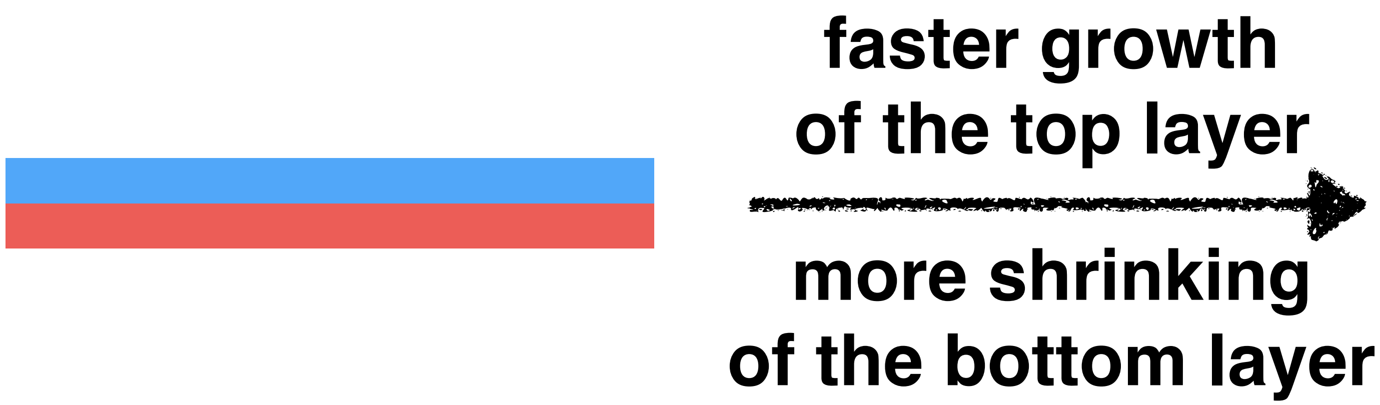
MAE 545: Lecture 10,11 (3/28, 3/30)

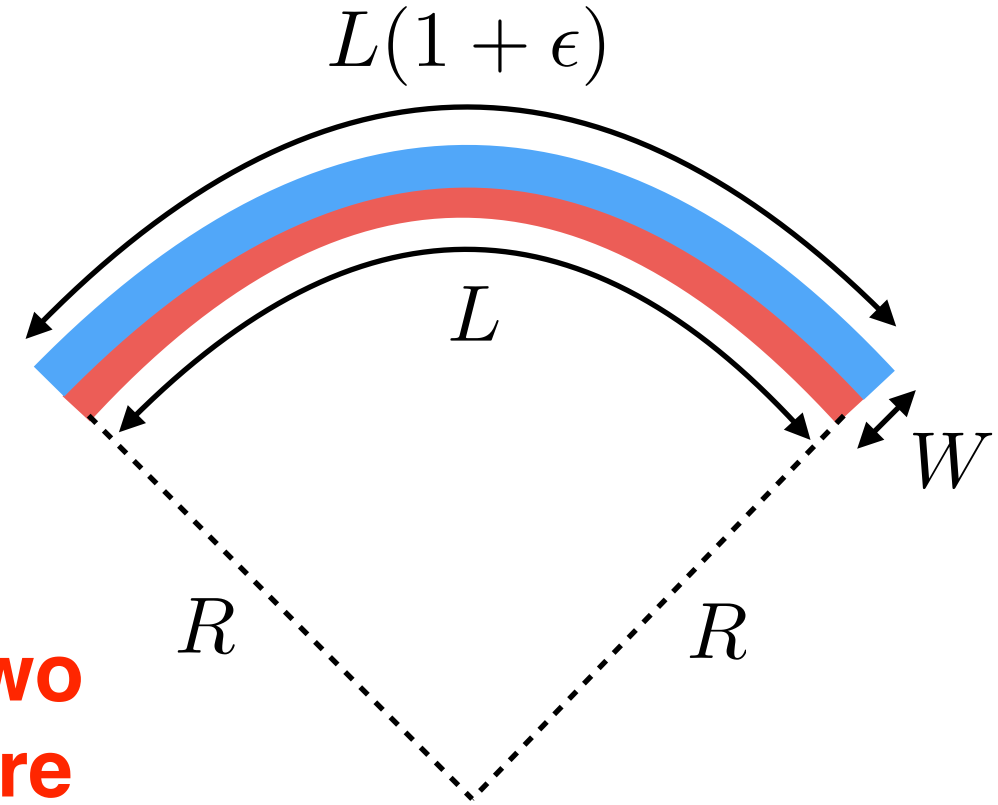
Helices and spirals

Helices in plants

How are helices formed?

Differential growth or differential shrinking produces spontaneous curvature





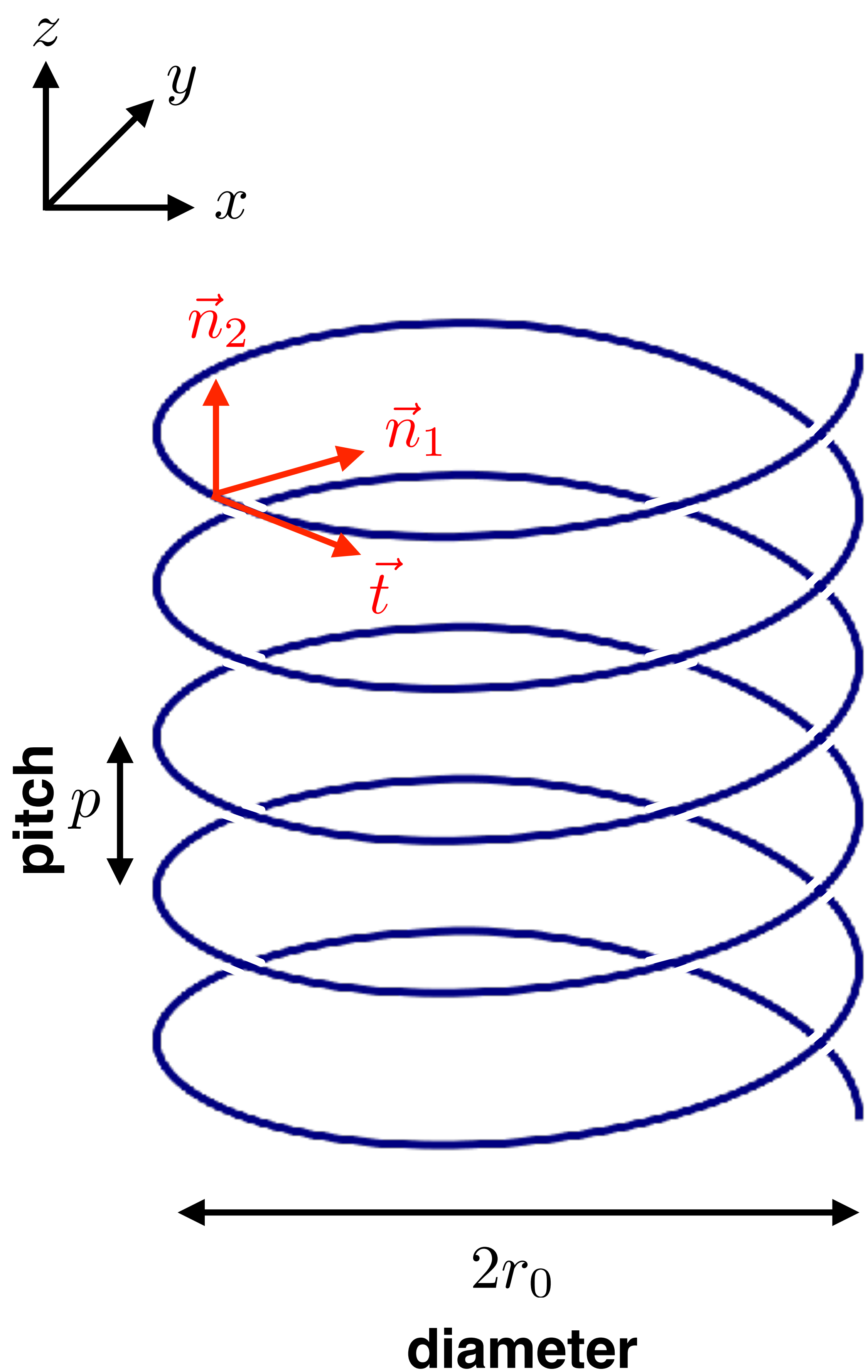
Differential growth (shrinking) of the two layers produces spontaneous curvature

$$\boxed{K = \frac{1}{R} = \frac{\epsilon}{W}}$$

$$\frac{L(1+\epsilon)}{L} = \frac{R+W}{R}$$

Filaments that are longer than $L > 2\pi R$ form helices to avoid steric interactions.

Helix



Mathematical description

$$\vec{r}(s) = \left(r_0 \cos(s/\lambda), r_0 \sin(s/\lambda), \frac{p}{2\pi\lambda} s \right)$$

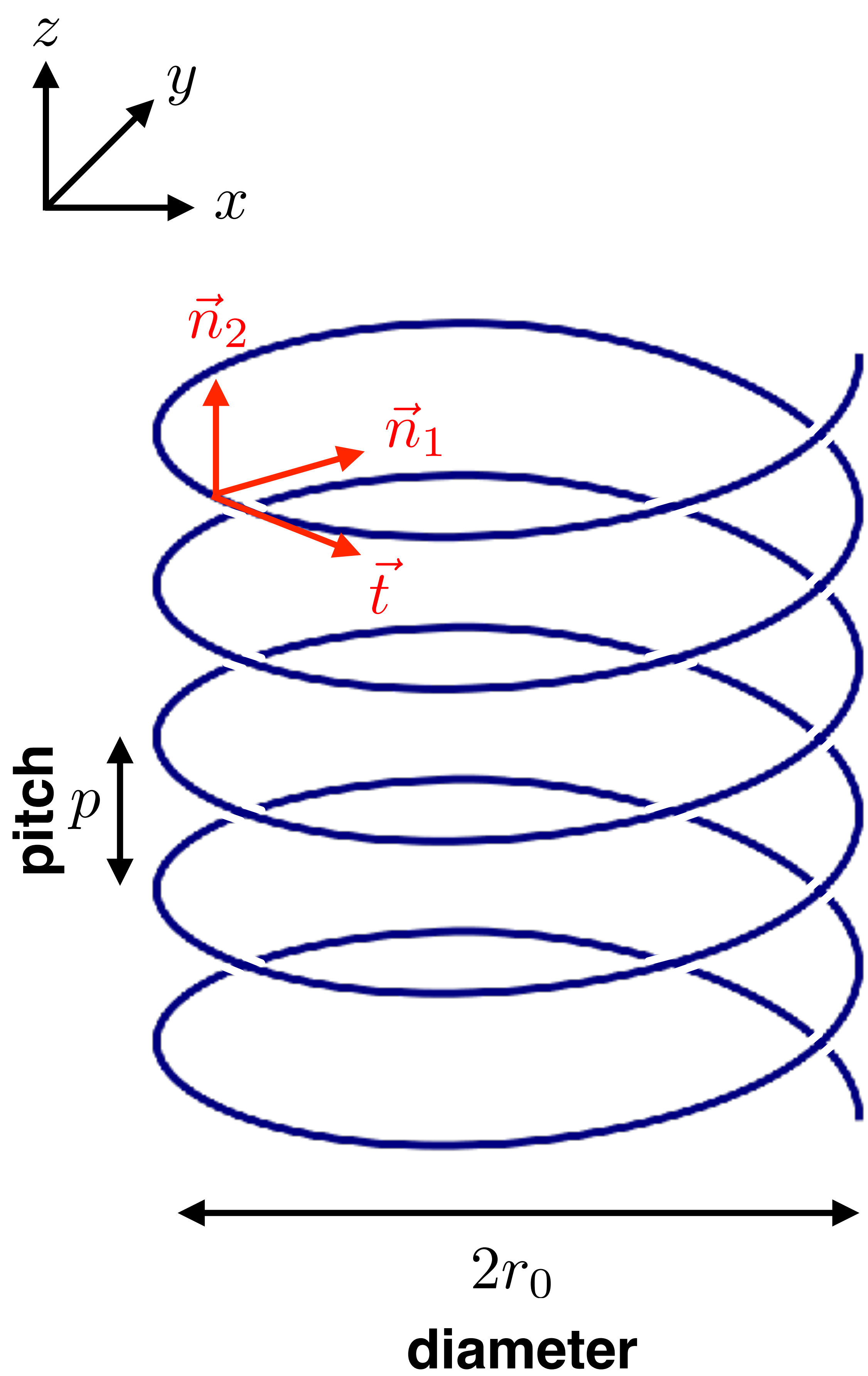
Set λ to fix the metric in natural parametrization:

