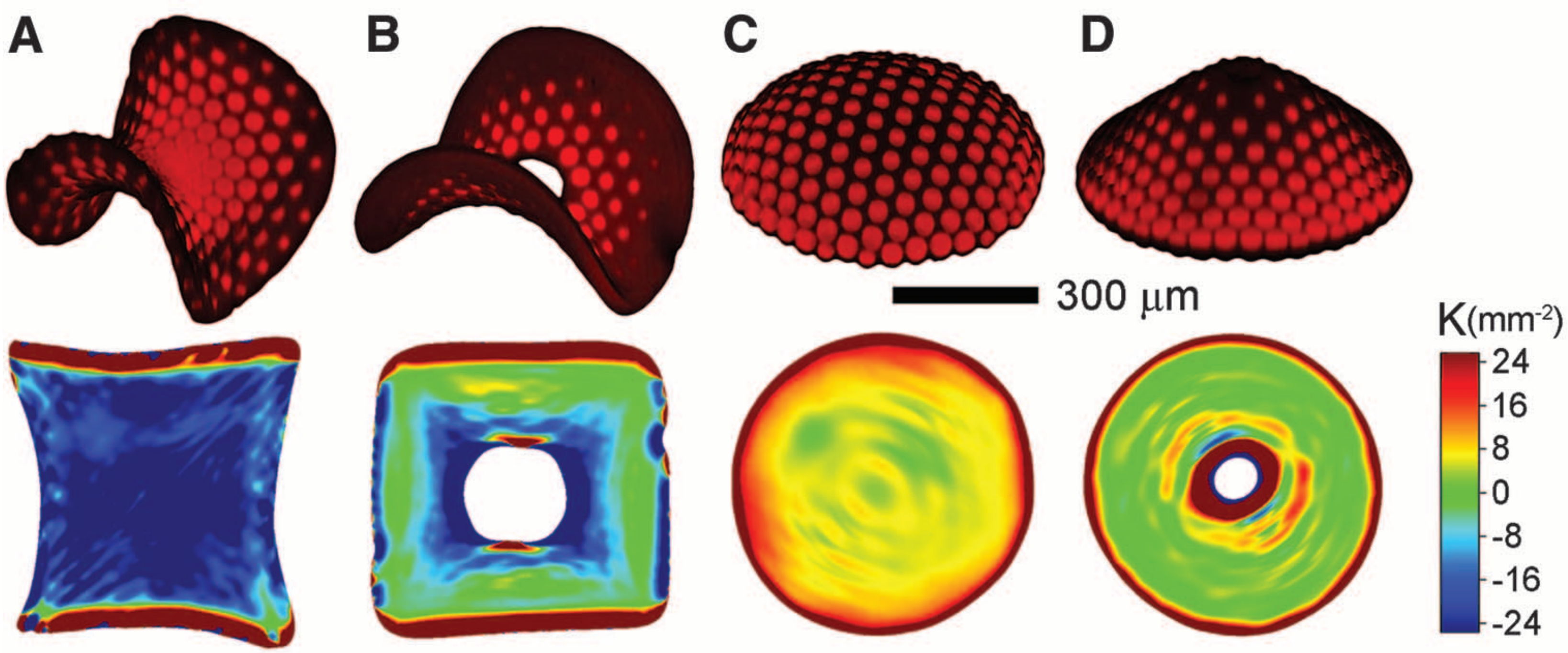
MAE 545: Lecture 21 (12/8)

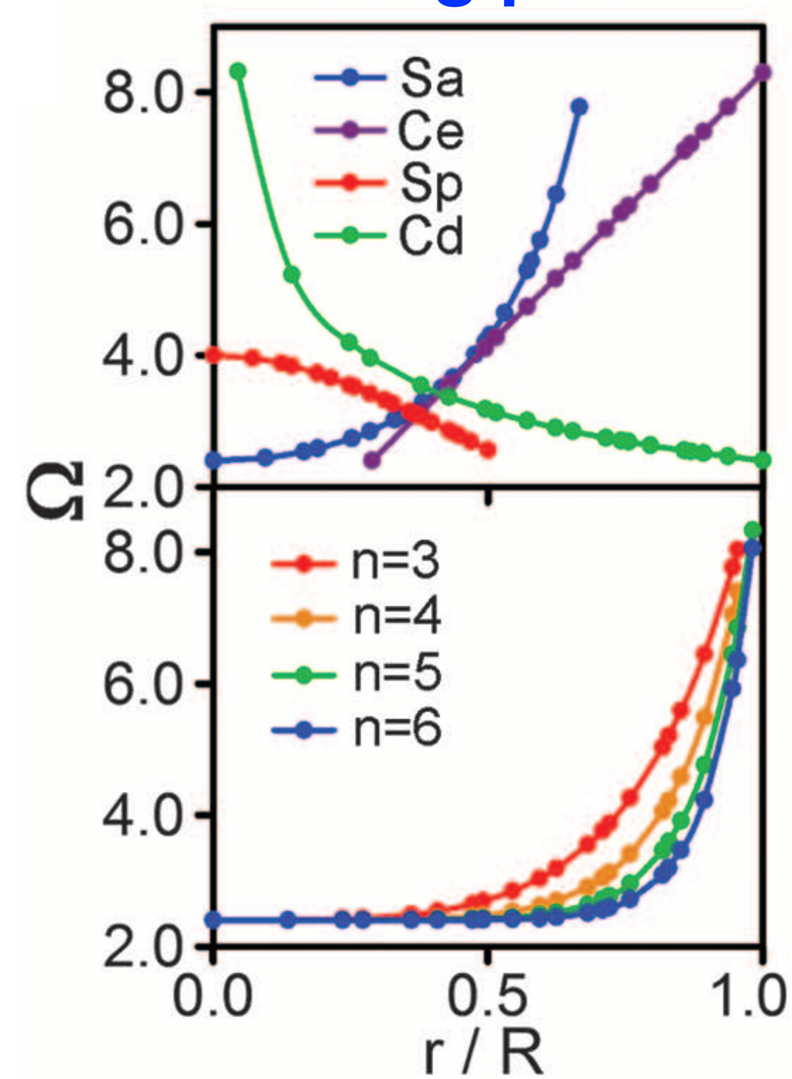
Helices, spirals and phyllotaxis

Shaping of gel membrane properties by halftone lithography

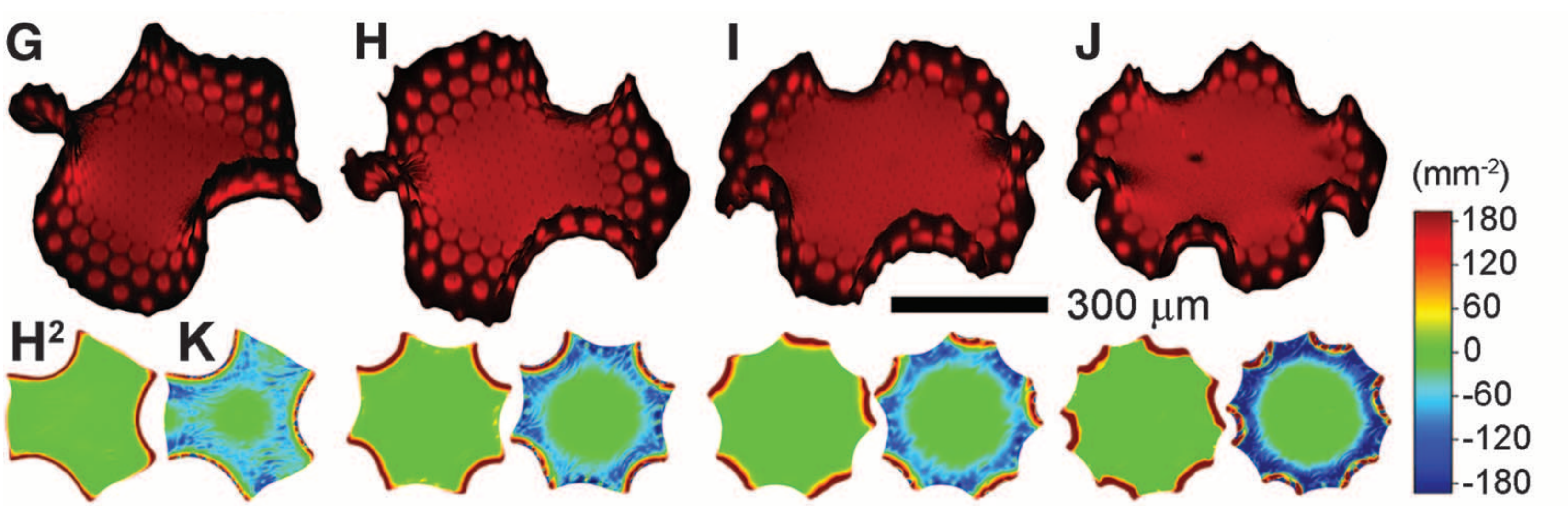
saddle (Sa) | cone with excess angle (Ce) | spherical cap (Sp) | cone with deficit angle (Cd)



swelling profiles



Enneper's minimal surfaces (H=0)



H - mean curvature

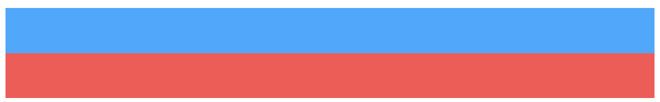
K - Gauss curvature

J. Kim et al., Science 335, 1201 (2012)

Helices in plants

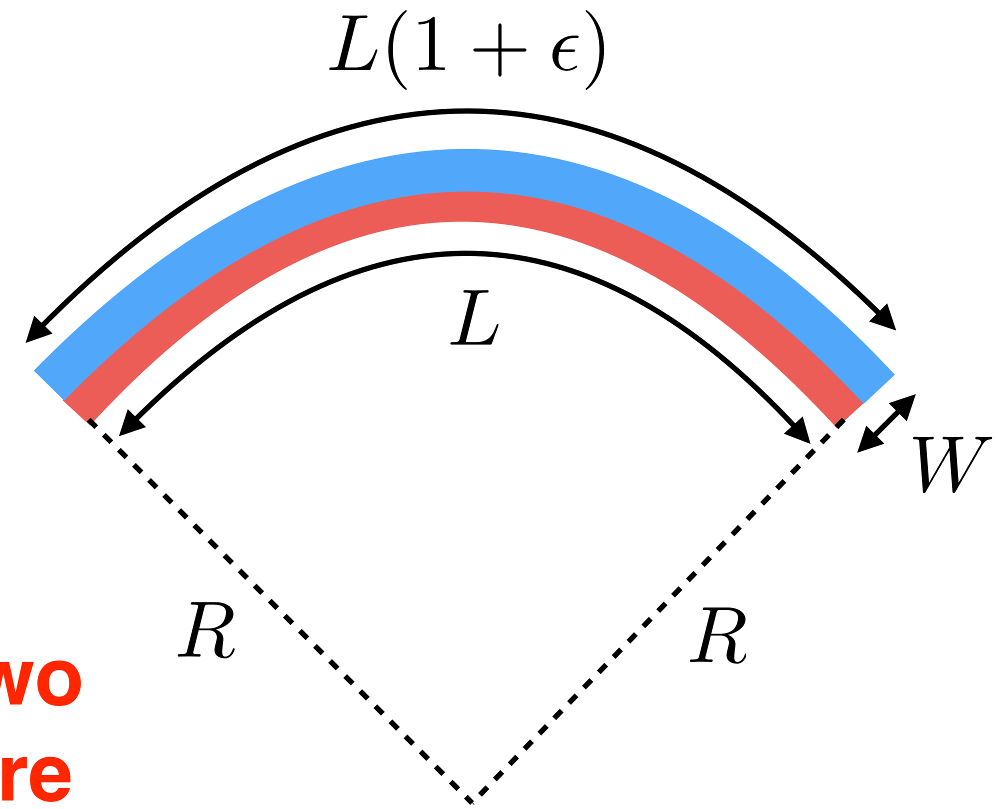
How are helices formed?

Differential growth or differential shrinking produces spontaneous curvature



faster growth of the top layer

more shrinking of the bottom layer



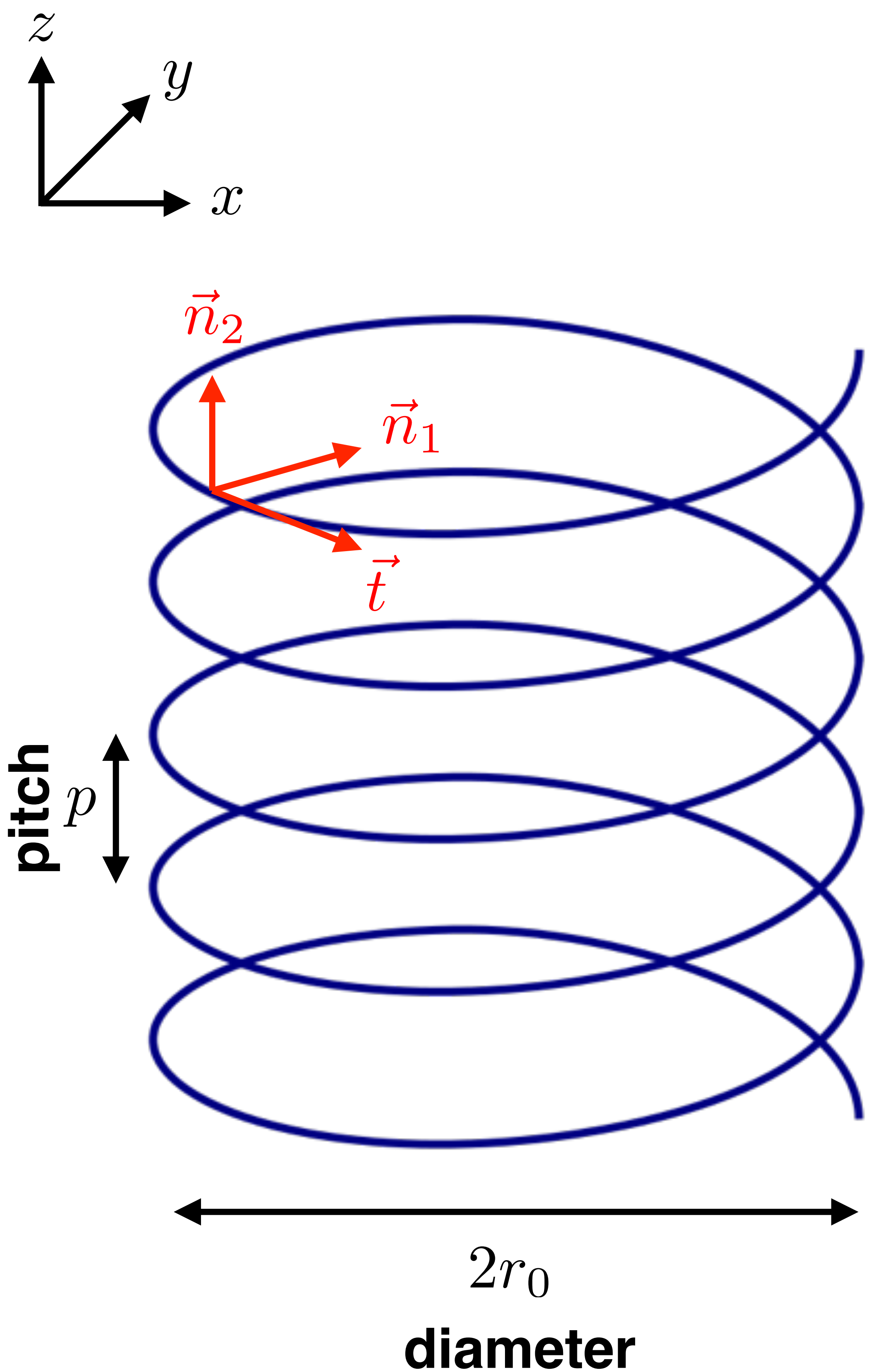
Differential growth (shrinking) of the two layers produces spontaneous curvature

$$K = \frac{1}{R} = \frac{\epsilon}{W}$$

$$\frac{L(1+\epsilon)}{L} = \frac{R+W}{R}$$

Filaments that are longer than $L > 2\pi R$ form helices to avoid steric interactions.

