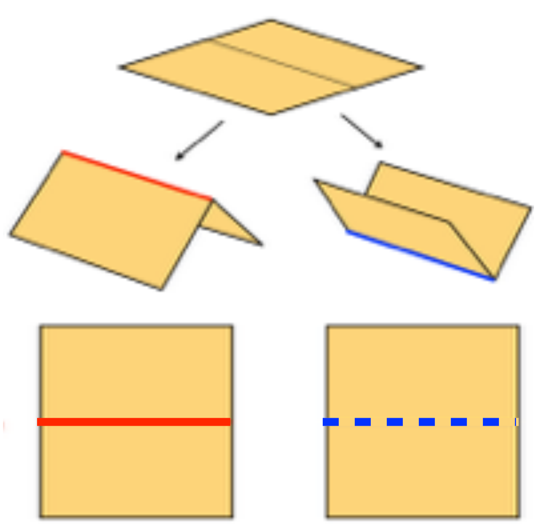
pattern of valley folds — intermediate layer — pattern of mountain folds



Randlett's flapping bird

width of the "cuts" determines the folding angle





mountain — valley

J.-H. Na et al, Adv. Mater. 27, 79 (2015)

Temperature controls swelling and thus the folding of origami