$$\vec{t}(s) = \frac{d\vec{r}}{ds} = \left(-\frac{r_0}{\lambda} \sin(s/\lambda), \frac{r_0}{\lambda} \cos(s/\lambda), \frac{p}{2\pi\lambda} \right)$$

$$g = \vec{t} \cdot \vec{t} = \frac{r_0^2}{\lambda^2} + \frac{p^2}{4\pi^2\lambda^2} = 1$$

$$\lambda = \sqrt{r_0^2 + (p/2\pi)^2}$$

Helix



Mathematical description

$$\vec{r}(s) = \left(r_0 \cos(s/\lambda), r_0 \sin(s/\lambda), \frac{p}{2\pi\lambda} s \right)$$

$$\lambda = \sqrt{r_0^2 + (p/2\pi)^2}$$

Tangent and normal vectors

$$\vec{t}(s) = \frac{d\vec{r}}{ds} = \left(-\frac{r_0}{\lambda} \sin(s/\lambda), \frac{r_0}{\lambda} \cos(s/\lambda), \frac{p}{2\pi\lambda} \right)$$

$$\vec{n}_1(s) = \big(-\cos(s/\lambda), -\sin(s/\lambda), 0 \big)$$

$$\vec{n}_2(s) = \left(\frac{p}{2\pi\lambda} \sin(s/\lambda), -\frac{p}{2\pi\lambda} \cos(s/\lambda), \frac{r_0}{\lambda} \right)$$

Helix curvatures

$$\vec{n}_1 \cdot \frac{d^2\vec{r}}{ds^2} = \frac{r_0}{\lambda^2} = \frac{r_0}{r_0^2 + (p/2\pi)^2} = K$$

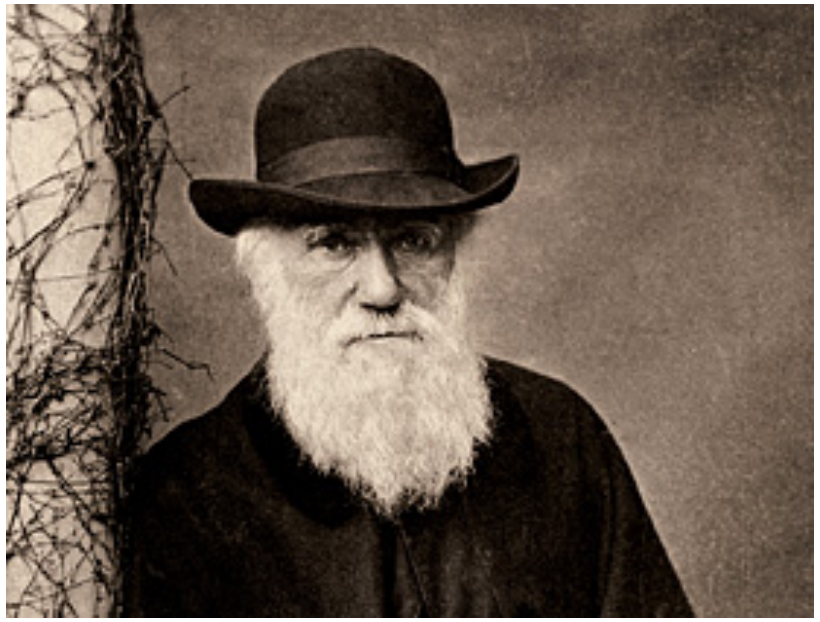
$$\vec{n}_2 \cdot \frac{d^2\vec{r}}{ds^2} = 0$$

Cucumber tendril climbing via helical coiling

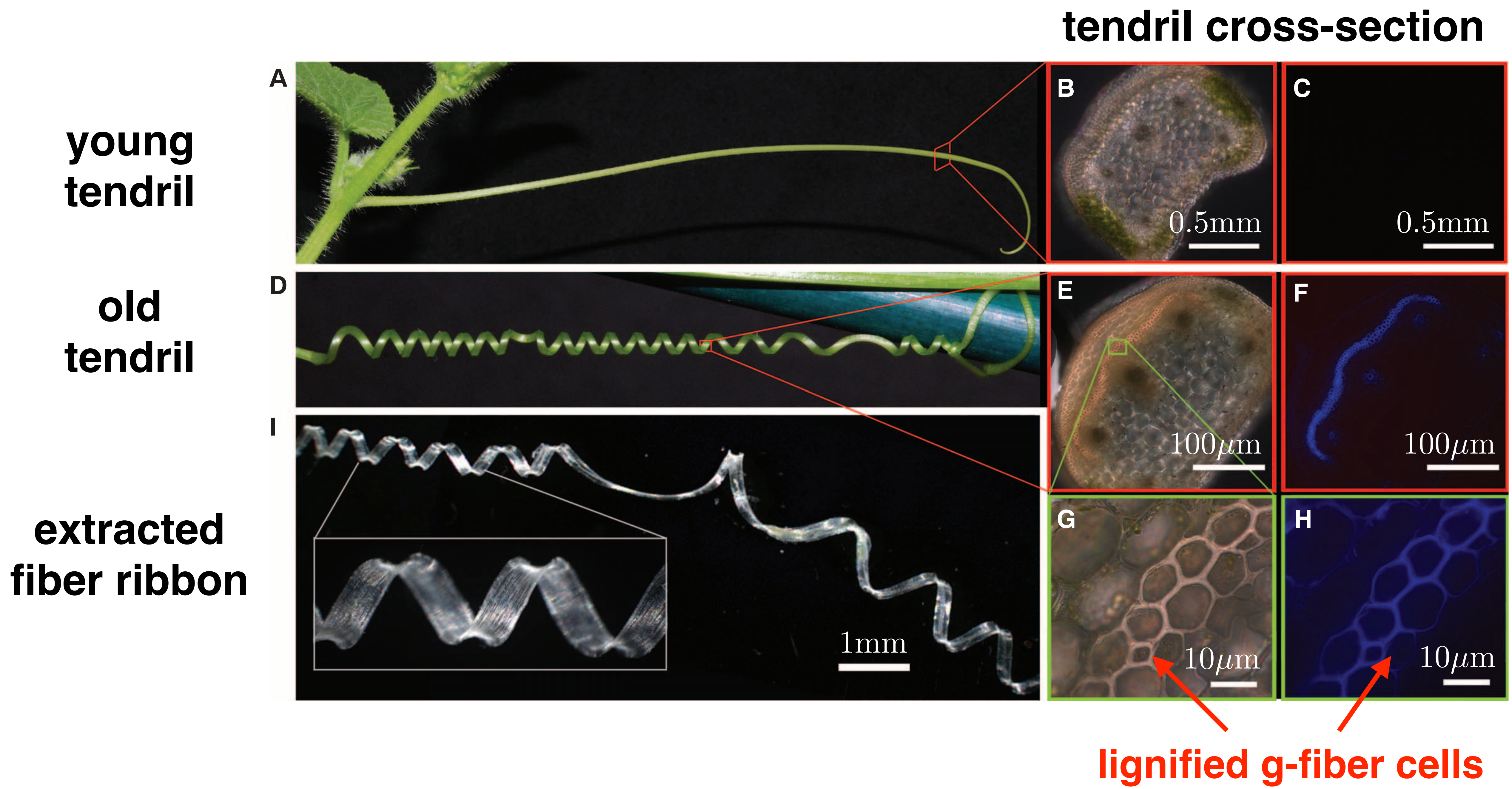
Cucumber tendrils want to pull themselves up above other plants in order to get more sunlight.

S. J. Gerbode et al., Science 337, 1087 (2012)

Already studied by Charles Darwin in 1865:



Helical coiling of cucumber tendril

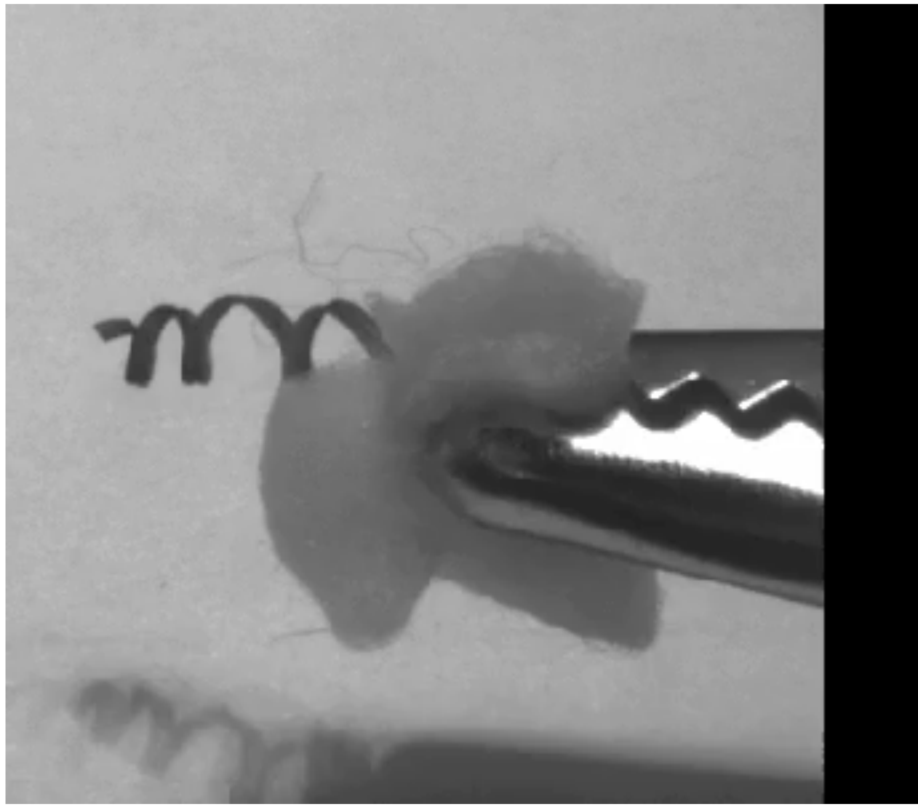


Coiling in older tendrils is due to a thin layer of stiff, lignified gelatinous fiber cells, which are also found in wood.