Helix



Mathematical description

$$\vec{r}(s) = \left(r_0 \cos(s/\lambda), r_0 \sin(s/\lambda), \frac{p}{2\pi\lambda} s \right)$$

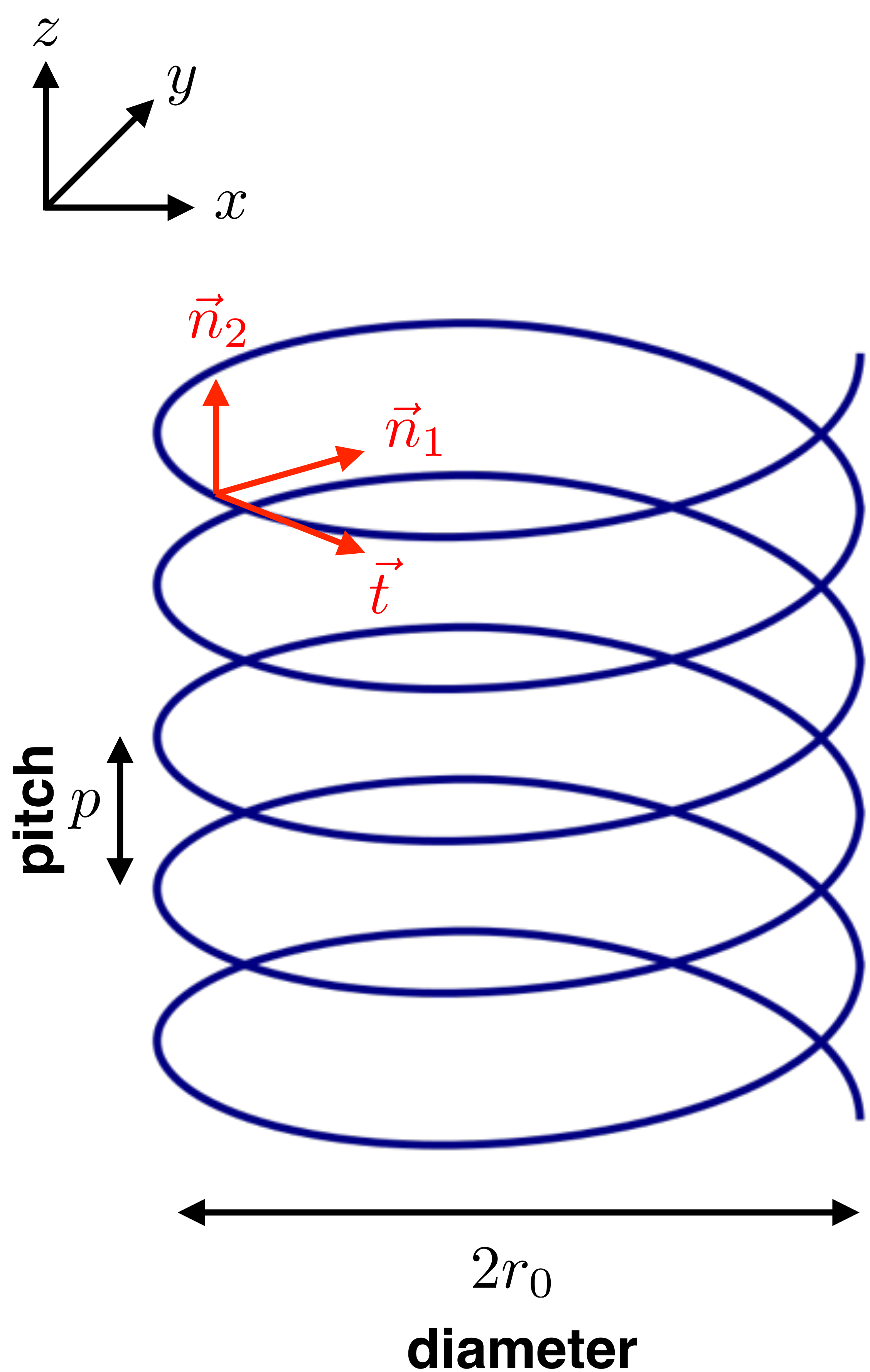
Set λ to fix the metric

$$\vec{t}(s) = \frac{d\vec{r}}{ds} = \left(-\frac{r_0}{\lambda} \sin(s/\lambda), \frac{r_0}{\lambda} \cos(s/\lambda), \frac{p}{2\pi\lambda} \right)$$

$$g = \vec{t} \cdot \vec{t} = \frac{r_0^2}{\lambda^2} + \frac{p^2}{4\pi^2\lambda^2} = 1$$

$$\lambda = \sqrt{r_0^2 + (p/2\pi)^2}$$

Helix



Mathematical description

$$\vec{r}(s) = \left(r_0 \cos(s/\lambda), r_0 \sin(s/\lambda), \frac{p}{2\pi\lambda} s \right)$$

$$\lambda = \sqrt{r_0^2 + (p/2\pi)^2}$$

Tangent and normal vectors

$$\vec{t}(s) = \frac{d\vec{r}}{ds} = \left(-\frac{r_0}{\lambda} \sin(s/\lambda), \frac{r_0}{\lambda} \cos(s/\lambda), \frac{p}{2\pi\lambda} \right)$$

$$\vec{n}_1(s) = \left(-\cos(s/\lambda), -\sin(s/\lambda), 0 \right)$$

$$\vec{n}_2(s) = \left(\frac{p}{2\pi\lambda} \sin(s/\lambda), -\frac{p}{2\pi\lambda} \cos(s/\lambda), \frac{r_0}{\lambda} \right)$$

Helix curvatures

$$\vec{n}_1 \cdot \frac{d^2\vec{r}}{ds^2} = \frac{r_0}{\lambda^2} = \frac{r_0}{r_0^2 + (p/2\pi)^2} = K$$

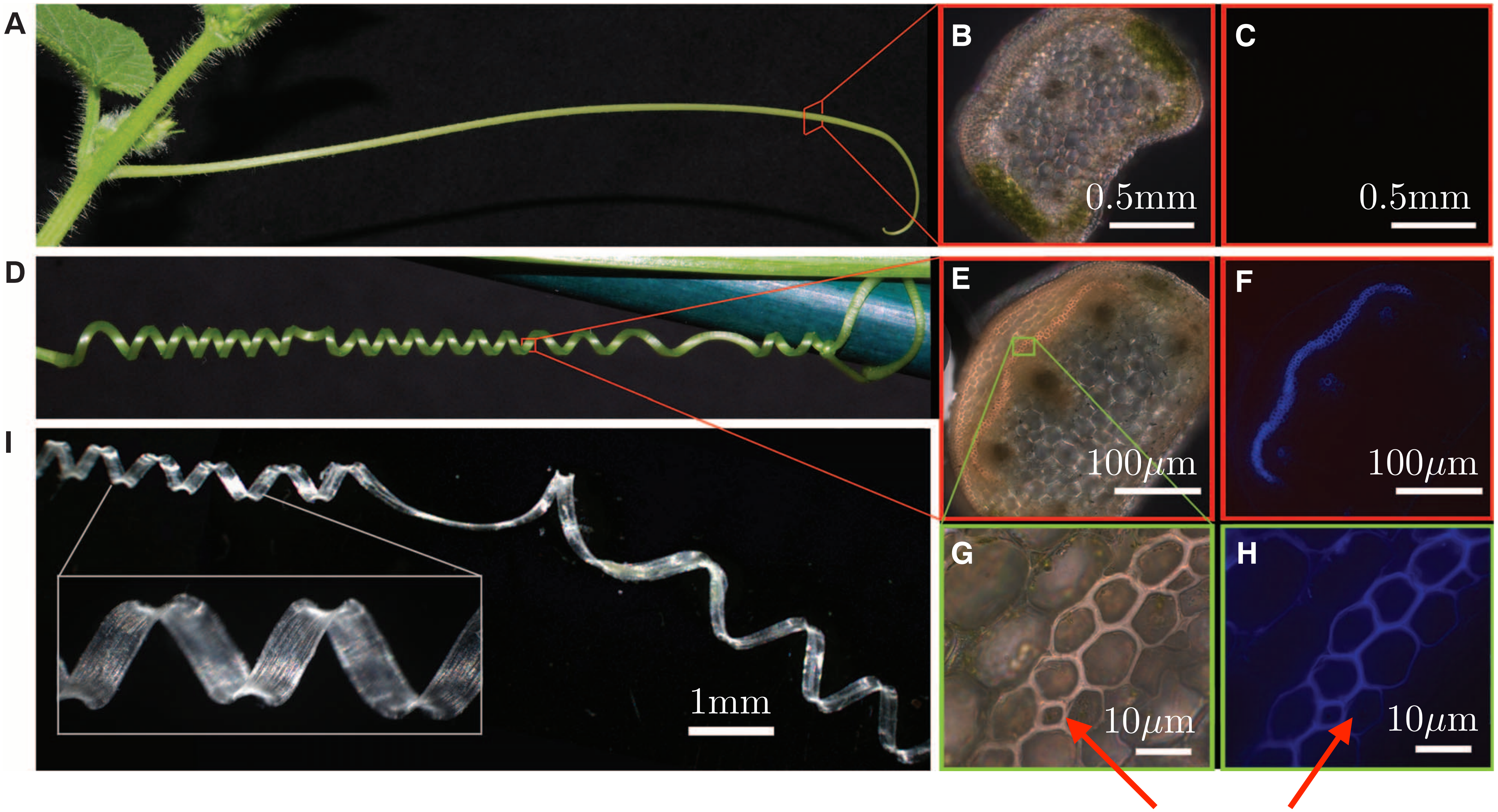
$$\vec{n}_2 \cdot \frac{d^2\vec{r}}{ds^2} = 0$$

Cucumber tendril climbing via helical coiling

Cucumber tendrils want to pull themselves up above other plants in order to get more sunlight.

S. J. Gerbode et al., Science 337, 1087 (2012)

Helical coiling of cucumber tendril



Coiling in older tendrils is due to a thin layer of stiff, lignified gelatinous fiber cells, which are also found in wood.

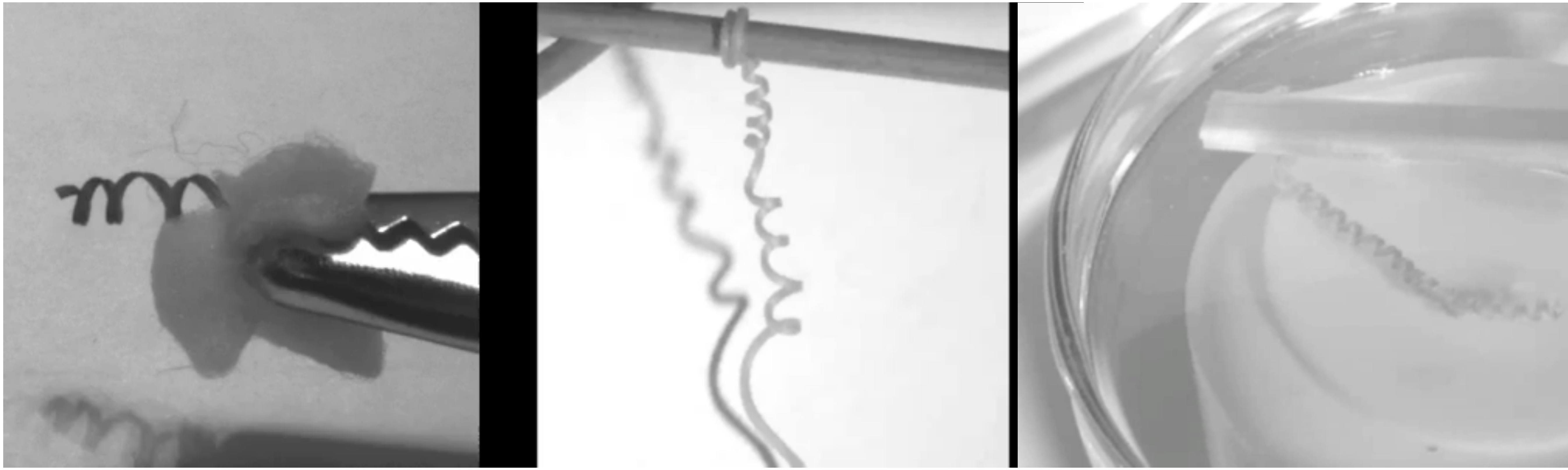
S. J. Gerbode et al., Science 337, 1087 (2012)

