MAE 545: Lecture 12 (4/4)

Spirals and phyllotaxis

Spirals in nature

shells

beaks

claws

horns

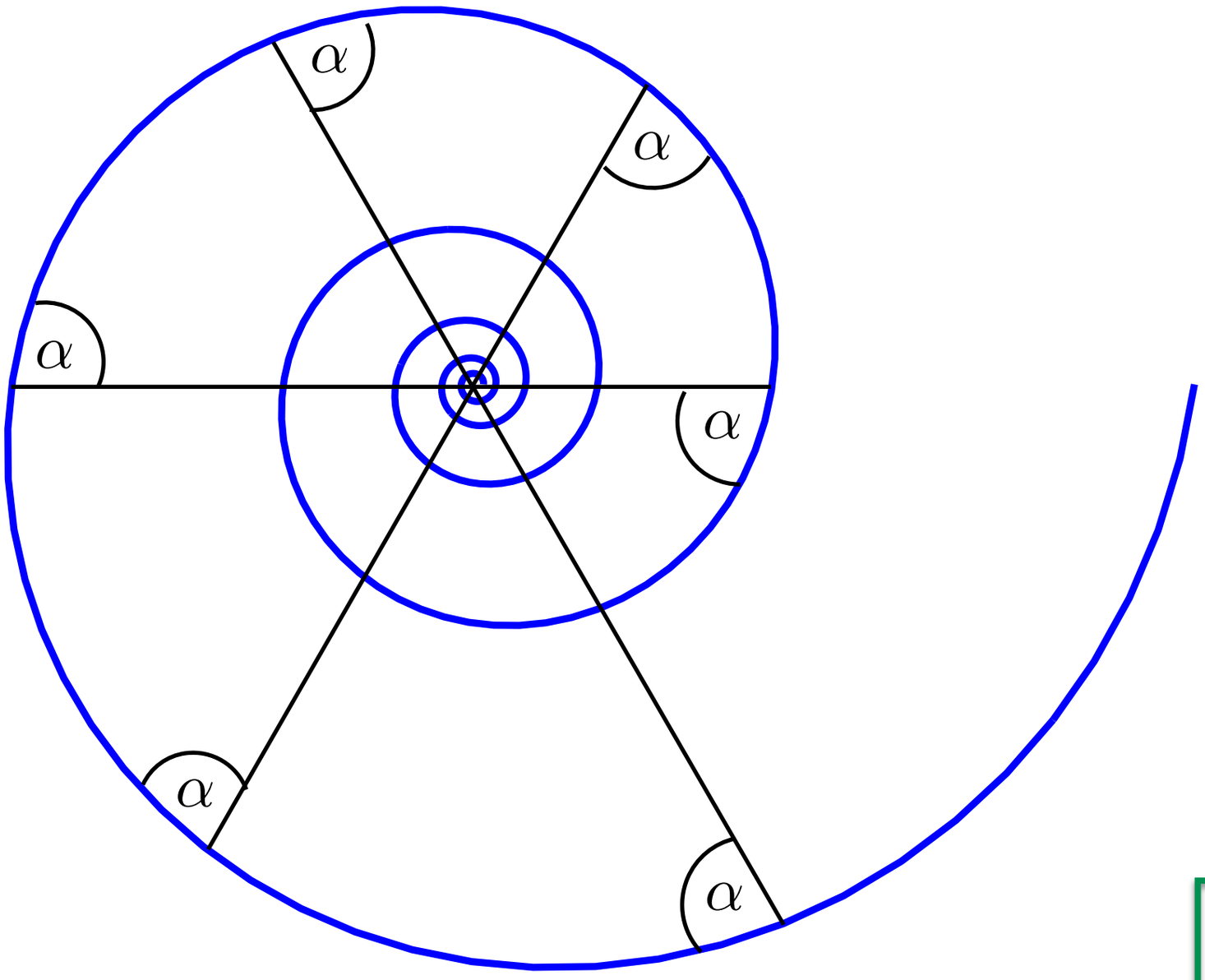
teeth

tusks

What simple mechanism could produce spirals?