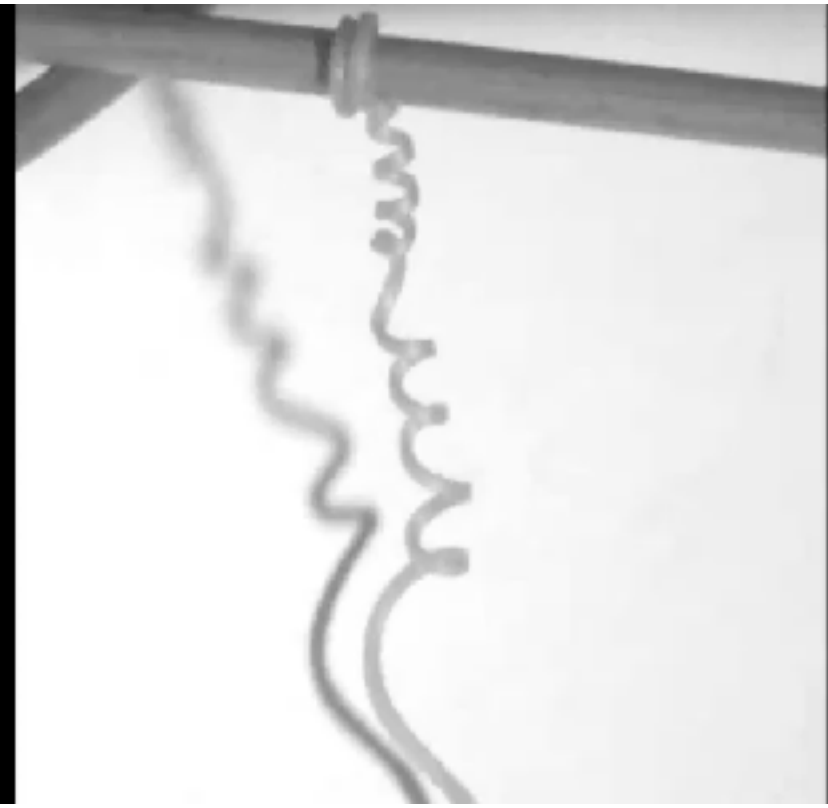
S. J. Gerbode et al., Science 337, 1087 (2012)

Helical coiling of cucumber tendril

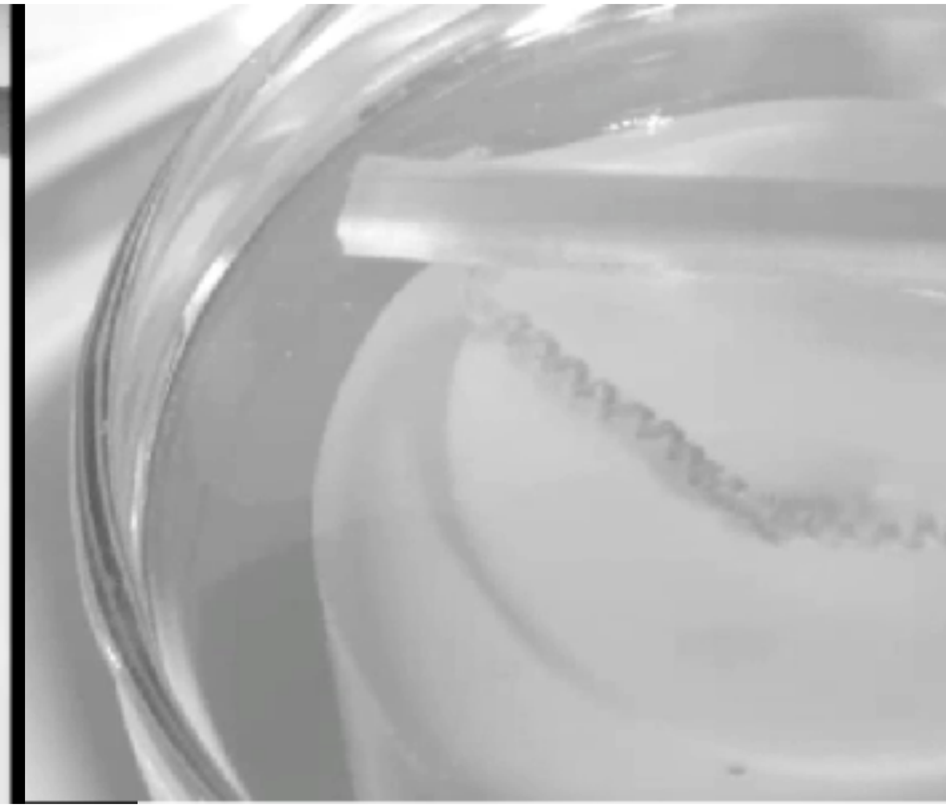
Drying of fibber ribbon increases coiling



Drying of tendril increases coiling

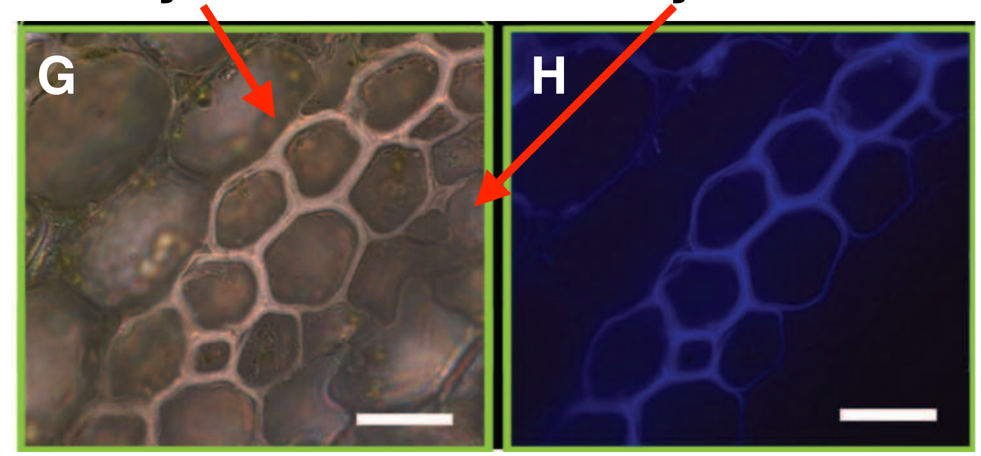


Rehydrating of tendril reduces coiling



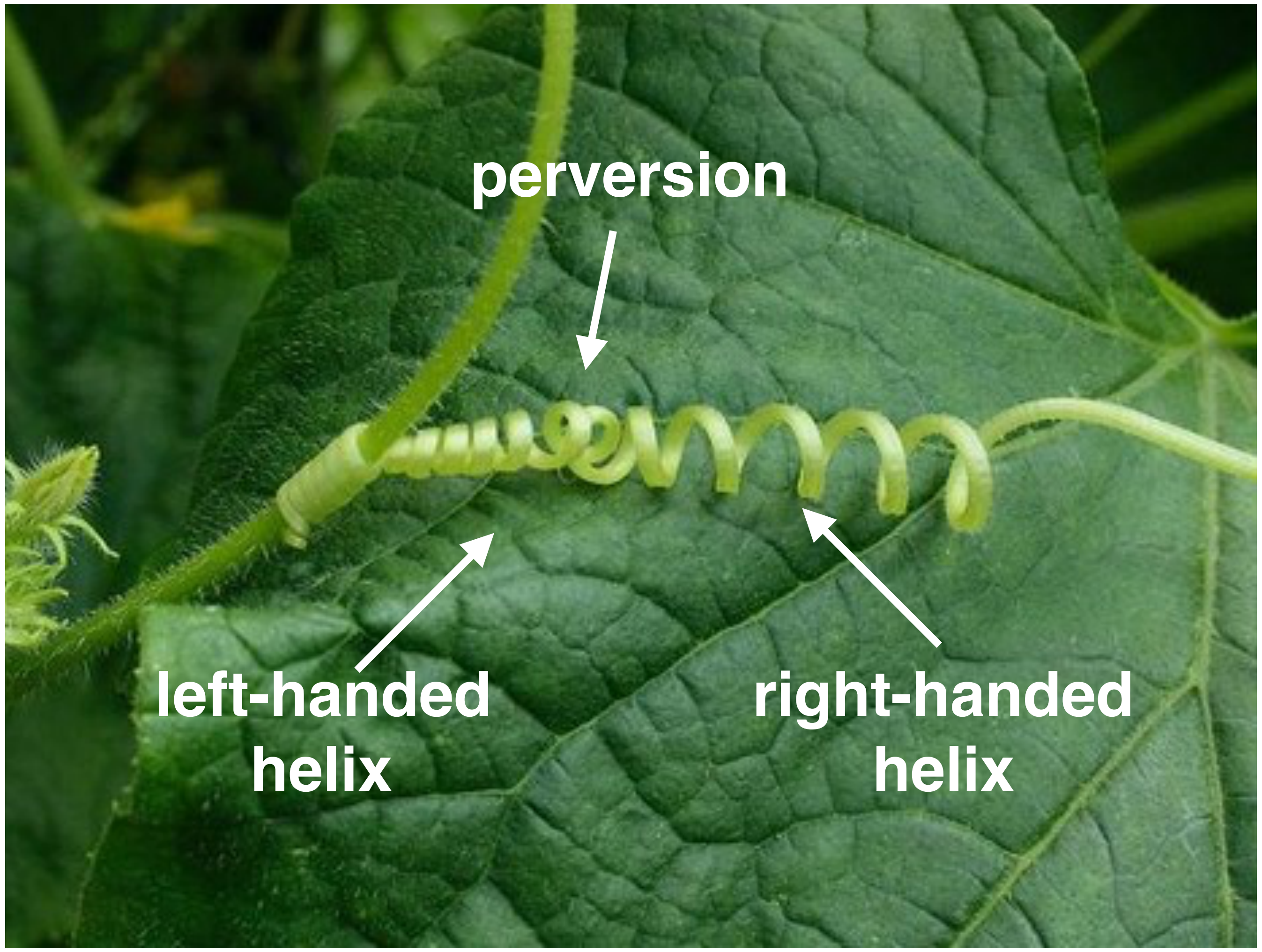
During the lignification of g-fiber cells water is expelled, which causes shrinking.

The inside layer is more lignified and therefore shrinks more and is also stiffer than the outside layer.



S. J. Gerbode et al., Science 337, 1087 (2012)

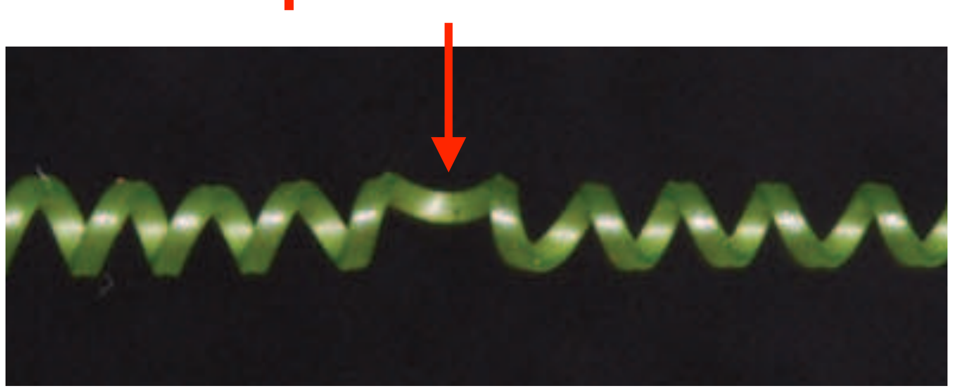
Coiling of tendrils in opposite directions



Ends of the tendril are fixed and cannot rotate. This constraints the linking number.