Helical coiling of cucumber tendril

Drying of fibber ribbon increases coiling

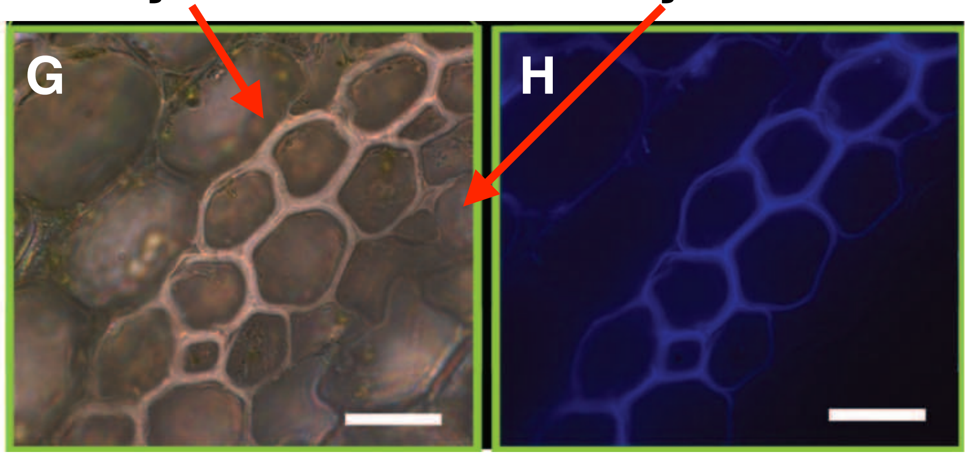
Drying of tendril increases coiling

Rehydrating of tendril increases coiling



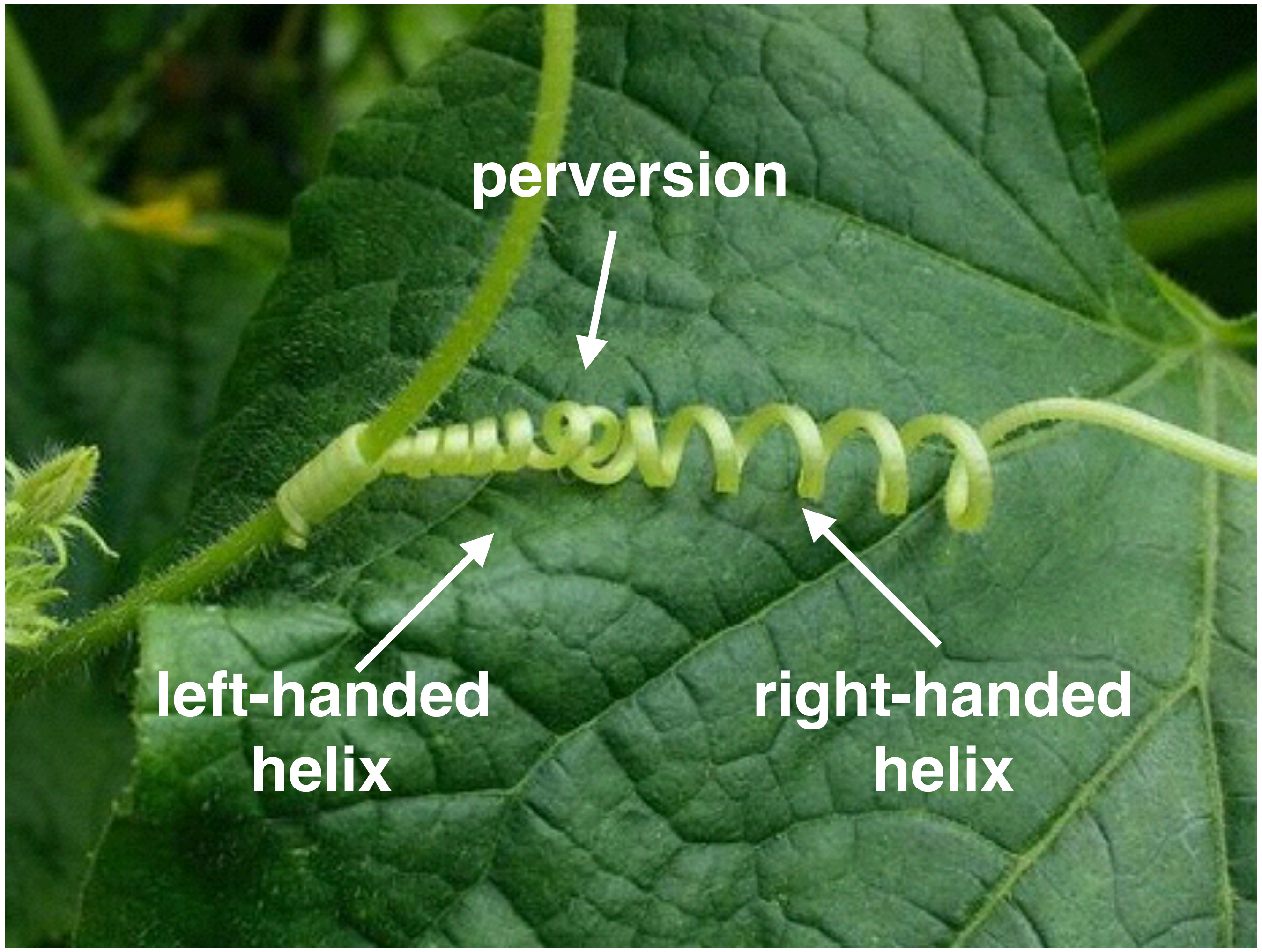
During the lignification of g-fiber cells water is expelled, which causes shrinking.

The inside layer is more lignified and therefore shrinks more and is also stiffer than the outside layer.

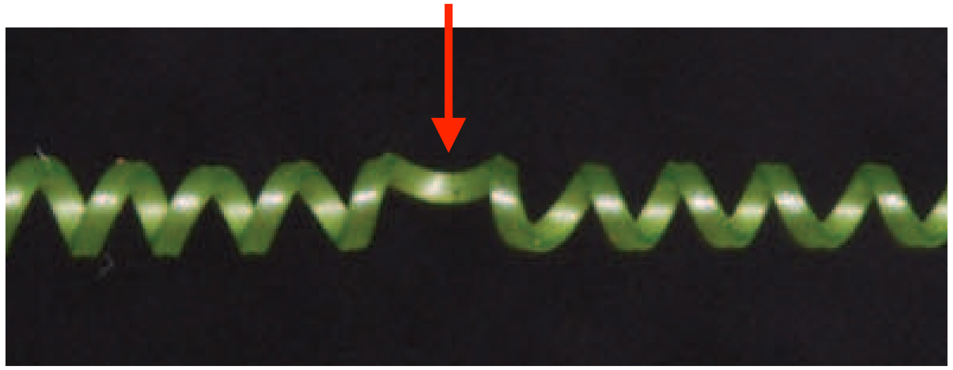


S. J. Gerbode et al., Science 337, 1087 (2012)

Coiling of tendrils in opposite directions



perversion



Ends of the tendril are fixed and cannot rotate. This constraints the linking number.

Link = Twist + Writhe

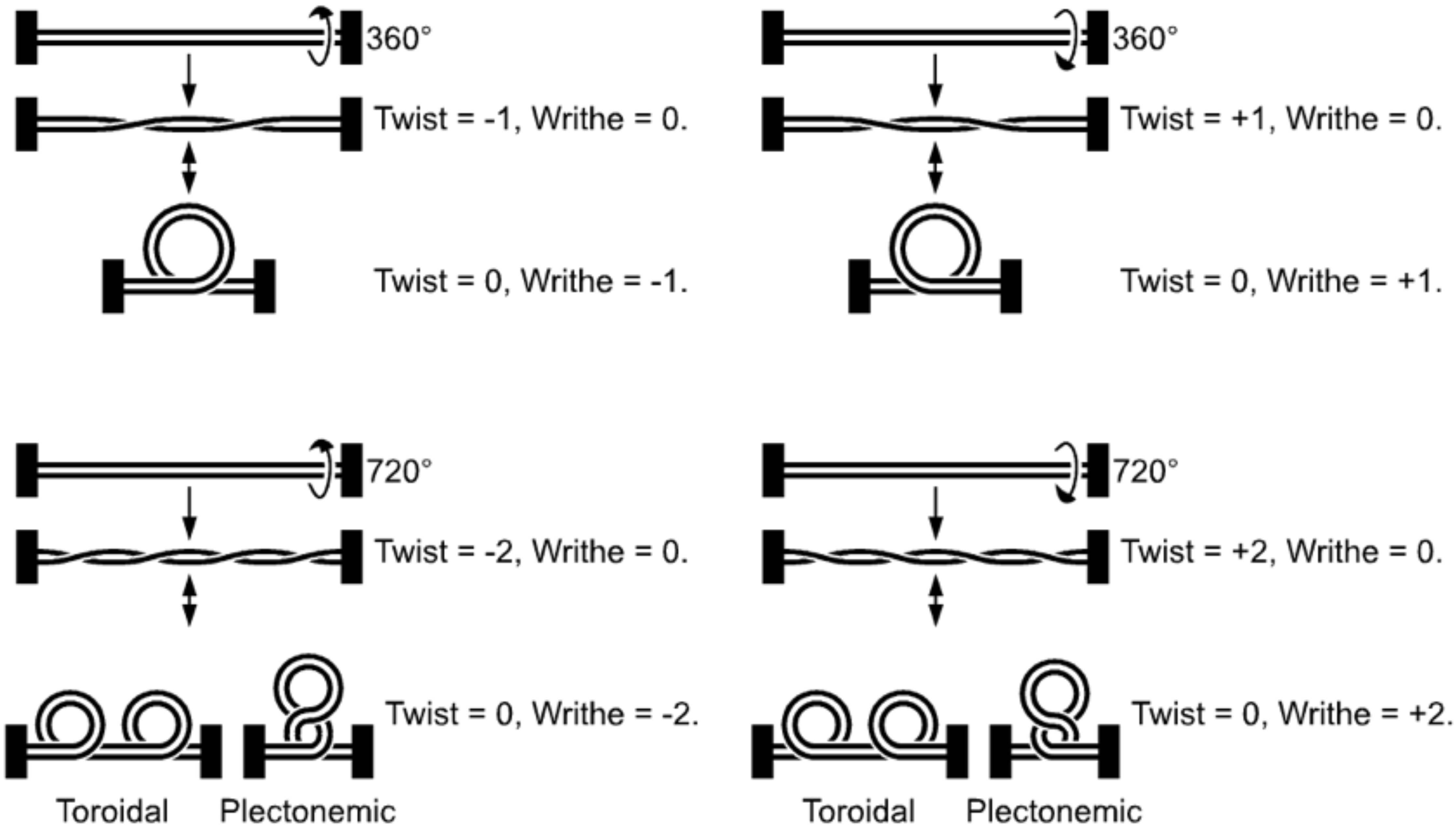
Coiling in the same direction increases Writhe, which needs to be compensated by the twist.

In order to minimize the twisting energy tendrils combine two helical coils of opposite handedness (=opposite Writhe).

Note: there is no bending energy when the curvature of two helices correspond to the spontaneous curvature due to the differential shrinking of fiber.

Twist, Writhe and Linking numbers

Ln=Tw+Wr	linking number: total number of turns of a particular end
Tw	twist: number of turns due to twisting the beam
Wr	writhe: number of crossings when curve is projected on a plane

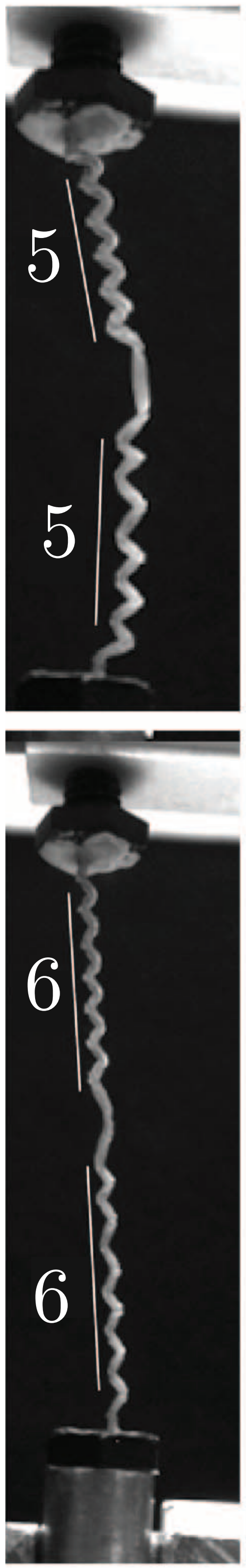


Overwinding of tendril coils

Old tendrils overwind when stretched.

relaxed

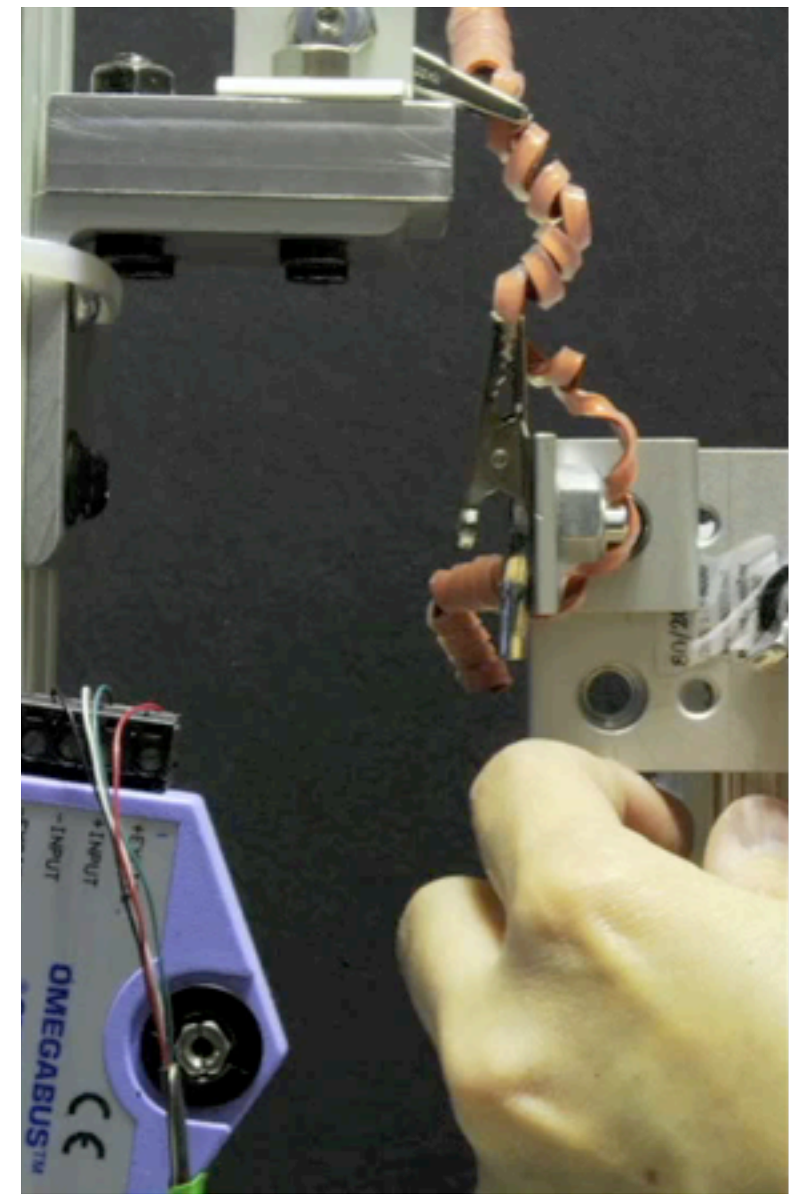
stretched



Rubber model unwinds when stretched.