Equiangular (logarithmic) spiral

$$\alpha = 82^\circ$$



in polar coordinates radius grows exponentially

$$r(\theta) = a^\theta = \exp^{(\theta \cot \alpha)}$$

$$\cot \alpha = \ln a$$

name logarithmic spiral:

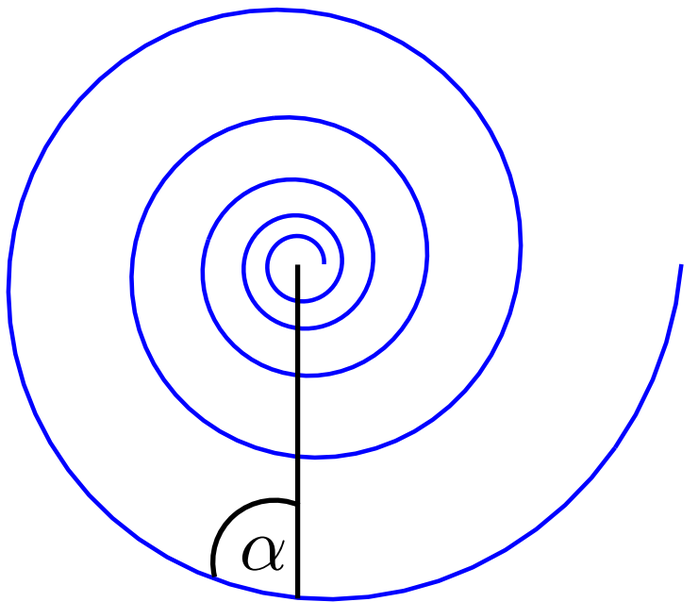
$$\theta = \frac{\ln r}{\ln a}$$

Ratio between growth velocities in the radial and azimuthal directions velocities is constant!

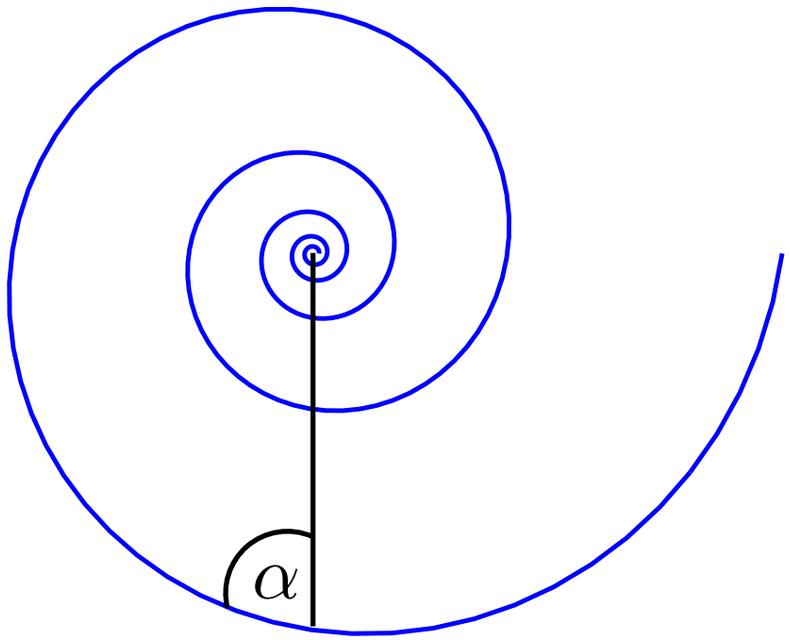
$$\cot \alpha = \frac{dr}{r d\theta} = \frac{dr/dt}{r d\theta/dt} = \frac{v_r}{v_\theta}$$