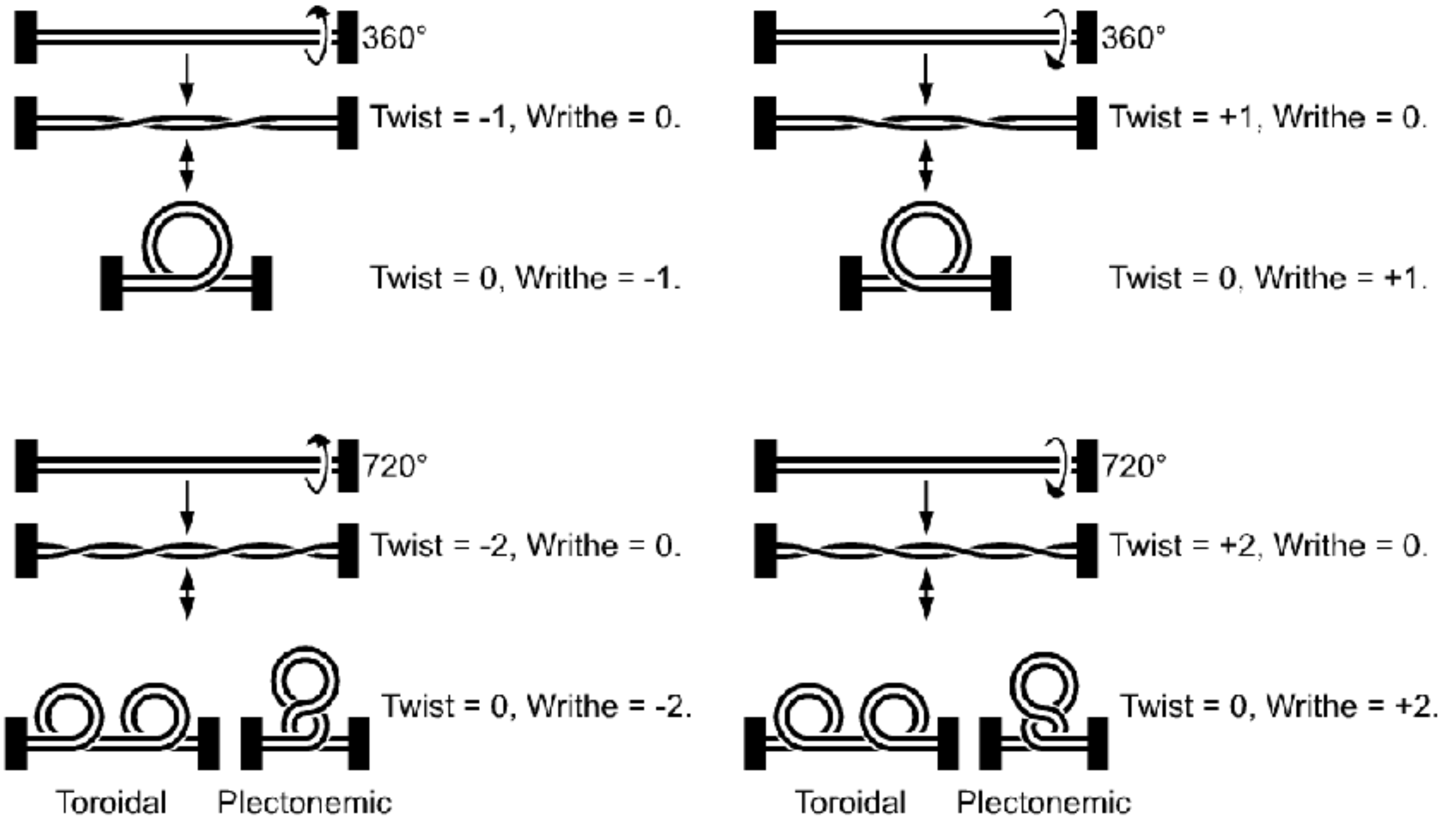
Link = Twist + Writhe

perversion

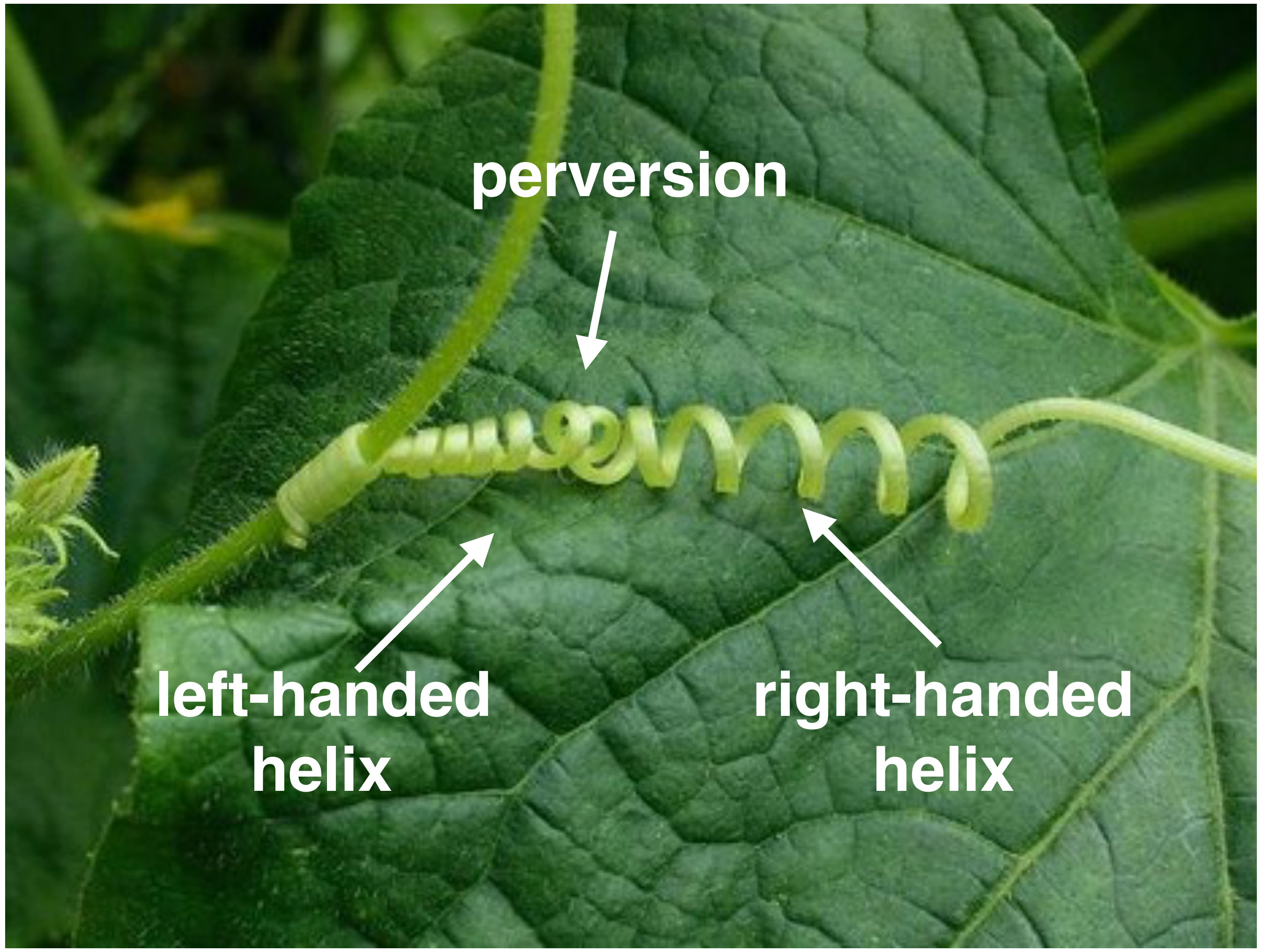


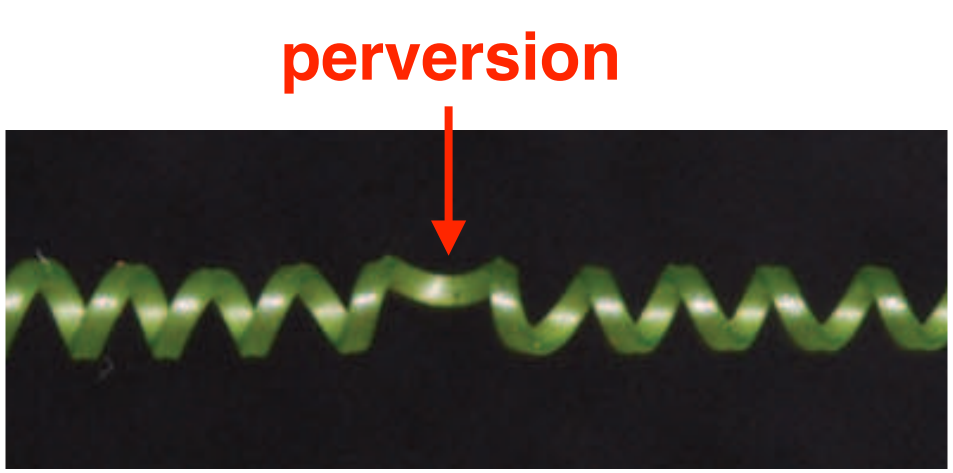
Twist, Writhe and Linking numbers

Ln=Tw+Wr	linking number: total number of turns of a particular end
Tw	twist: number of turns due to twisting the beam
Wr	writhe: number of crossings when curve is projected on a plane



Coiling of tendrils in opposite directions





Ends of the tendril are fixed and cannot rotate. This constraints the linking number.

Link = Twist + Writhe

Coiling in the same direction increases Writhe, which needs to be compensated by the twist.

In order to minimize the twisting energy tendrils combine two helical coils of opposite handedness (=opposite Writhe).

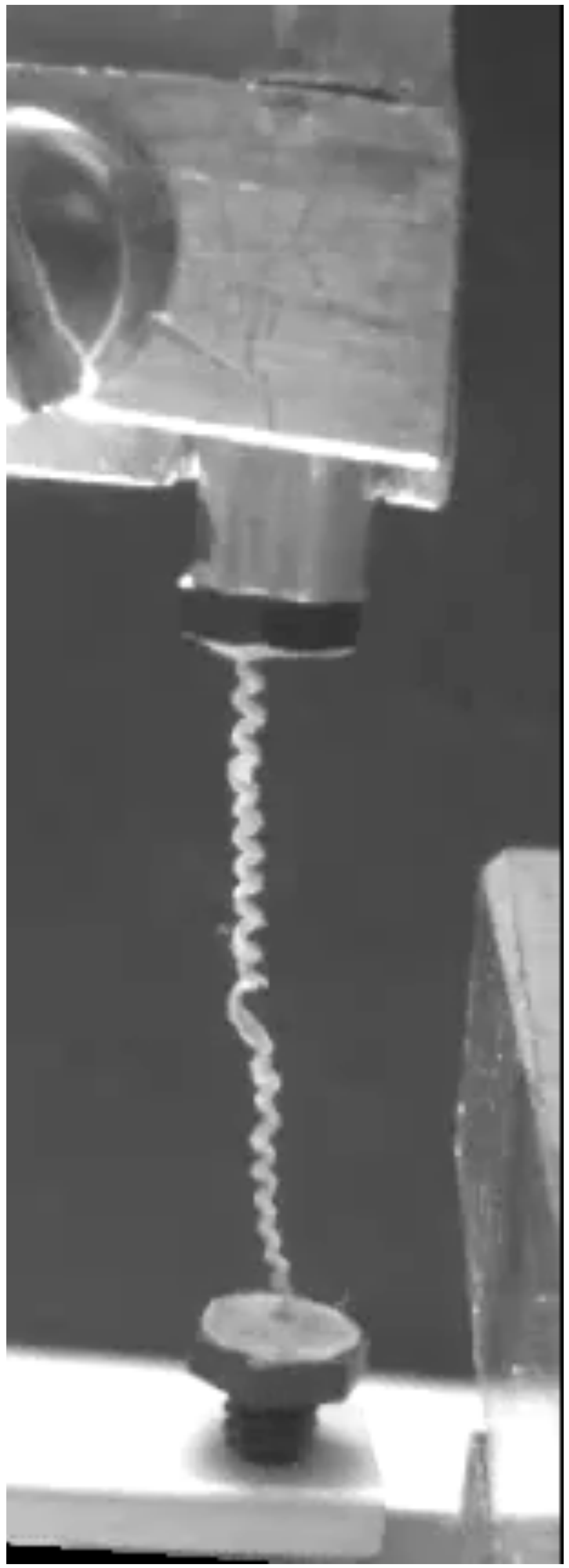
Note: there is no bending energy when the curvature of two helices correspond to the spontaneous curvature due to the differential shrinking of fiber.