relaxed

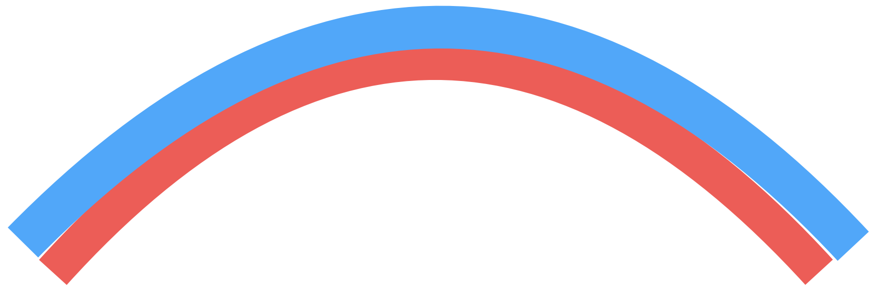
stretched



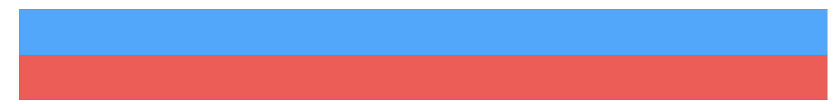
S. J. Gerbode et al., Science 337, 1087 (2012)

Overwinding of tendril coils

Preferred curved state



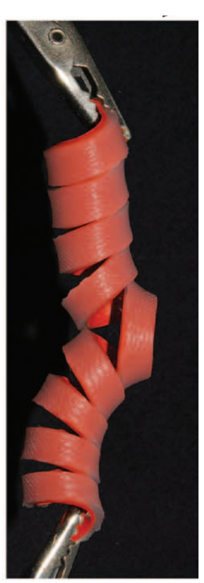
Flattened state



In tendrils the red inner layer is much stiffer then the outside blue layer.

High bending energy cost associated with stretching of the stiff inner layer!

Tendrils try to keep the preferred curvature when stretched!

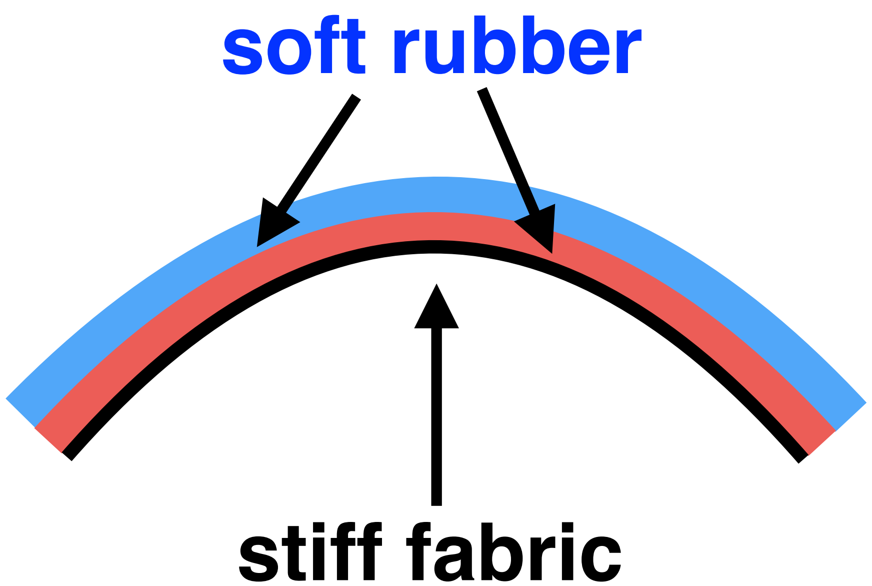


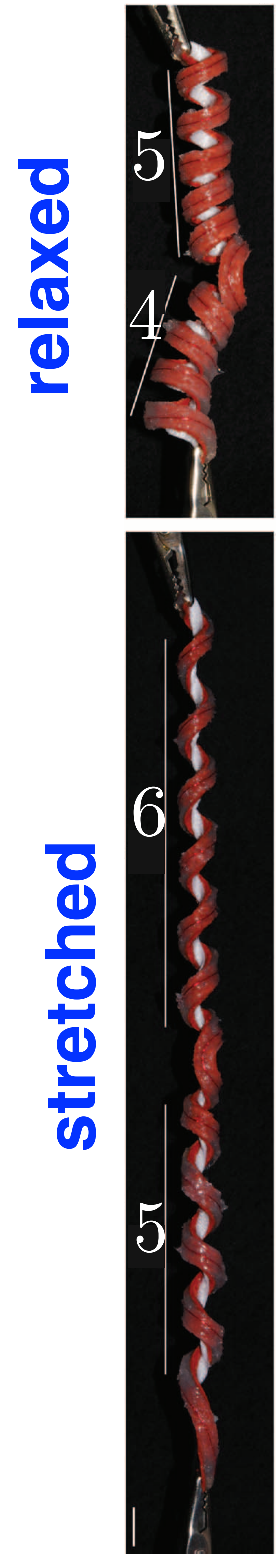
In rubber models both layers have similar stiffness.

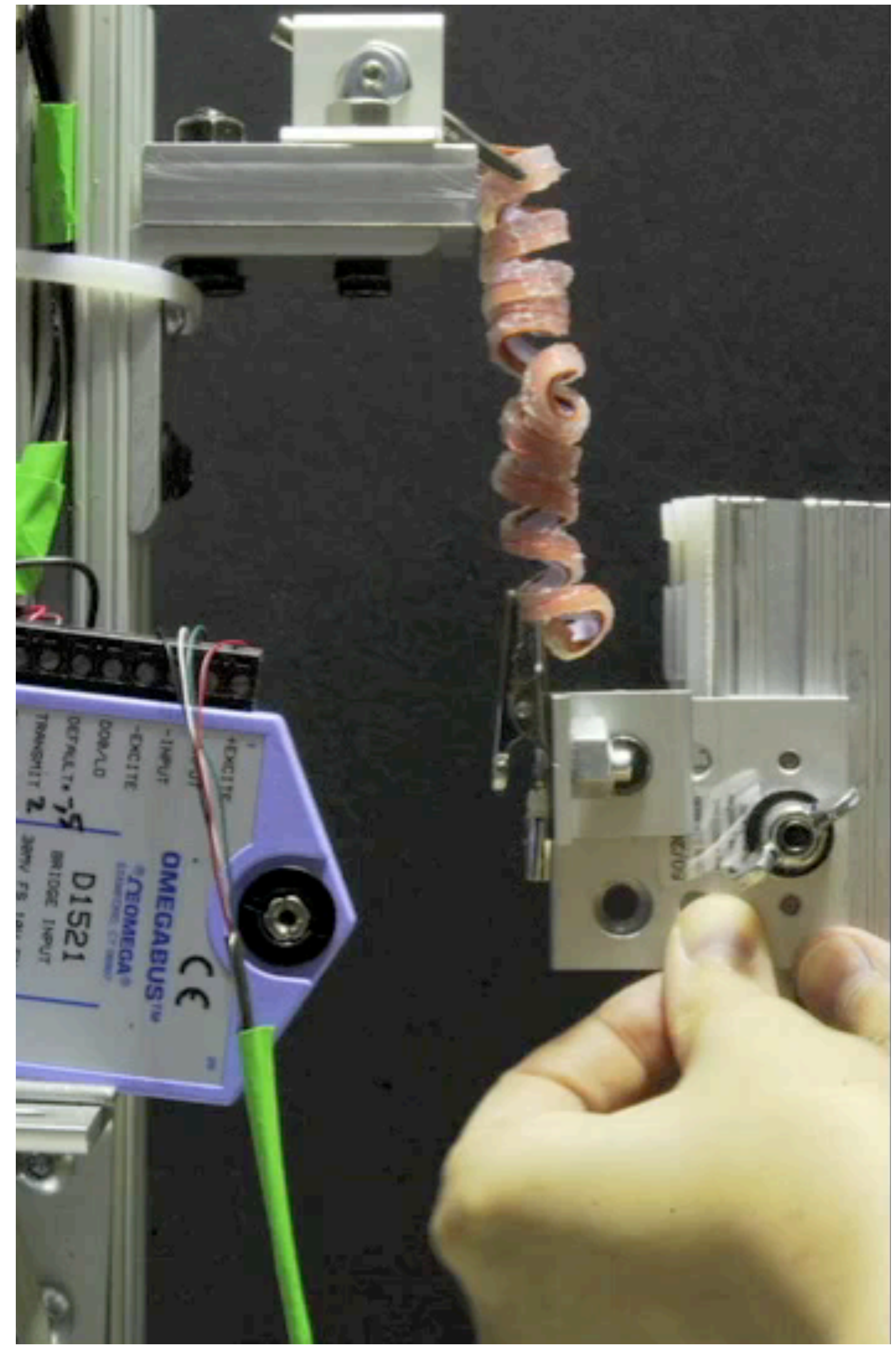
Small bending energy.

S. J. Gerbode et al., Science 337, 1087 (2012)

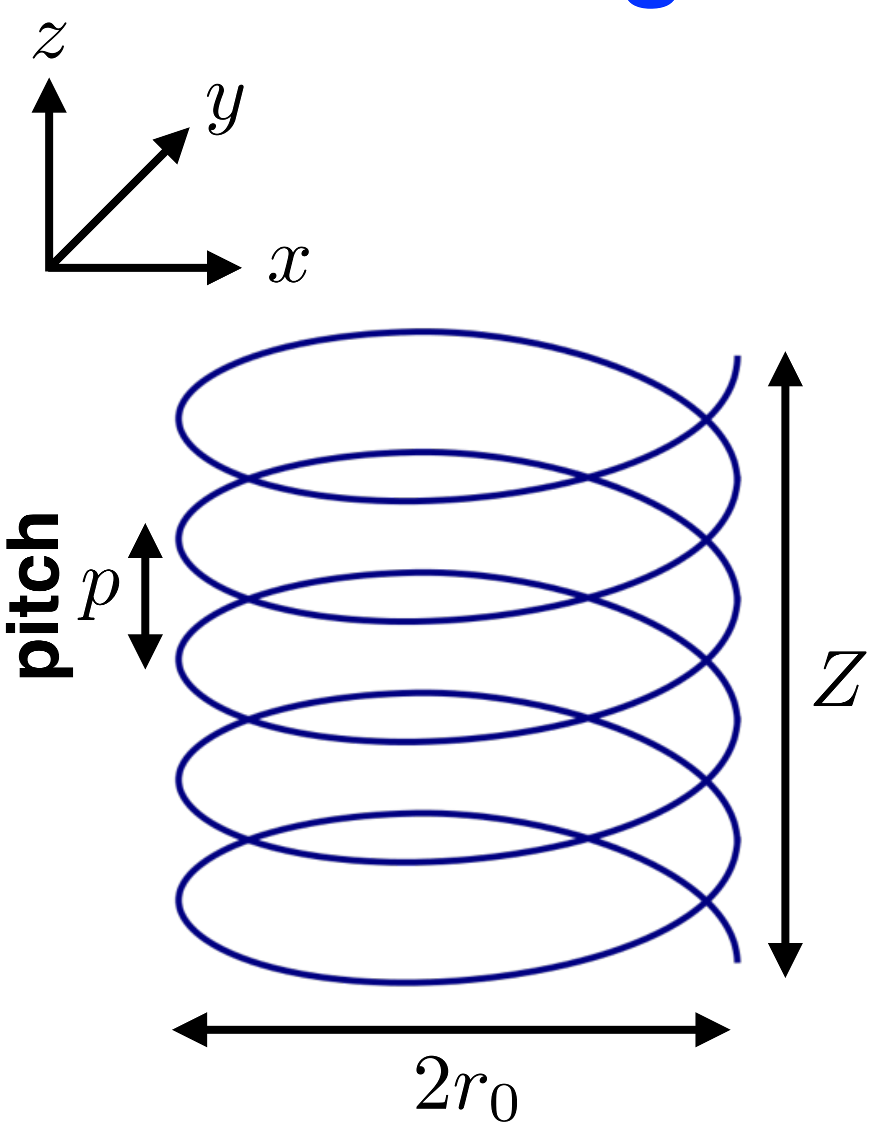
Overwinding of rubber models with an additional stiff fabric on the inside layers







Overwinding of helix with infinite bending modulus



diameter

L **length of the helix backbone**

$N = \dfrac{Z}{p}$ **number of loops**

Mathematical description

$$\vec{r}(s) = \left(r_0 \cos(s/\lambda), r_0 \sin(s/\lambda), \frac{p}{2\pi\lambda} s \right)$$

$$\lambda = \sqrt{r_0^2 + (p/2\pi)^2} \qquad Z = pN = p(L/2\pi\lambda)$$

Infinite bending modulus fixes the helix curvature during stretching

$$K = \frac{r_0}{r_0^2 + (p/2\pi)^2}$$

Helix pitch and radius

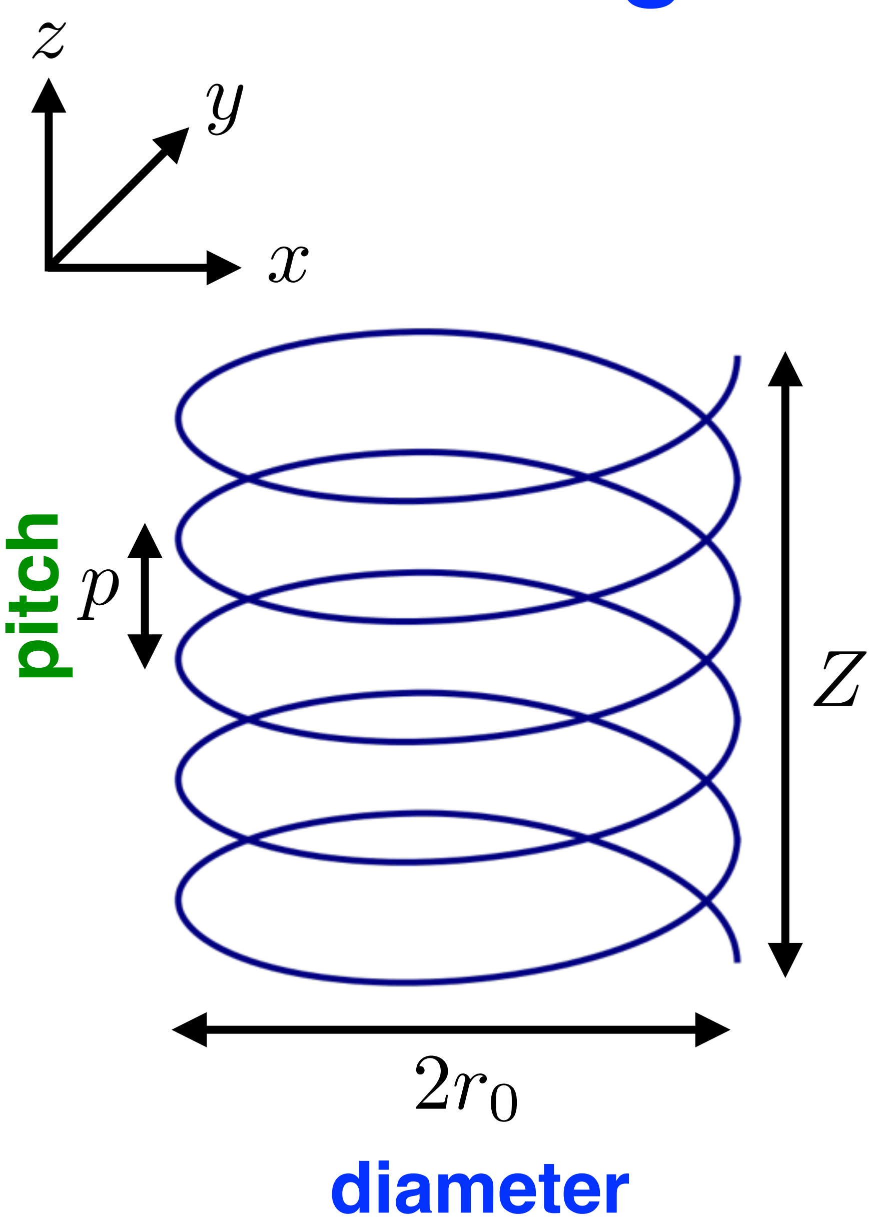
$$r_0 = \frac{1}{K}\left(1 - \frac{Z^2}{L^2}\right)$$

$$p = \frac{2\pi Z}{KL}\sqrt{1 - \frac{Z^2}{L^2}}$$

Number of loops

$$N = \frac{Z}{p} = \frac{KL}{2\pi\sqrt{1-(Z/L)^2}}$$

Overwinding of helix with infinite bending modulus



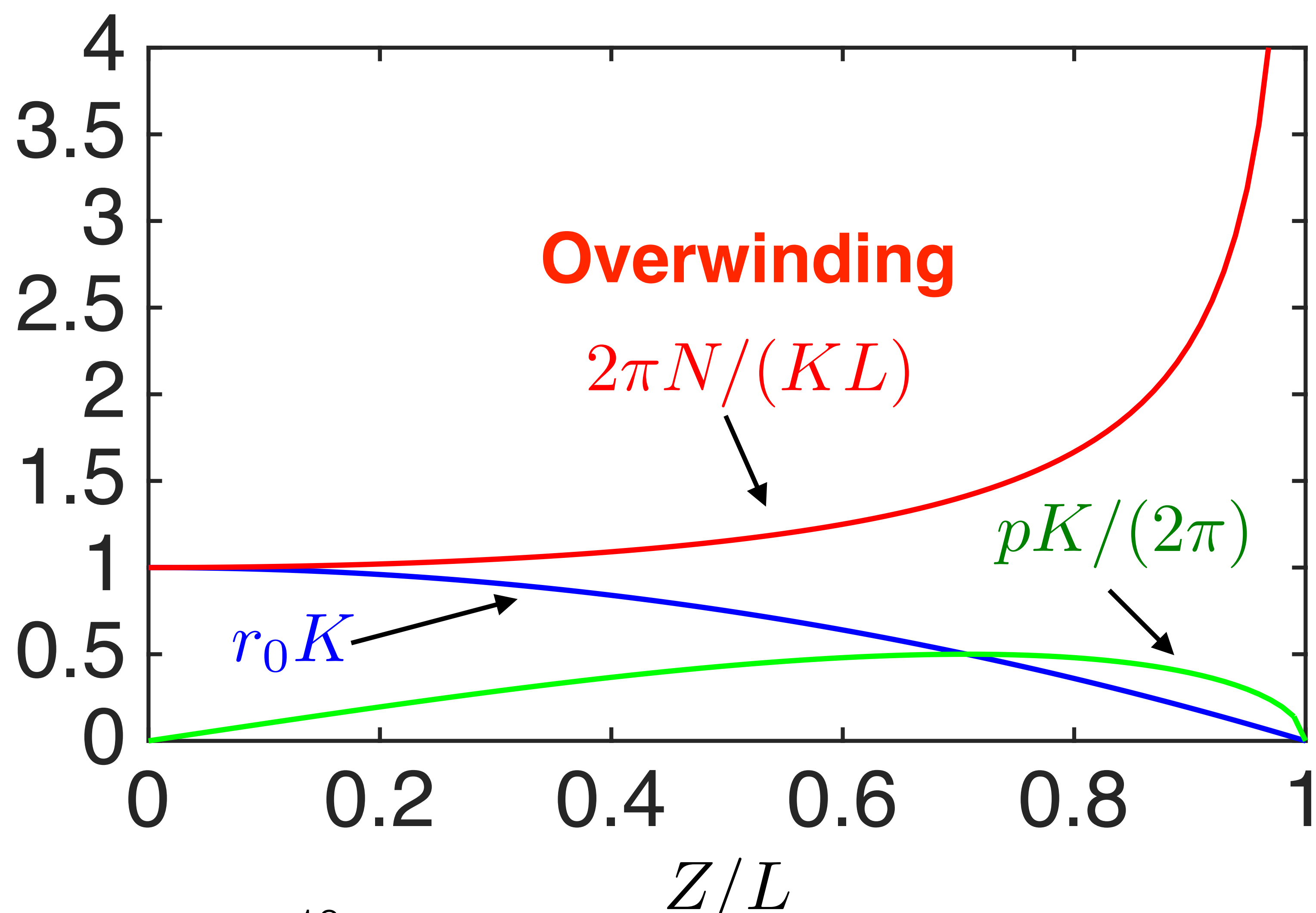
Helix pitch and radius

$$r_0 = \frac{1}{K}\left(1 - \frac{Z^2}{L^2}\right)$$

$$p = \frac{2\pi Z}{KL}\sqrt{1 - \frac{Z^2}{L^2}}$$

Number of loops

$$N = \frac{Z}{p} = \frac{KL}{2\pi\sqrt{1-(Z/L)^2}}$$



L **length of the helix backbone**

$N = \frac{Z}{p}$ **number of loops**

Spirals in nature

shells

beaks

claws

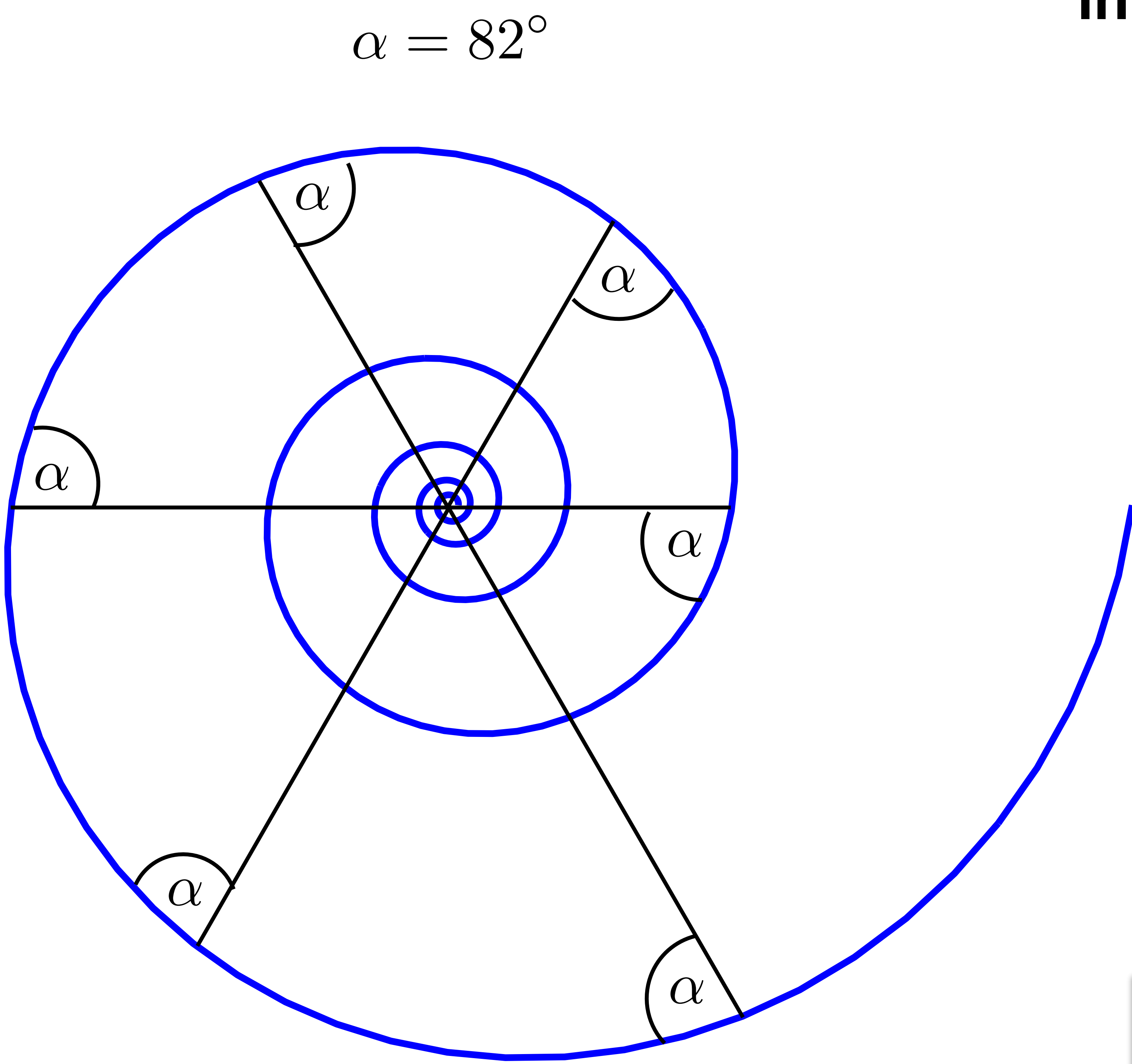
horns

teeth

tusks

What simple mechanism could produce spirals?

Equiangular (logarithmic) spiral



in polar coordinates radius grows exponentially

$$r(\theta) = a^{\theta} = \exp^{(\theta \cot \alpha)}$$

$$\cot \alpha = \ln a$$

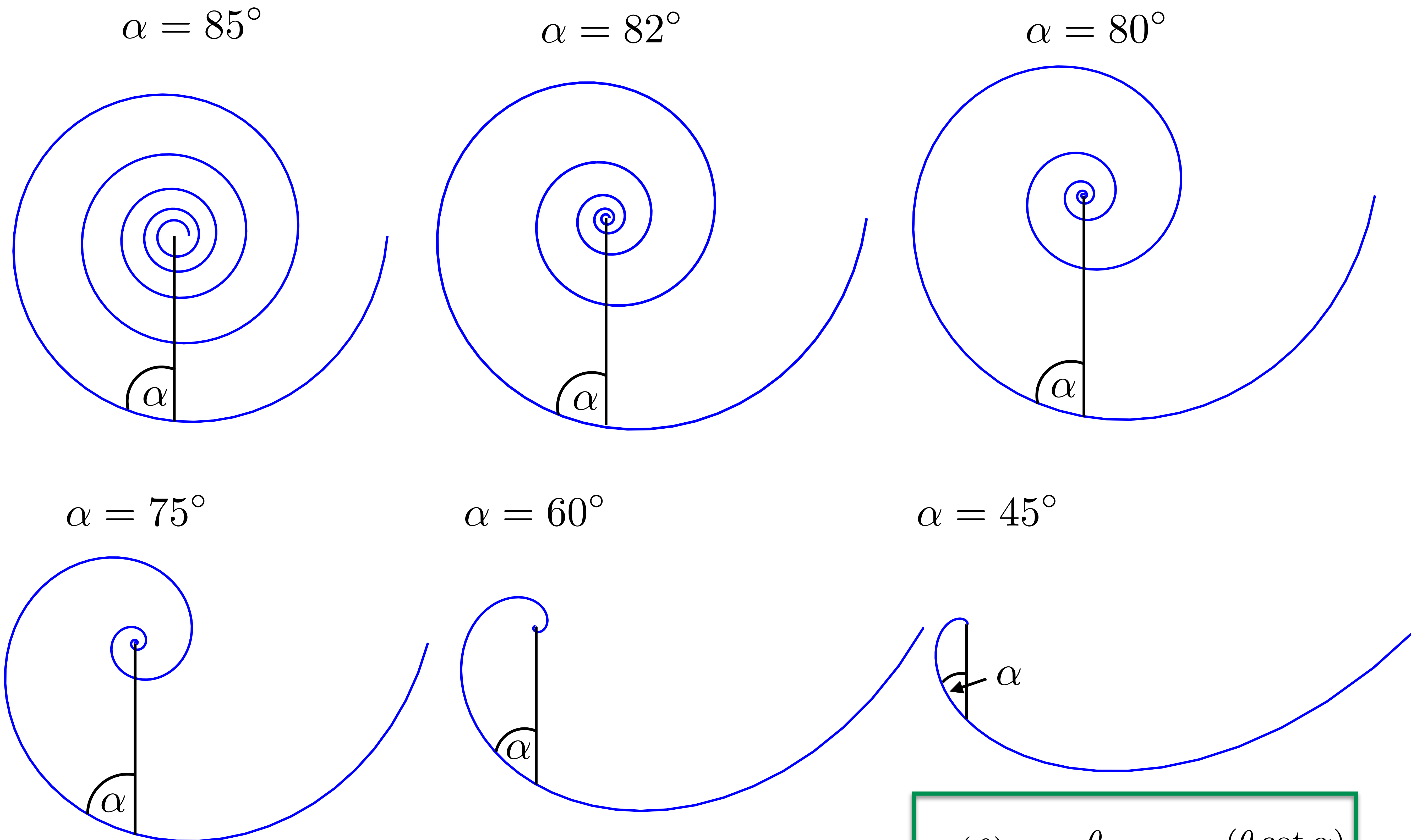
name logarithmic spiral:

$$\theta = \frac{\ln r}{\ln a}$$

Ratio between growth velocities in the radial and azimuthal directions velocities is constant!