Overwinding of tendril coils

Old tendrils overwind when stretched.

relaxed

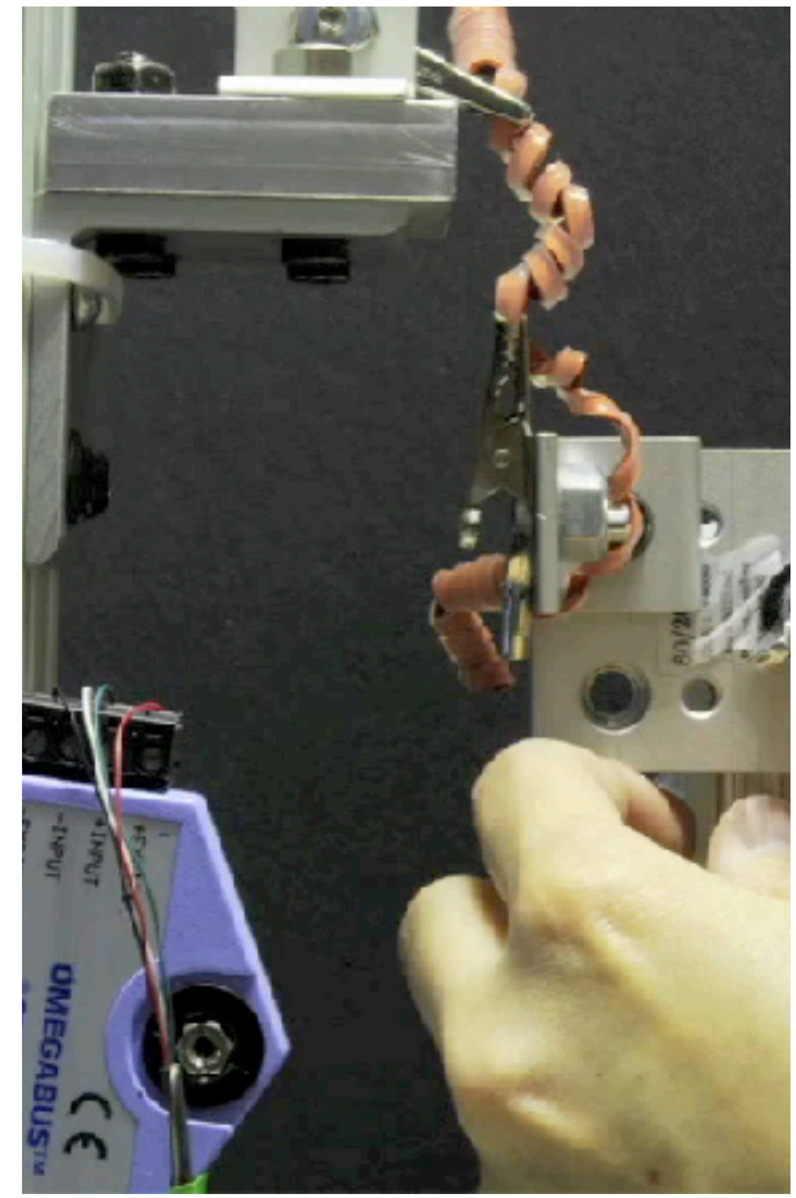
stretched



Rubber model unwinds when stretched.

relaxed

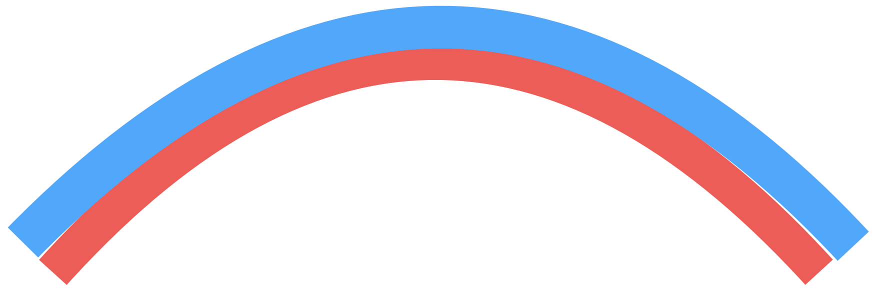
stretched



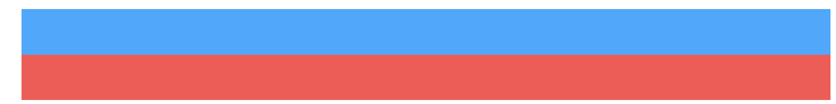
S. J. Gerbode et al., Science 337, 1087 (2012)

Overwinding of tendril coils

Preferred curved state



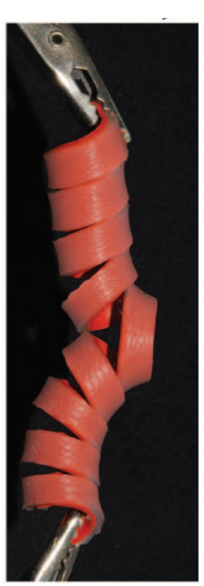
Flattened state



In tendrils the red inner layer is much stiffer then the outside blue layer.

High bending energy cost associated with stretching of the stiff inner layer!

Tendrils try to keep the preferred curvature when stretched!



In rubber models both layers have similar stiffness.

Small bending energy.

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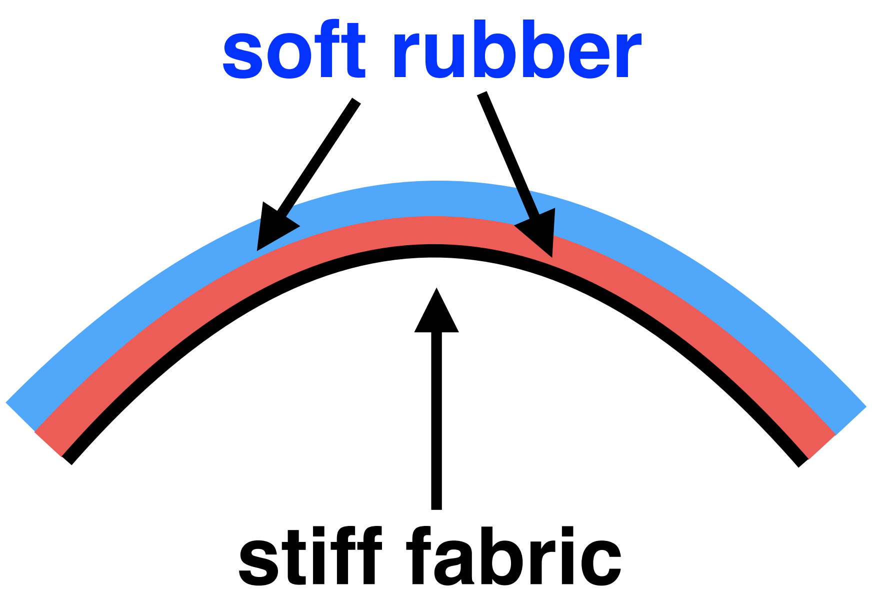
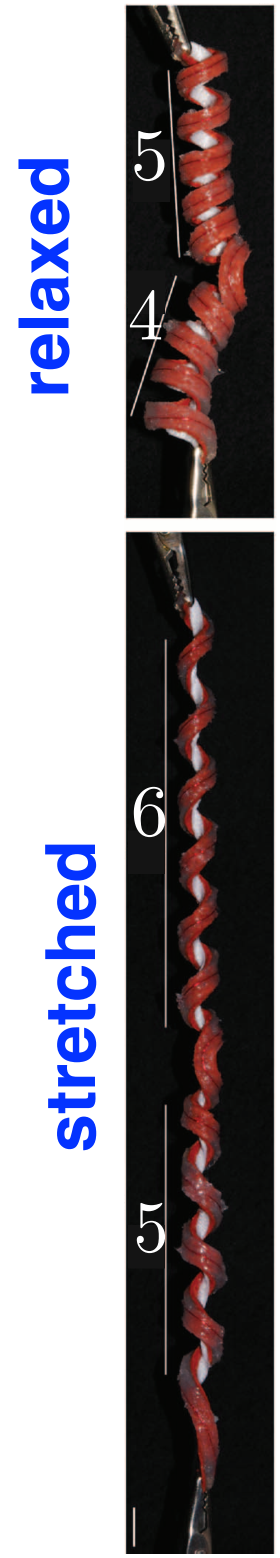
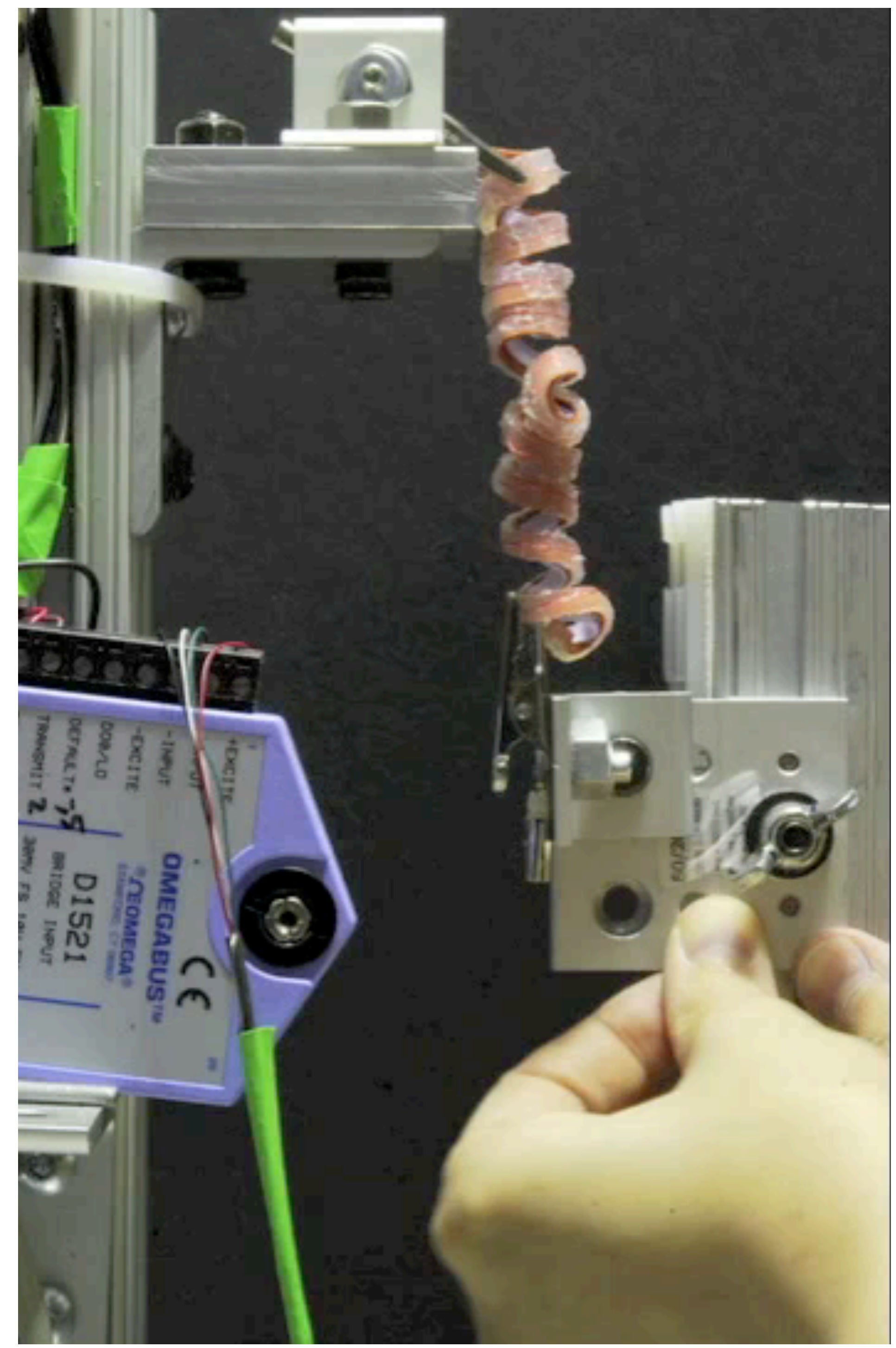
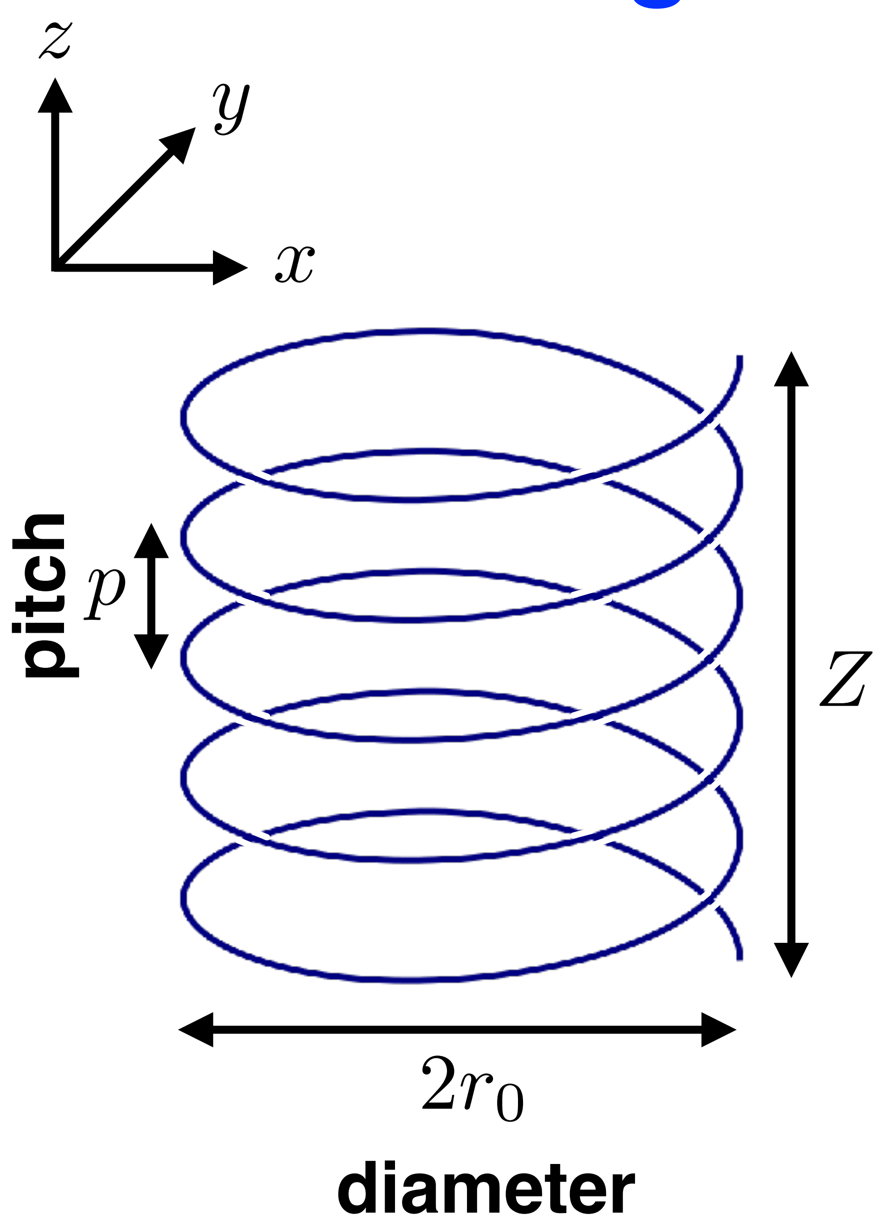
Overwinding of rubber models with an additional stiff fabric on the inside layers







Overwinding of helix with infinite bending modulus



Mathematical description

$$\vec{r}(s) = \left(r_0 \cos(s/\lambda), r_0 \sin(s/\lambda), \frac{p}{2\pi\lambda} s \right)$$

$$\lambda = \sqrt{r_0^2 + (p/2\pi)^2} \qquad Z = pN = p(L/2\pi\lambda)$$

Infinite bending modulus fixes the helix curvature during stretching

$$K = \frac{r_0}{r_0^2 + (p/2\pi)^2}$$

Helix pitch and radius

$$r_0 = \frac{1}{K}\left(1 - \frac{Z^2}{L^2}\right)$$

$$p = \frac{2\pi Z}{KL}\sqrt{1 - \frac{Z^2}{L^2}}$$

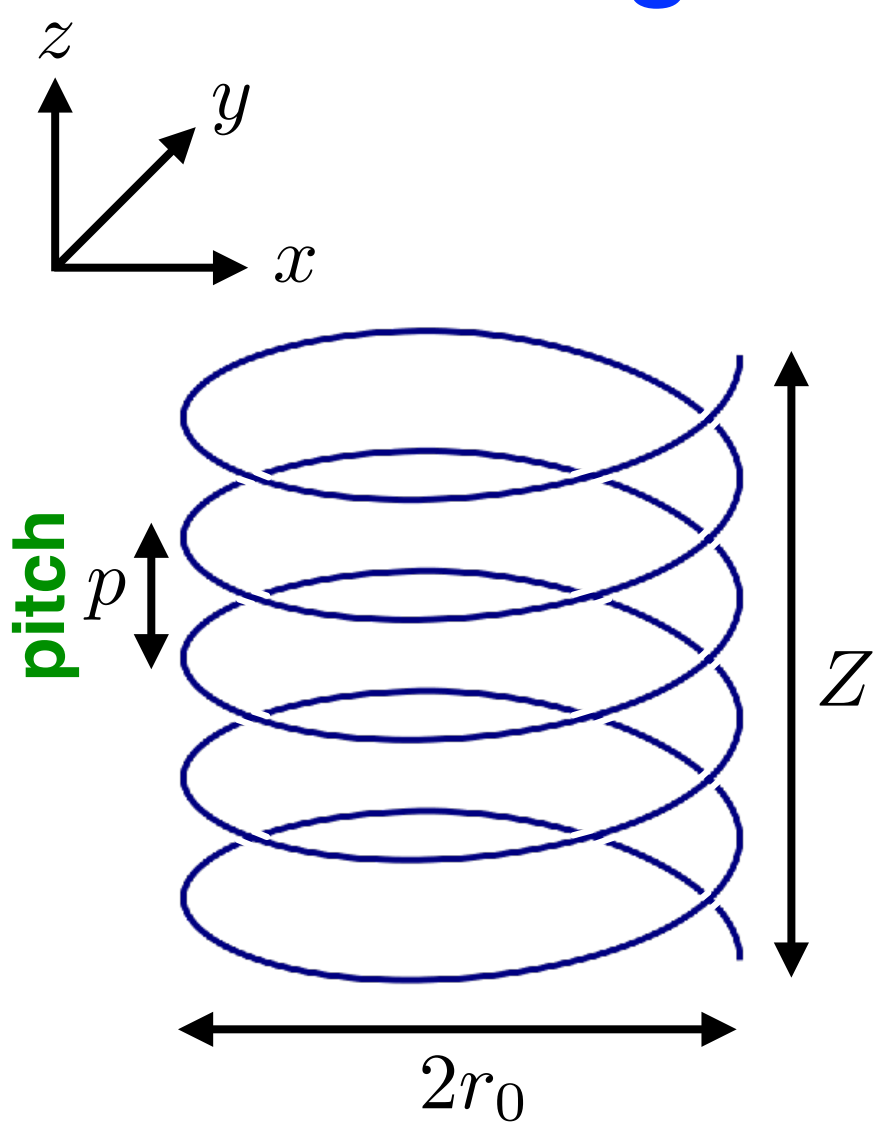
Number of loops

$$N = \frac{Z}{p} = \frac{KL}{2\pi\sqrt{1-(Z/L)^2}}$$

L **length of the helix backbone**

$N = \frac{Z}{p}$ **number of loops**

Overwinding of helix with infinite bending modulus



Helix pitch and radius

$$r_0 = \frac{1}{K}\left(1 - \frac{Z^2}{L^2}\right)$$

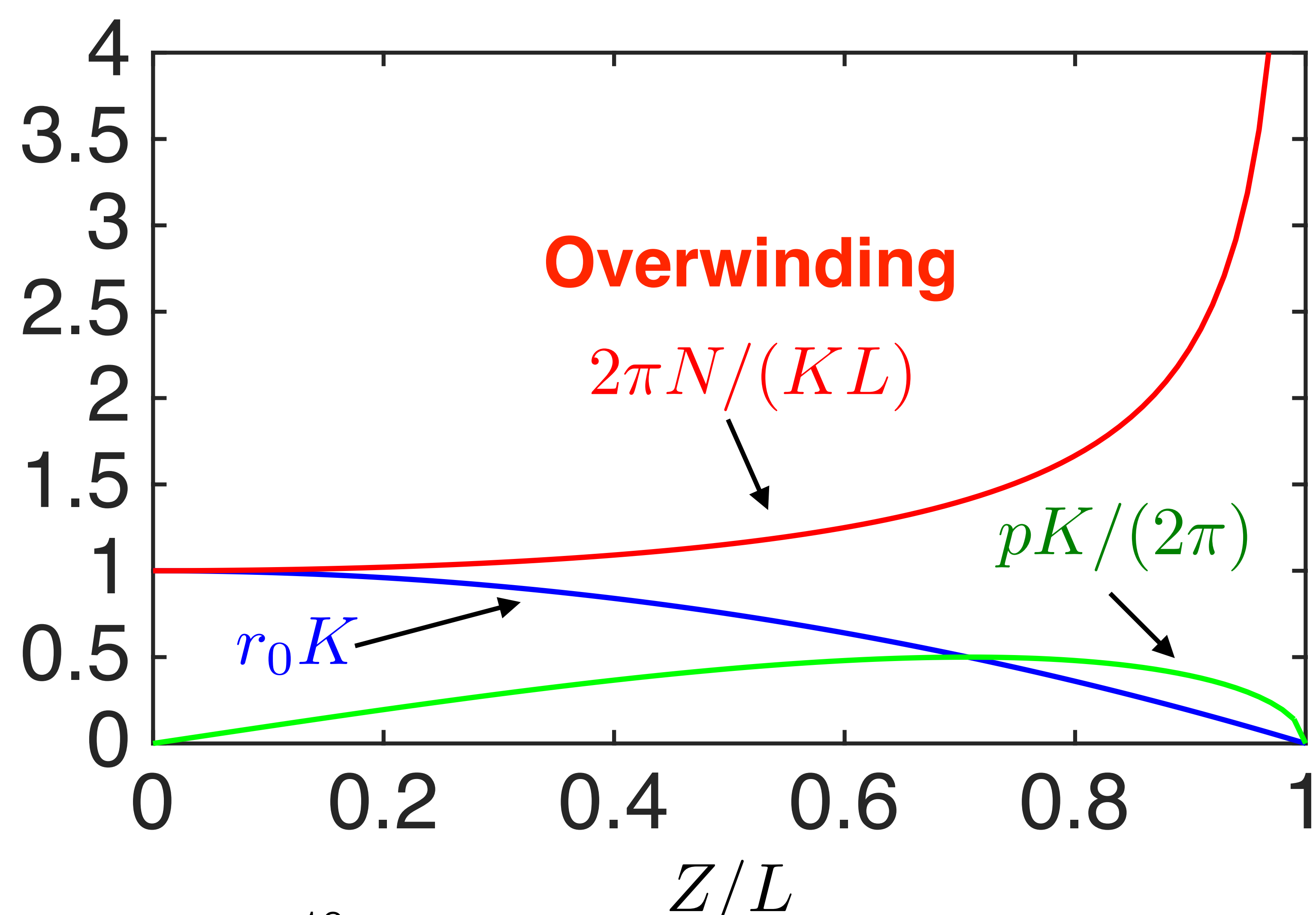
$$p = \frac{2\pi Z}{KL}\sqrt{1 - \frac{Z^2}{L^2}}$$