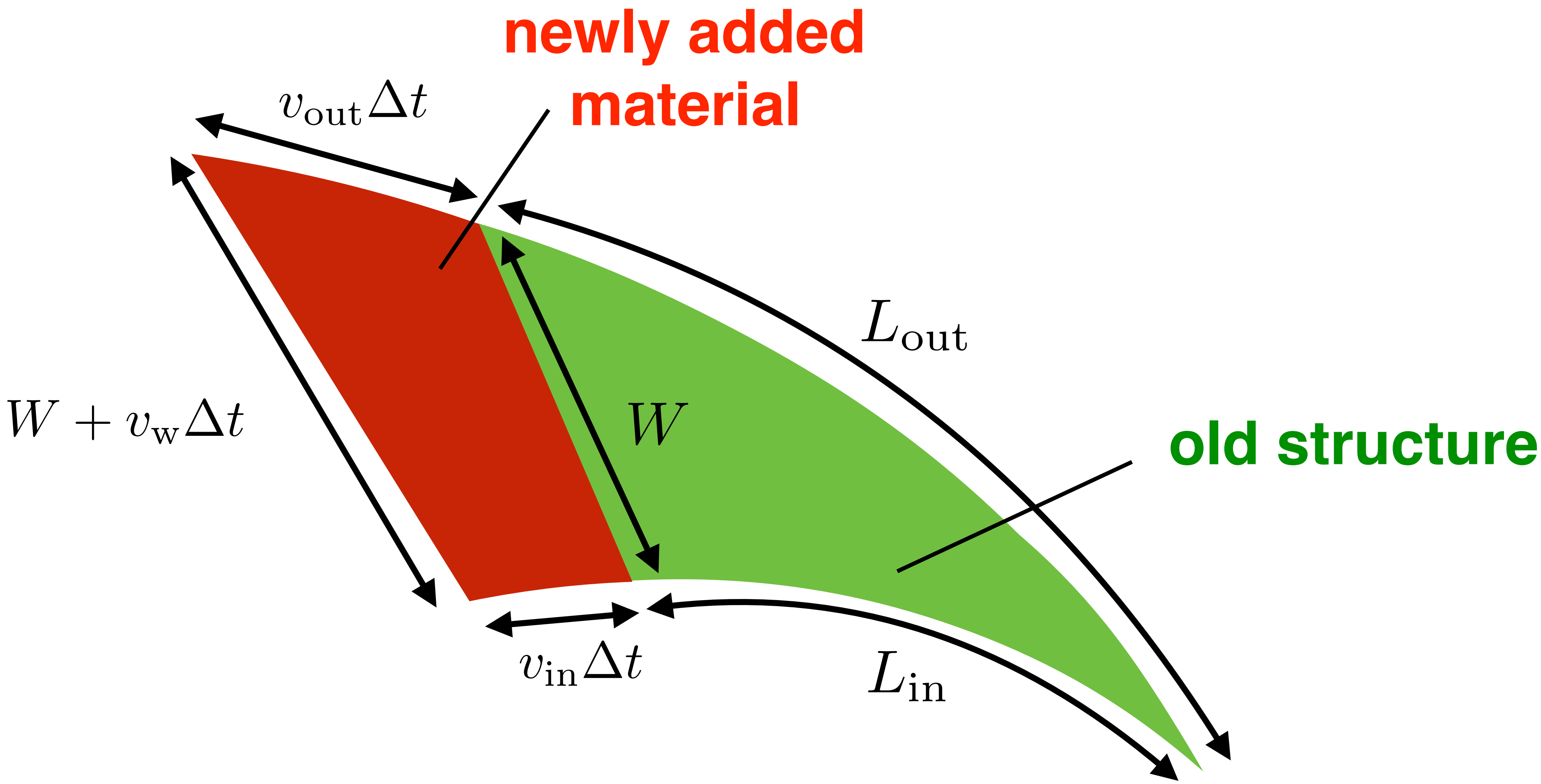
$$\cot \alpha = \frac{dr}{r d\theta} = \frac{dr/dt}{r d\theta/dt} = \frac{v_r}{v_\theta}$$

Equiangular (logarithmic) spiral



$$r(\theta) = a^{\theta} = \exp^{(\theta \cot \alpha)}$$

Growth of spiral structures



New material is added at a constant ratio of growth velocities, which produces spiral structure with side lengths and the width in the same proportions.

$$v_{\text{out}} : v_{\text{in}} : v_W = L_{\text{out}} : L_{\text{in}} : W$$

Note: growth with constant width (v_W=0) produces helices

Growth of spiral structures

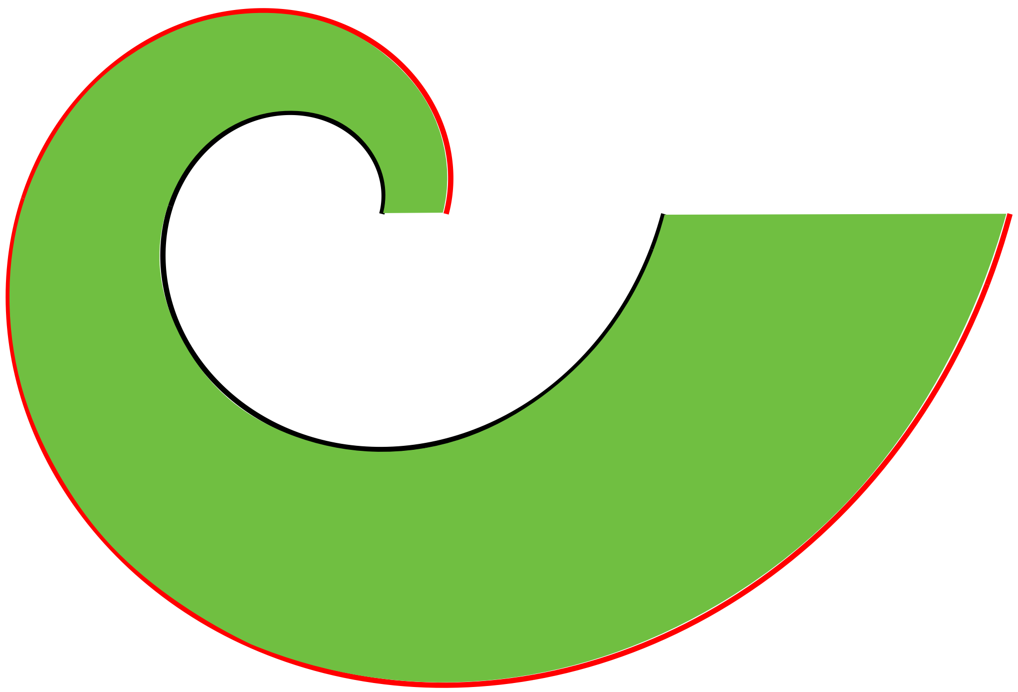
Assume the following spiral profiles of the outer and inner layers:

$$r_{\text{out}}(\theta) = e^{\theta \cot \alpha}$$

$$r_{\text{in}}(\theta) = \lambda e^{\theta \cot \alpha}$$

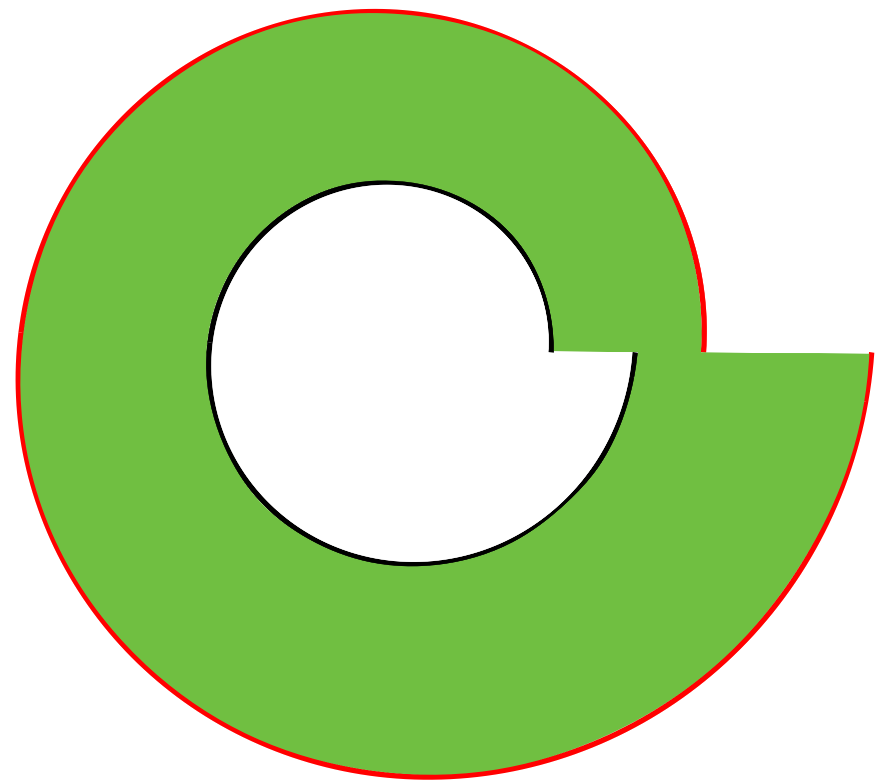
$$\lambda e^{2\pi \cot \alpha} > 1$$

$\lambda = 0.5, \ \alpha = 75^\circ$



$$\lambda e^{2\pi \cot \alpha} < 1$$

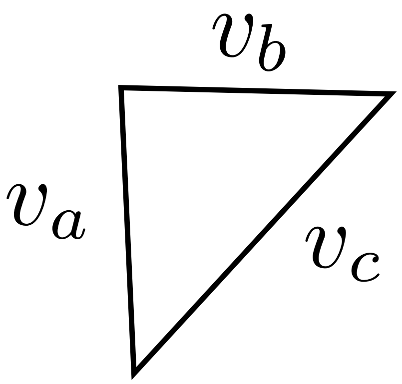
$\lambda = 0.5, \ \alpha = 86^\circ$



In some shells the inner layer does not grow at all

3D spirals

3D spiral of ram's horns is due to the triangular cross-section of the horn, where each side grows with a different velocity.



Shells of mollusks are often conical

Phyllotaxis

Phyllotaxis is classification of leaves on a plant stem

maize

distichous pattern

leaves alternating every 180^{0}

Coleus sp.

decussate pattern

pairs of leaves at 90^{0}

Veronicastrum virginicum

whorled pattern

3 or more leaves originating from the same node (180^{0})