Number of loops

$$N = \frac{Z}{p} = \frac{KL}{2\pi\sqrt{1-(Z/L)^2}}$$

L **length of the helix backbone**

$N = \frac{Z}{p}$ **number of loops**



Spirals in nature

shells

beaks

claws

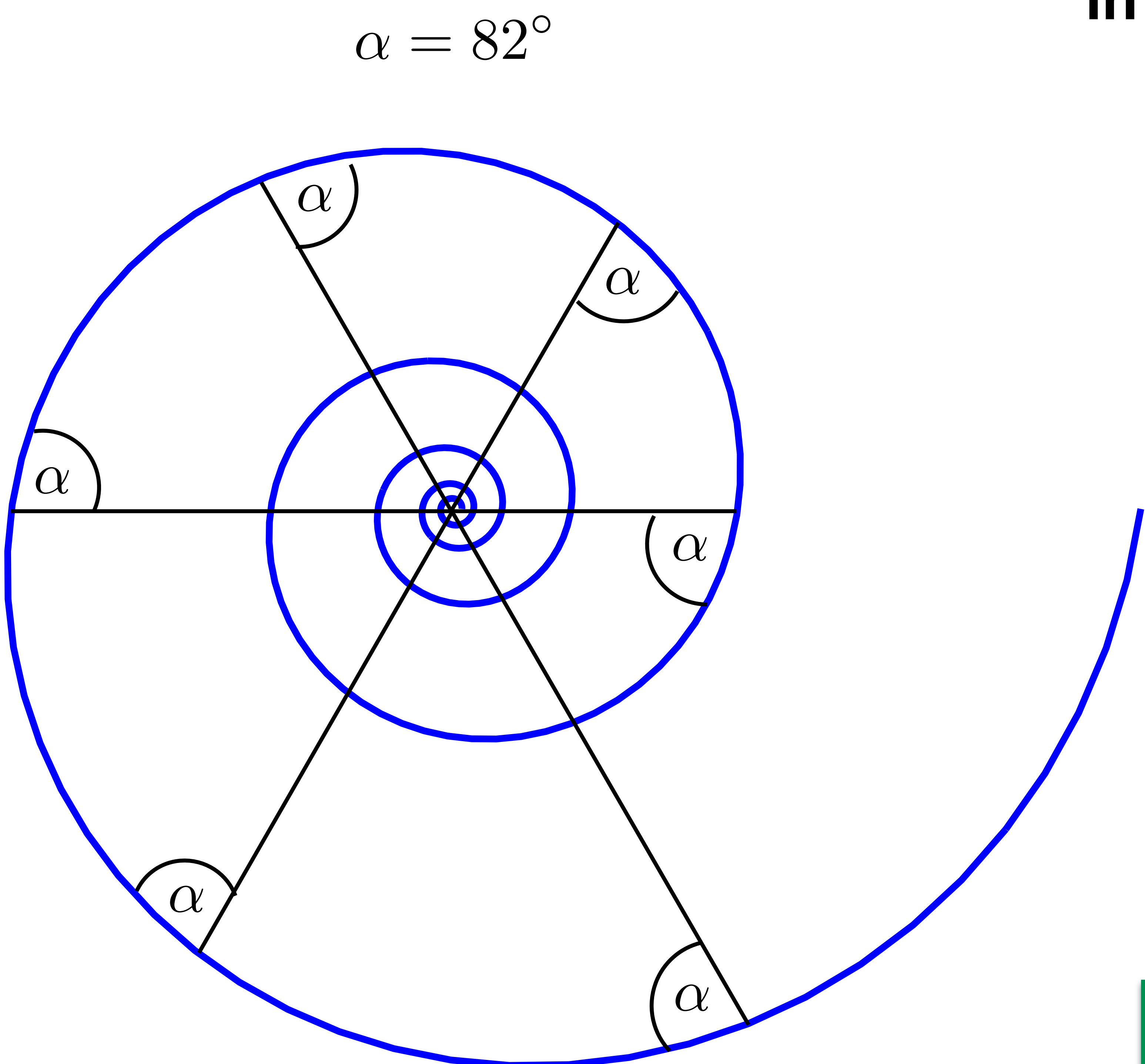
horns

teeth

tusks

What simple mechanism could produce spirals?

Equiangular (logarithmic) spiral



in polar coordinates radius grows exponentially

$$r(\theta) = a^{\theta} = \exp^{(\theta \cot \alpha)}$$

$$\cot \alpha = \ln a$$

name logarithmic spiral:

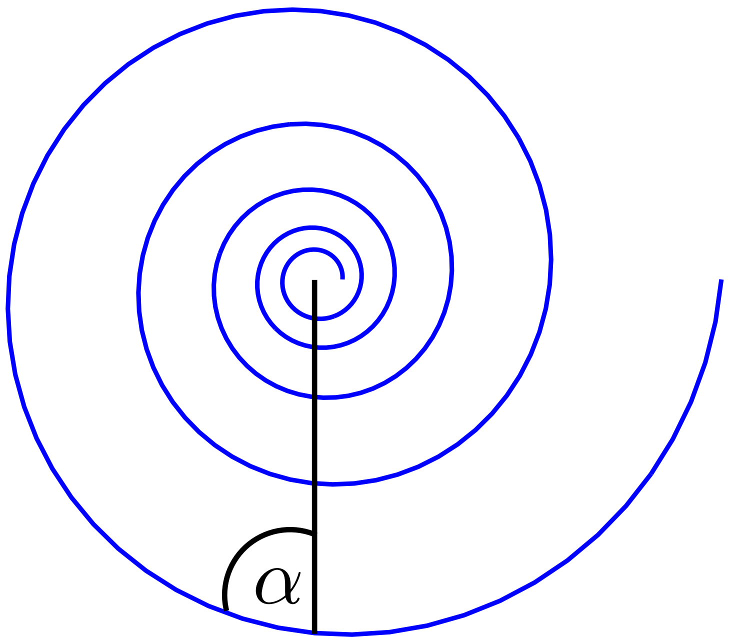
$$\theta = \frac{\ln r}{\ln a}$$

Ratio between growth velocities in the radial and azimuthal directions velocities is constant!

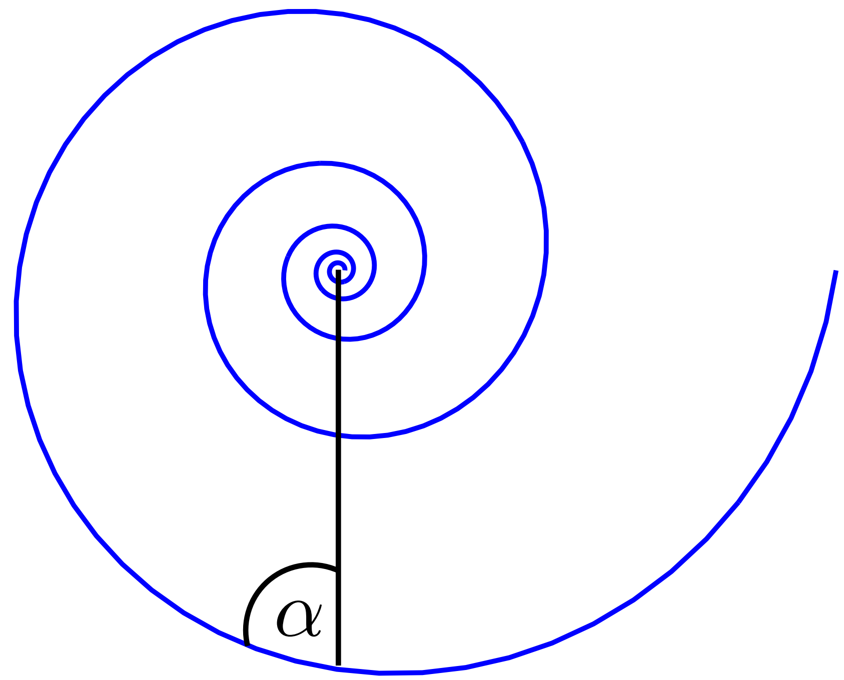
$$\cot \alpha = \frac{dr}{r d\theta} = \frac{dr/dt}{r d\theta/dt} = \frac{v_r}{v_\theta}$$