sunflower

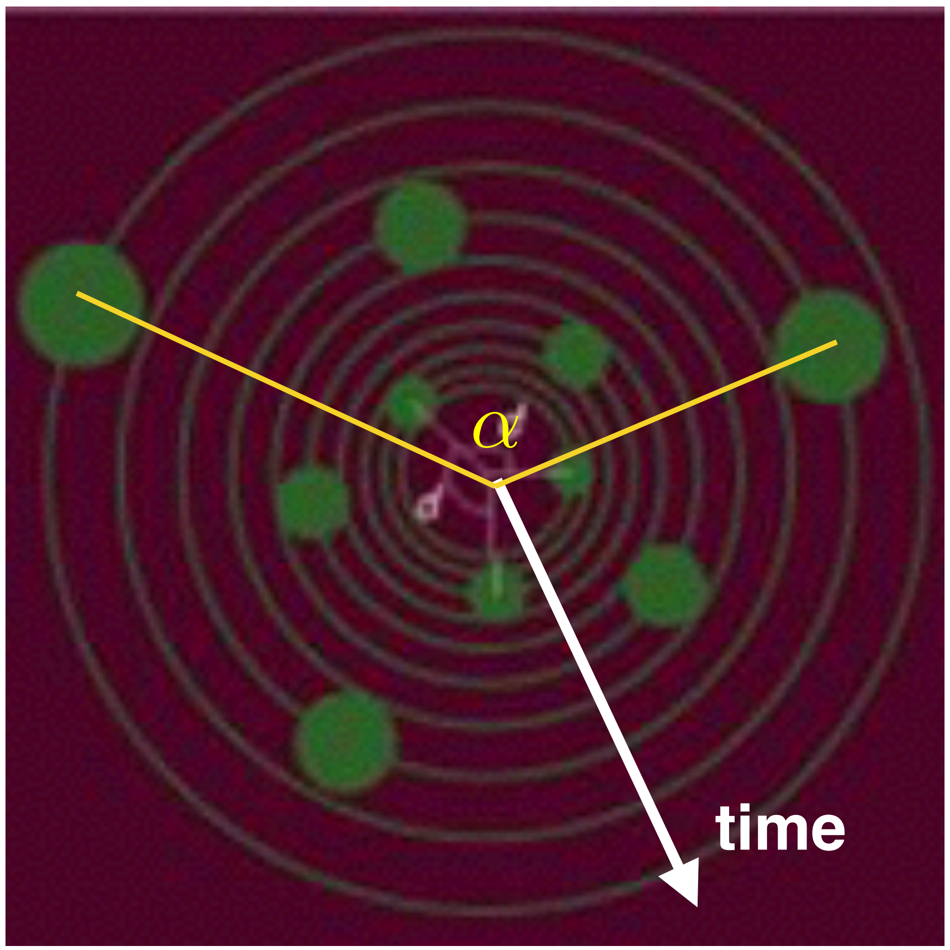
alternate (spiral) pattern

successive leaves at 137.5^{0}

http://www.sciteneg.com/PhiTaxis/PHYLLOTAXIS.htm

Spiral phyllotaxis

schematic description of leaves arrangement



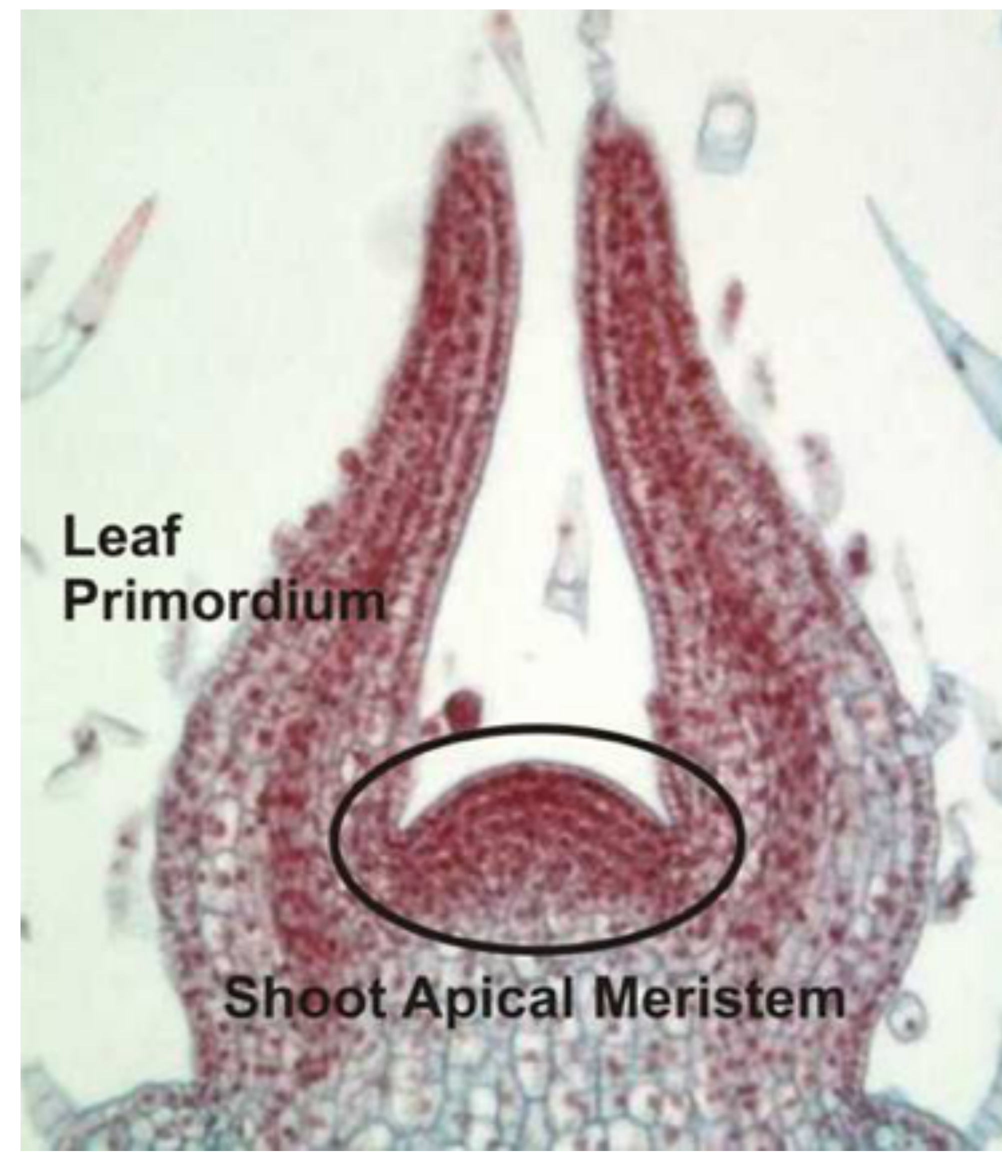
$\alpha \approx 137.5^\circ$

florets (petals)

floral primordia

leaves

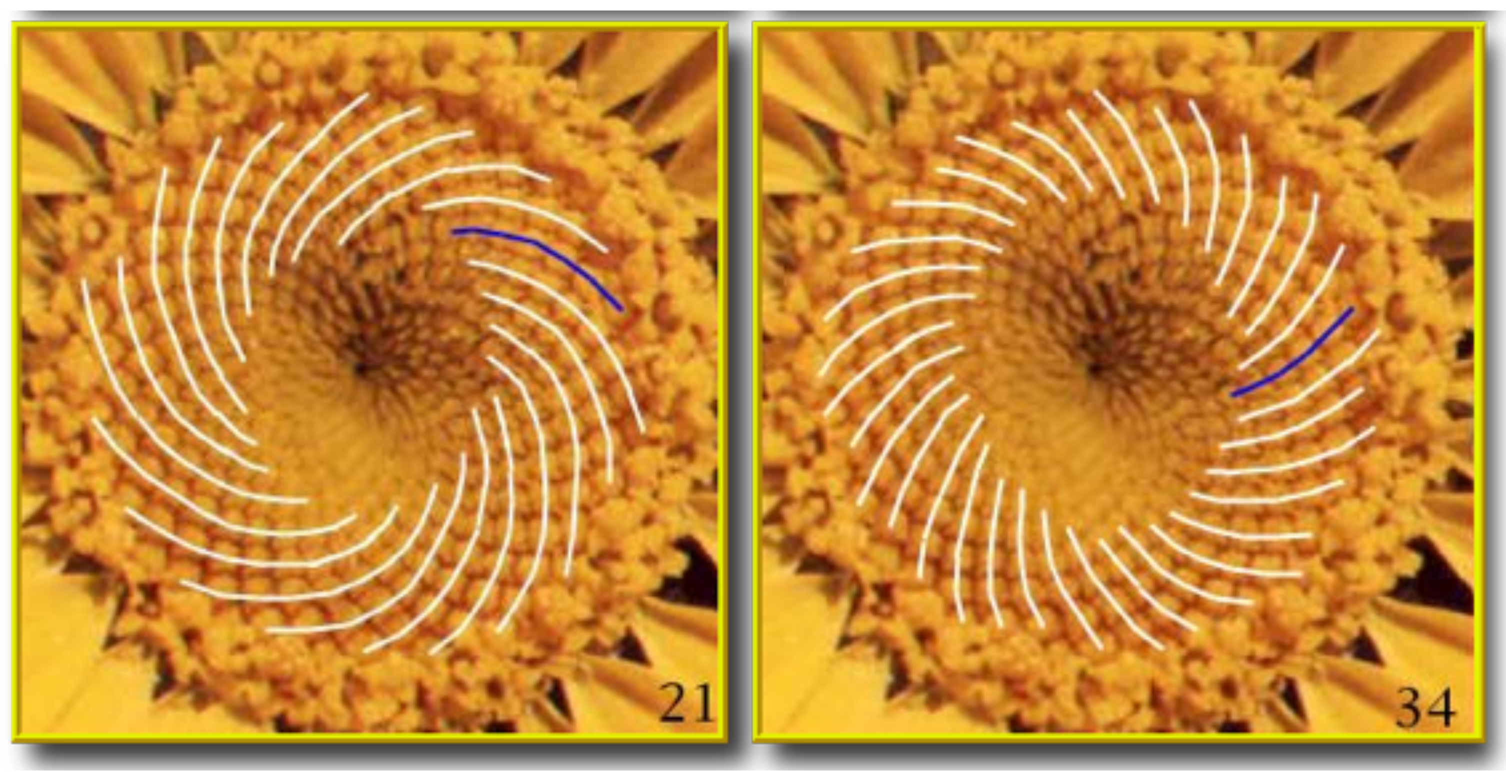
leaves grow from the apical meristem, which also gives rise to petals, sepals, etc.



Parastichy numbers

21 left-handed spirals

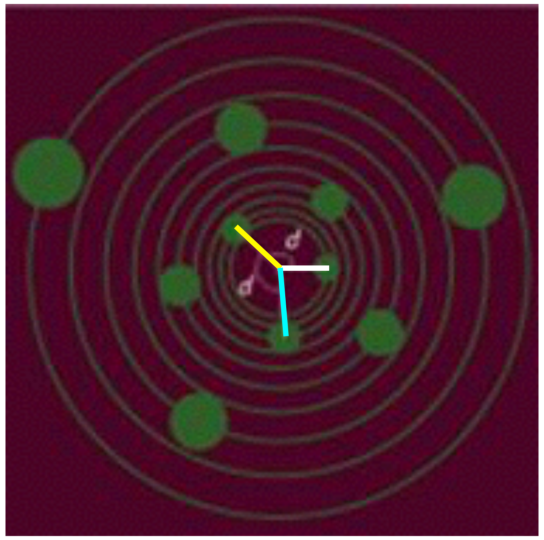
34 right-handed spirals



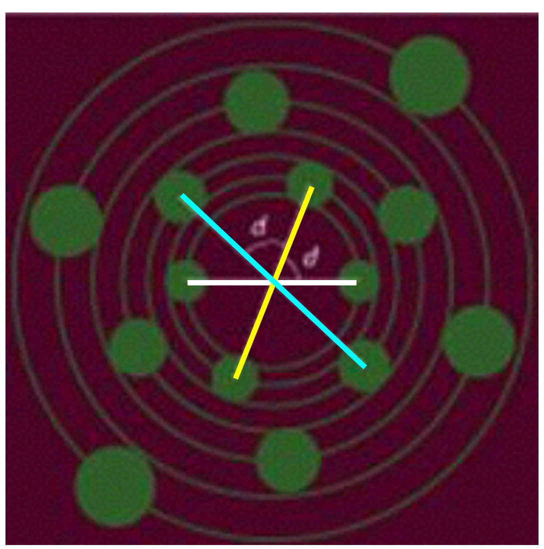
Parastichy numbers (21,34)

Parastichy numbers

spiral phyllotaxis

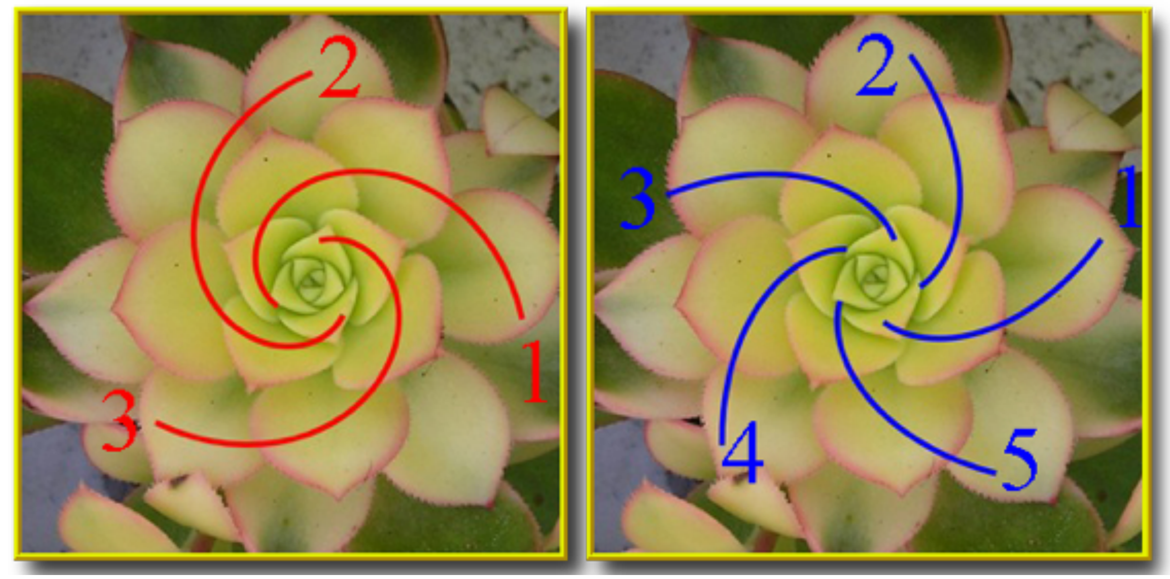


multijugate phyllotaxis

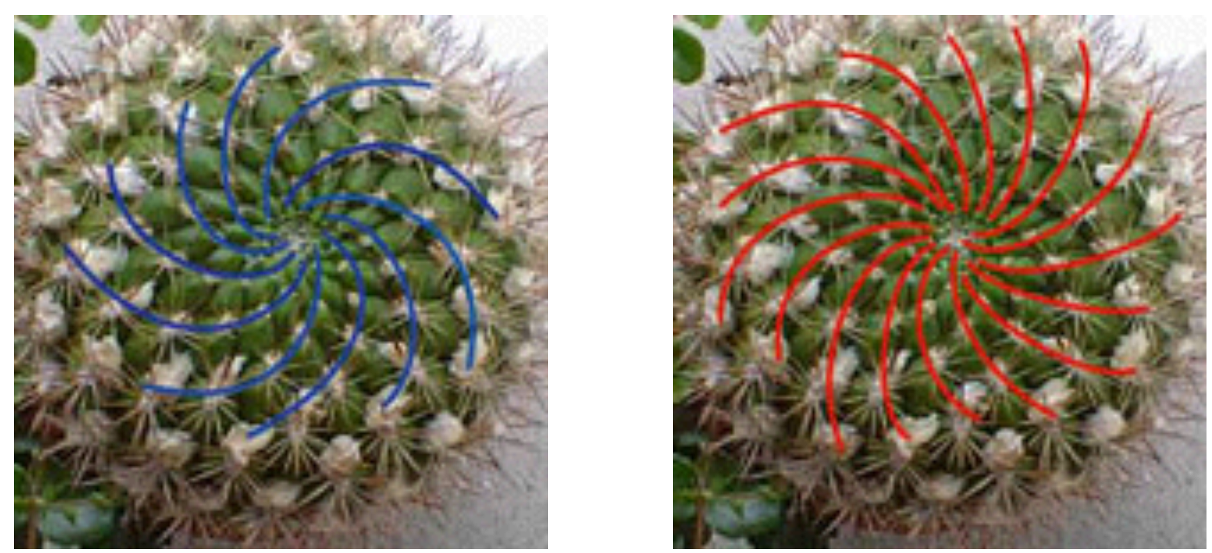


(e.g. 2 new leaves are added at the same time)

succulent plant (3,5)

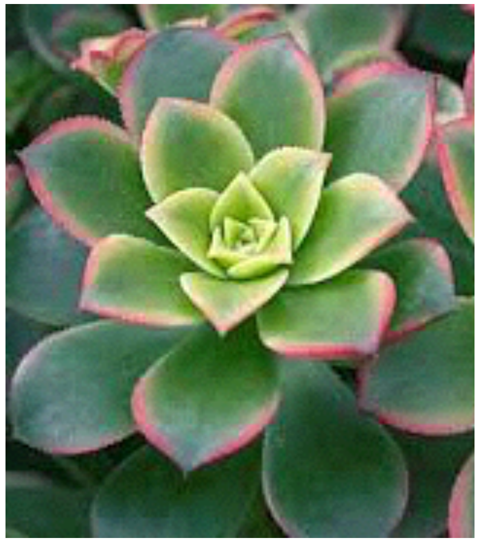


Gymnocalycium (10,16)=2(5,8)



Parastichy numbers

aonium (2,3)



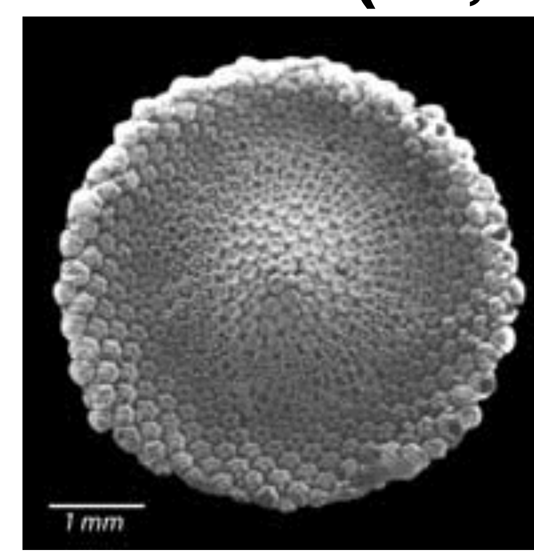
succulent plant (3,5)

aloe (5,8)

pince cone (8,13)

sunflower (21,34)

artichoke (34,55)



Parastichy numbers very often correspond to successive Fibonacci numbers!