Equiangular (logarithmic) spiral

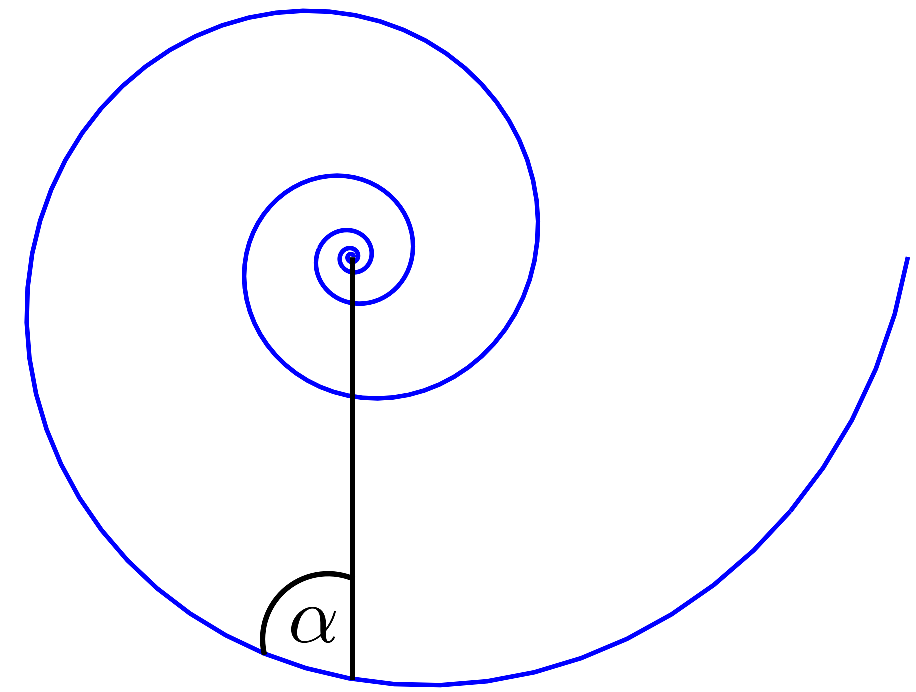
$\alpha = 85^\circ$



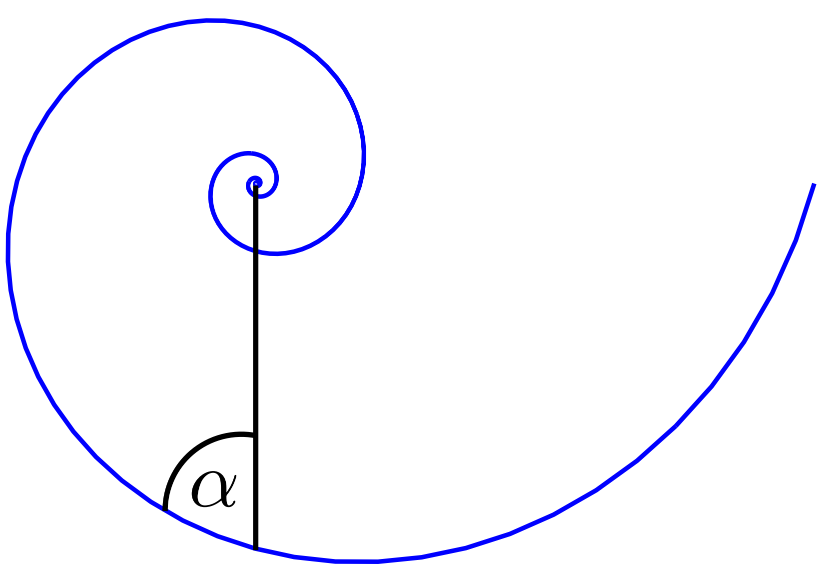
$\alpha = 82^\circ$



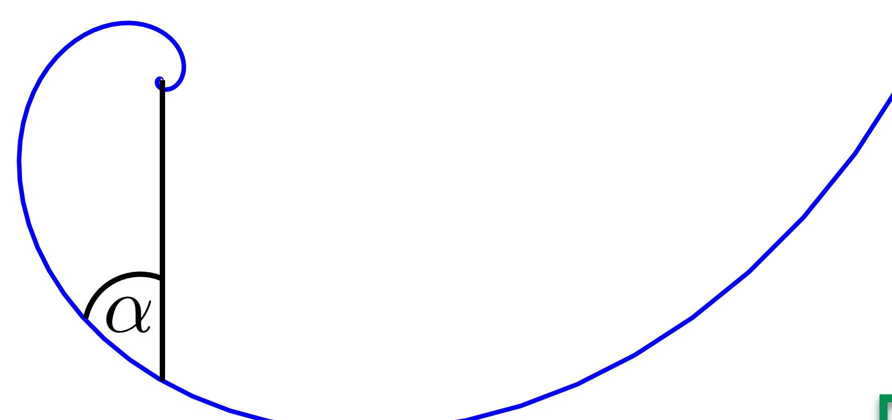
$\alpha = 80^\circ$



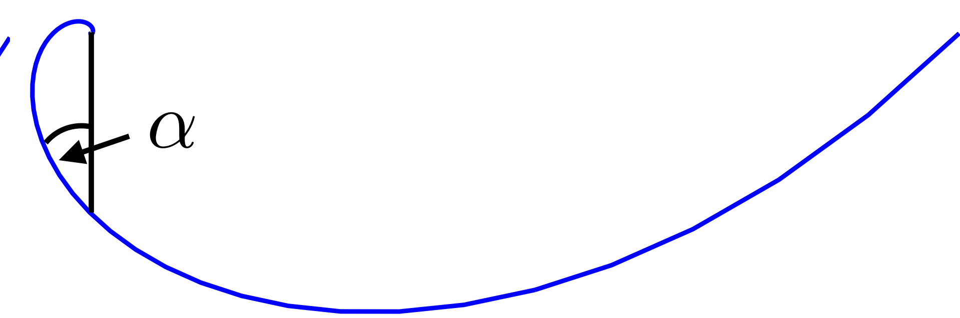
$\alpha = 75^\circ$



$\alpha = 60^\circ$

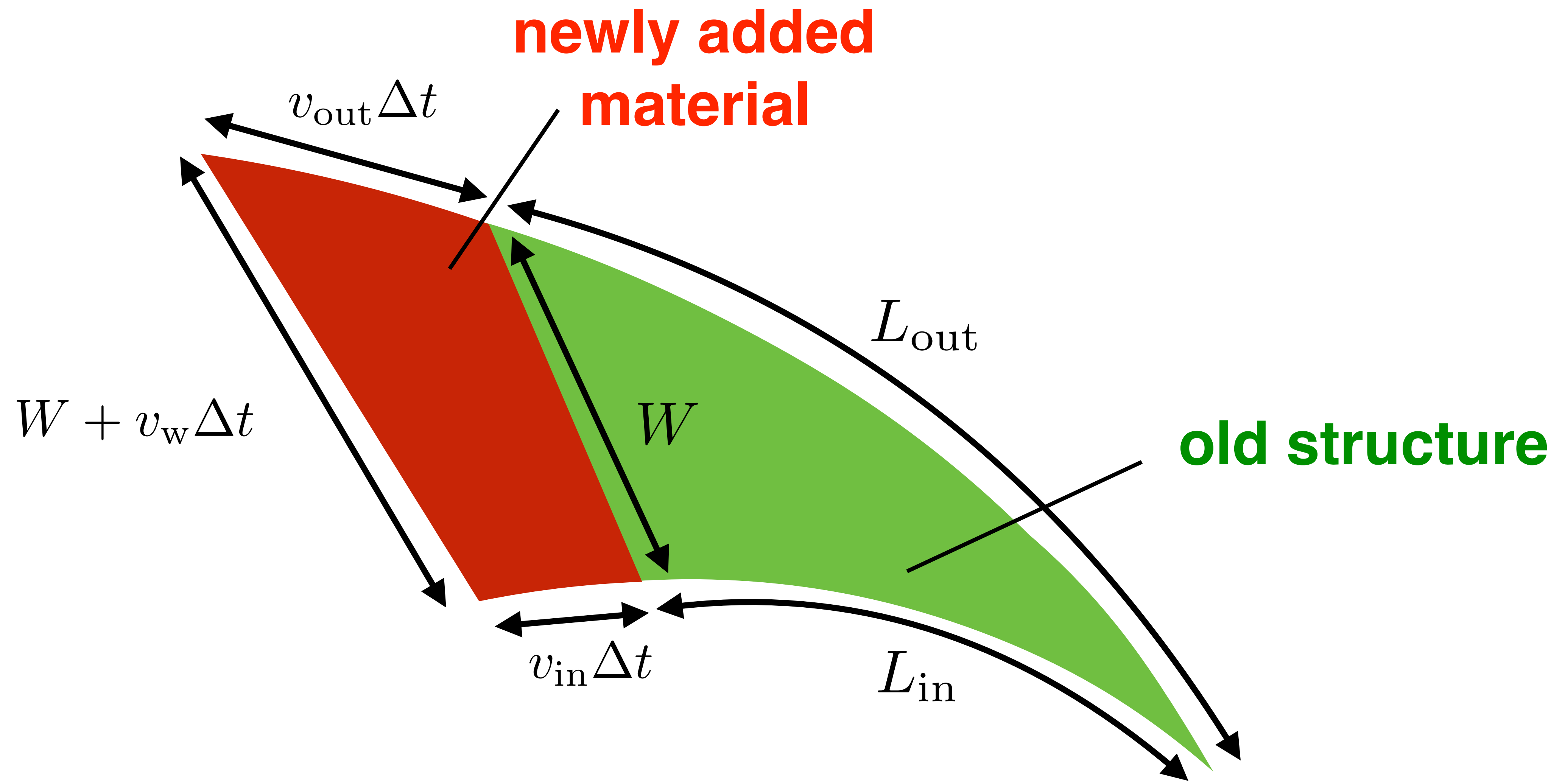


$\alpha = 45^\circ$



$$r(\theta) = a^{\theta} = \exp^{(\theta \cot \alpha)}$$

Growth of spiral structures



New material is added at a constant ratio of growth velocities, which produces spiral structure with side lengths and the width in the same proportions.

$$v_{\rm out} : v_{\rm in} : v_W = L_{\rm out} : L_{\rm in} : W$$

Note: growth with constant width (v_W=0) produces helices

Growth of spiral structures

Assume the following spiral profiles of the outer and inner layers:

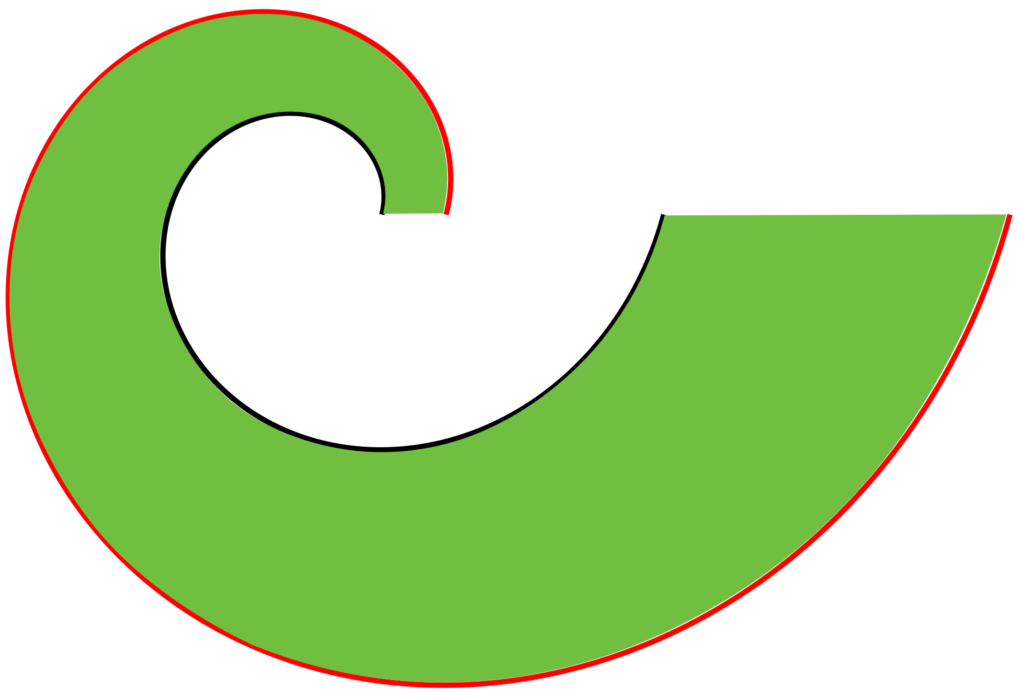
$$r_{\text{out}}(\theta) = e^{\theta \cot \alpha}$$

$$r_{\text{in}}(\theta) = \lambda e^{\theta \cot \alpha}$$

$$\lambda < 1$$

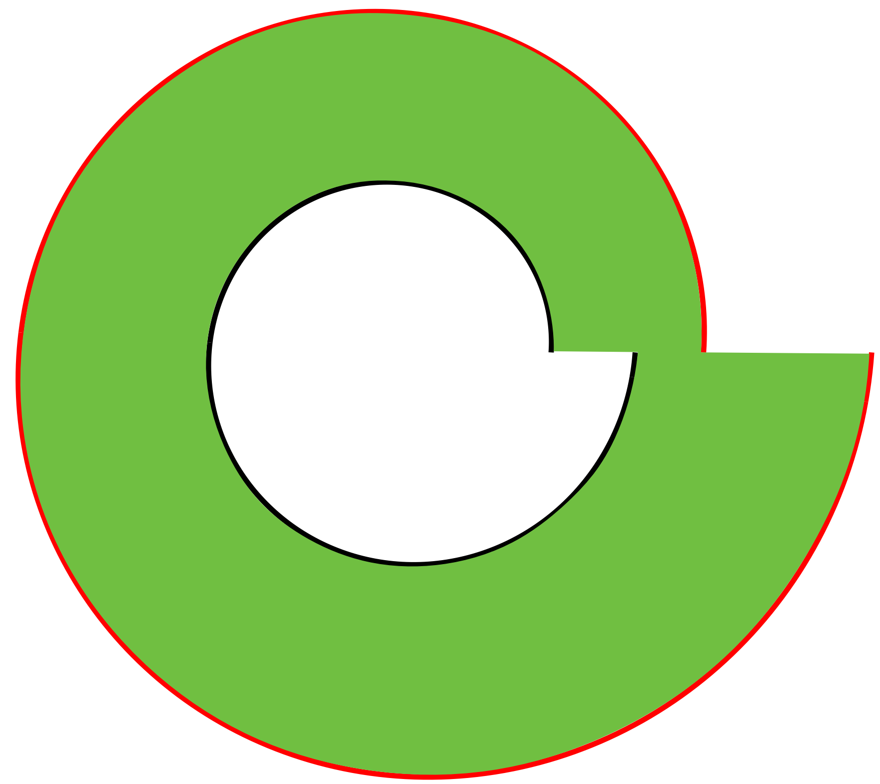
$$\lambda e^{2\pi \cot \alpha} > 1$$

$\lambda = 0.5, \ \alpha = 75^\circ$



$$\lambda e^{2\pi \cot \alpha} < 1$$

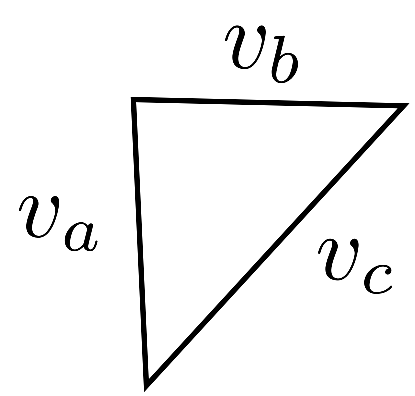
$\lambda = 0.5, \ \alpha = 86^\circ$



In some shells the inner layer does not grow at all

3D spirals

3D spiral of ram's horns is due to the triangular cross-section of the horn, where each side grows with a different velocity.



Shells of mollusks are often conical