Fibonacci numbers

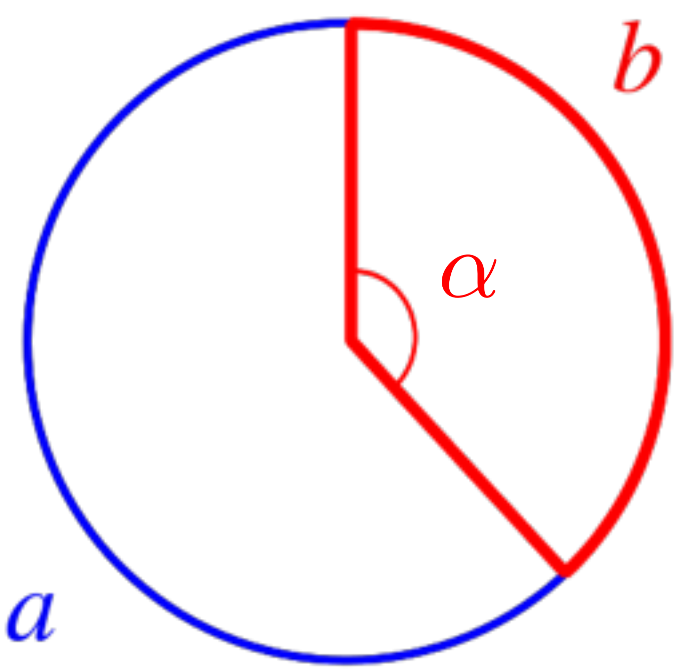
$$F_1 = 1$$
$$F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

Golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$

$$F_n = \frac{1}{\sqrt{5}}\left[\varphi^n - (1-\varphi)^n\right]$$

Sequence of Fibonacci numbers
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

Golden angle



divide perimeter in golden ratio $\frac{a+b}{a} = \frac{a}{b} \longrightarrow \frac{a}{b} = \varphi$

$$\alpha = 360^\circ \frac{b}{(a+b)} = \frac{360^\circ}{\varphi^2} \approx 137.5^\circ$$

In spiral phyllotaxis successive leaves grow at approximately Golden angle!

Non-Fibonacci parastichy numbers

Statistics for pine trees in Norway

95% Fibonacci numbers
4% Lucas numbers
1% not properly formed

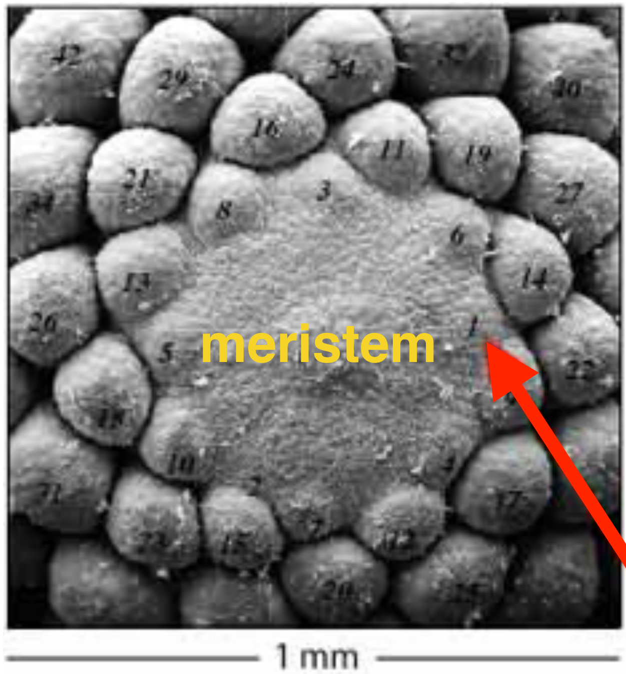
Lucas numbers

$$L_1 = 1$$
$$L_2 = 3$$
$$L_n = L_{n-1} + L_{n-2}$$

Sequence of Lucas numbers
1, 3, 4, 7, 11, 18, 29, 47, 76

Spiral phyllotaxis

Norway spruce

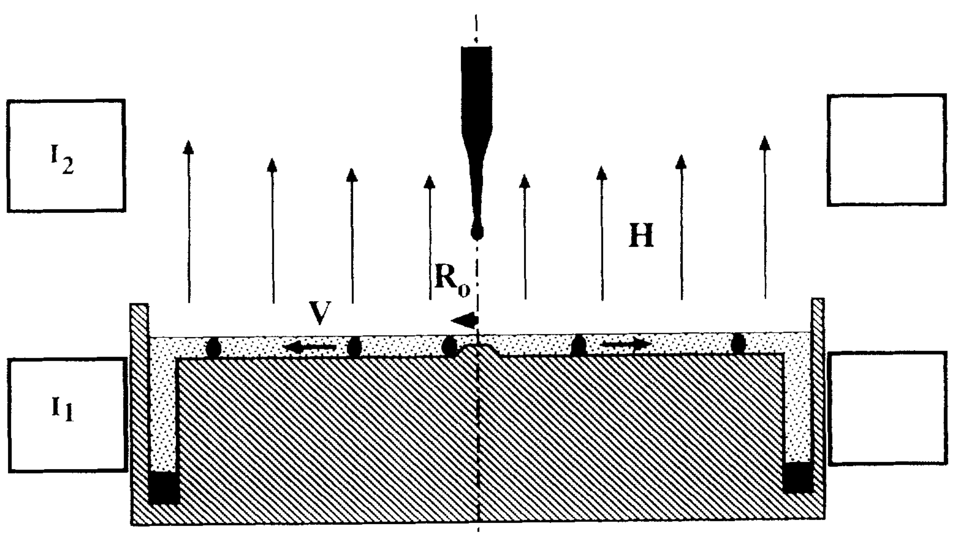


new primordial

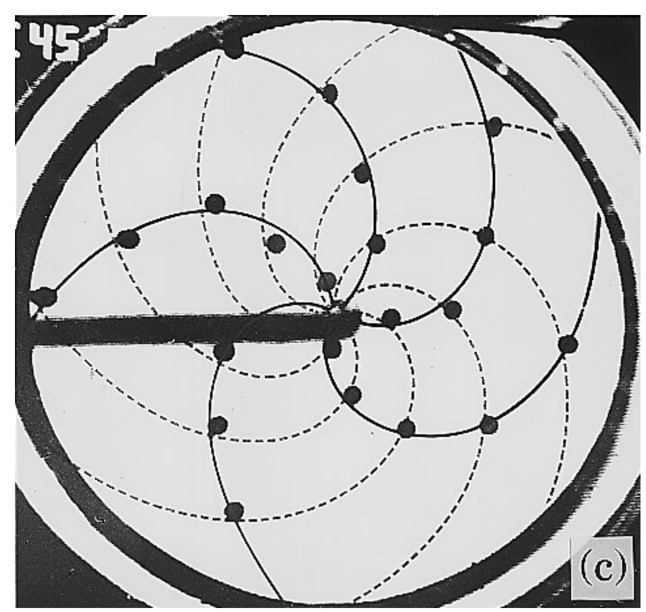
New primordia start growing at the site where plant hormone auxin is depleted.

Auxin hormones are released by growing primordia. New primordium wants to be as far apart as possible from the existing primordia.

Mechanical analog with magnetic repelling particles



magnetic field drives particles away from the center

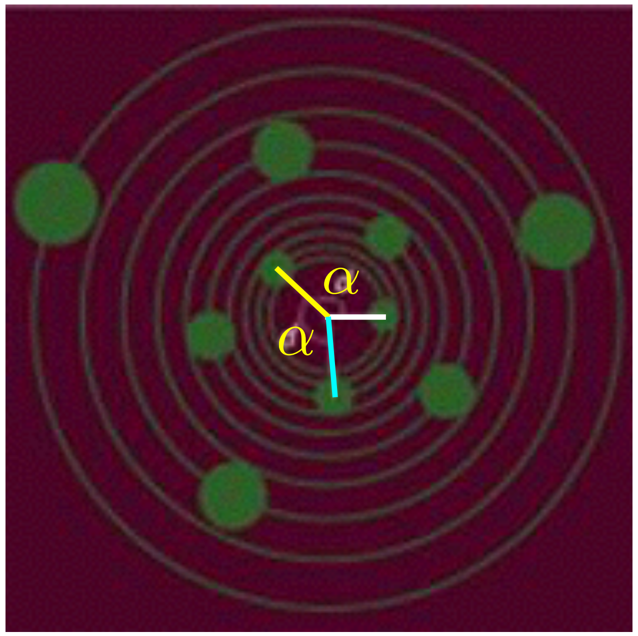


Parastichy numbers (5,8)

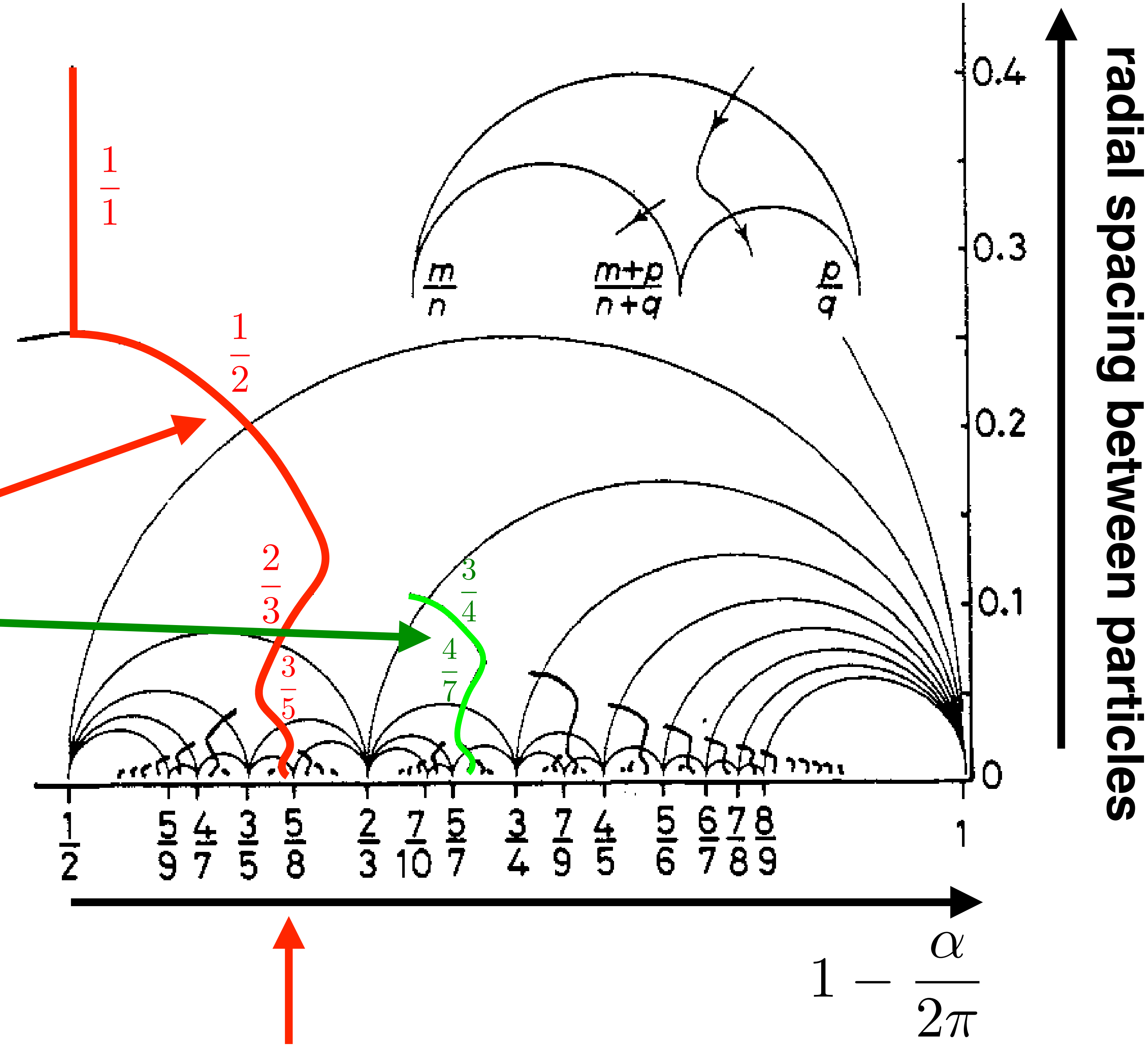
particles repel via magnetic dipol interactions

S. Douady and Y. Couder, PRL 68, 2098 (1992)

Energy minimization between repelling particles



Local energy minima for repelling particles



Fibonacci numbers

Lucas numbers

golden angle

As the plant is growing it is gradually reducing the time delay between formation of new primordia. The spiral patterns then go sequentially through all the Fibonacci parastichies.

Occasional excursions to the neighbor local minima produce Lucas parastichy numbers.

L. Levitov, PRL 66, 224 (1991)
L. Levitov, EPL 14, 533 (1991)

Further reading