Equiangular (logarithmic) spiral

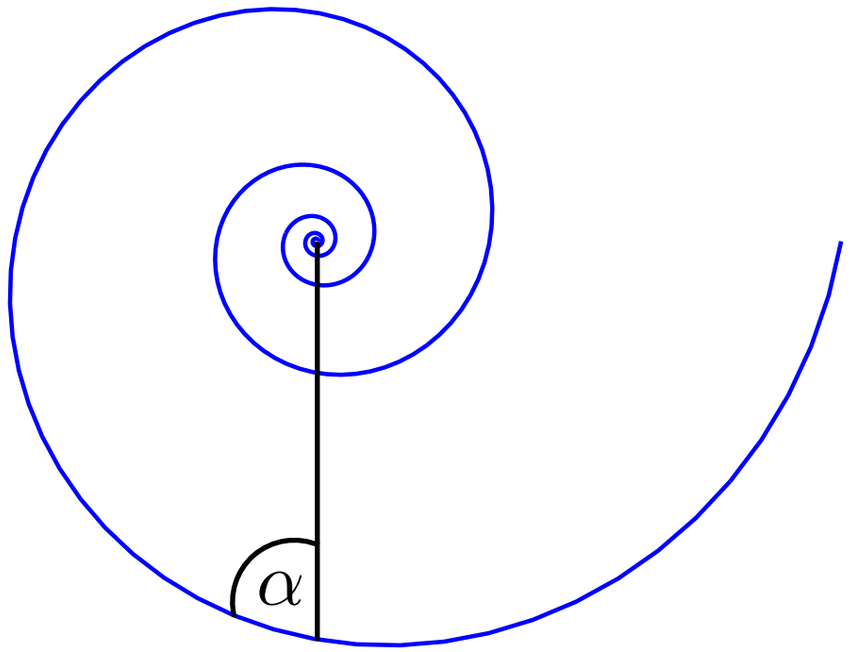
$\alpha = 85°$



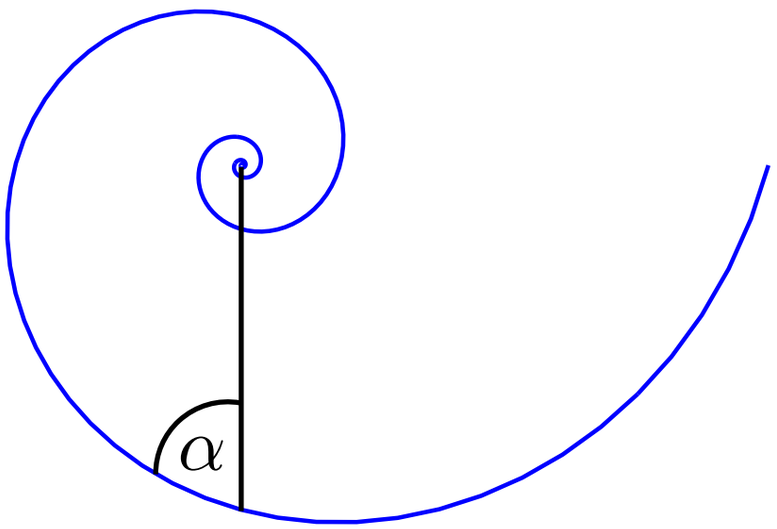
$\alpha = 82°$



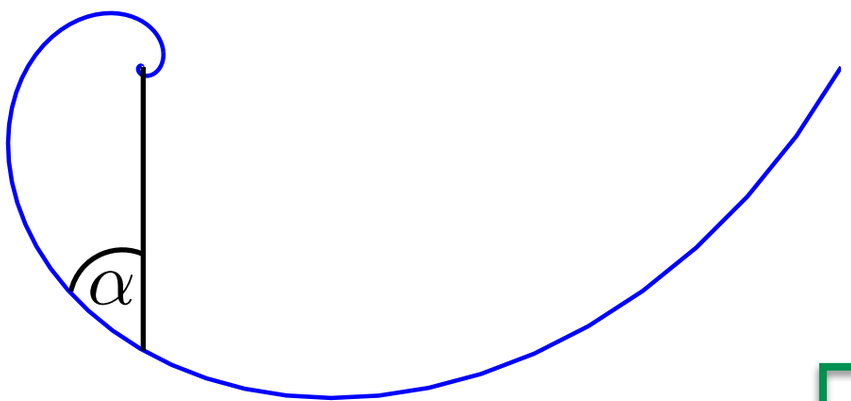
$\alpha = 80°$



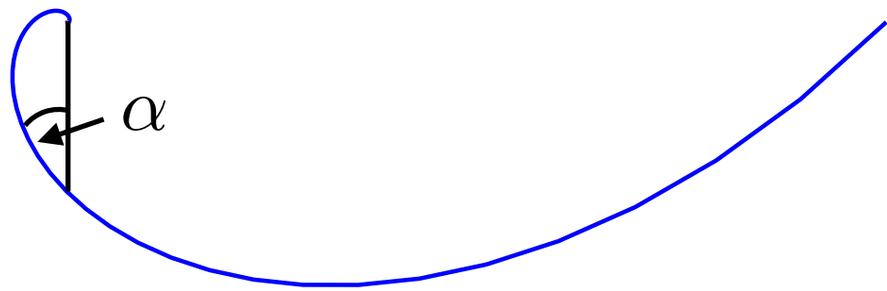
$\alpha = 75°$



$\alpha = 60°$

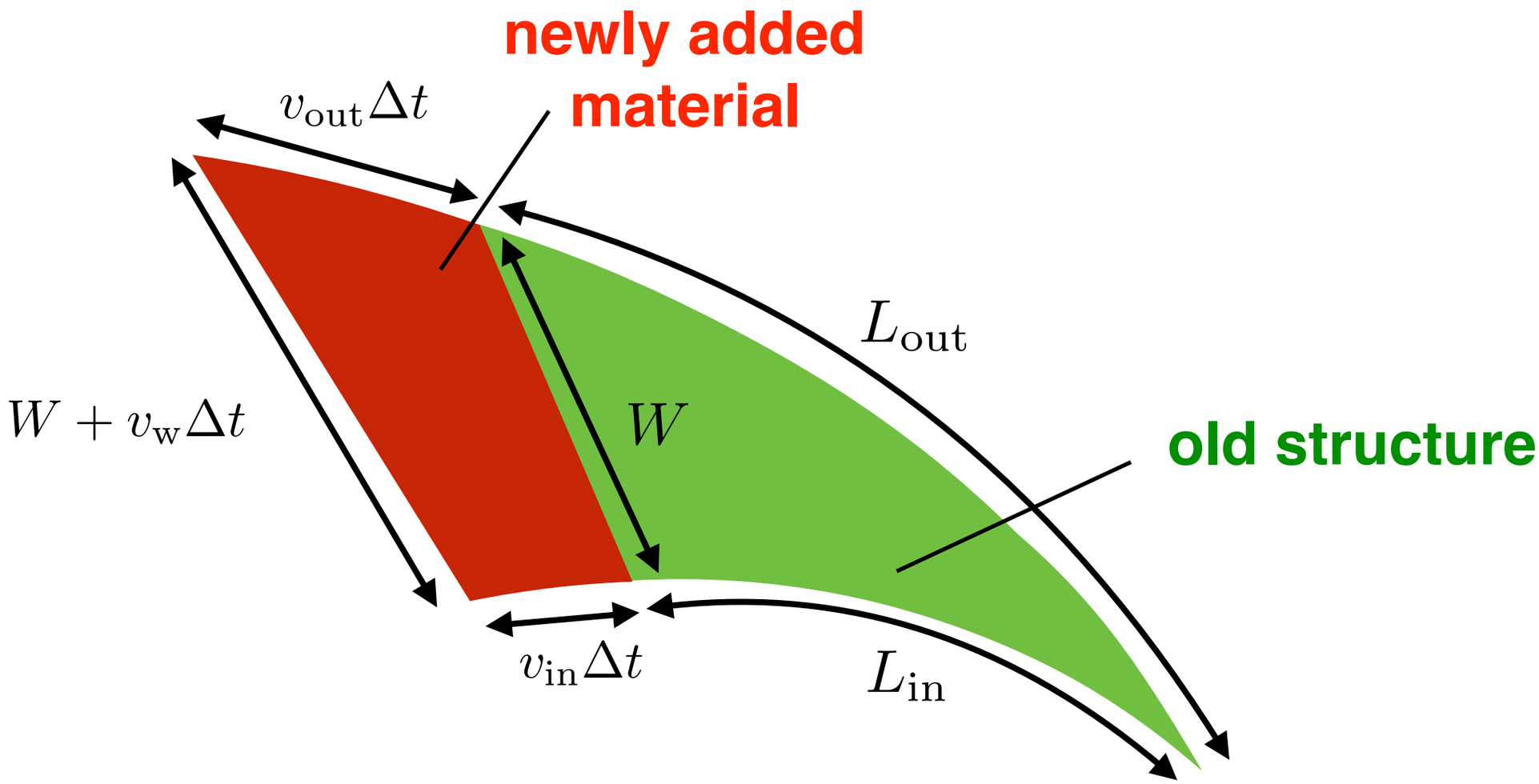


$\alpha = 45°$



$$r(\theta) = a^{\theta} = \exp^{(\theta \cot \alpha)}$$

Growth of spiral structures



New material is added at a constant ratio of growth velocities, which produces spiral structure with side lengths and the width in the same proportions.

$$v_{\mathrm{out}} : v_{\mathrm{in}} : v_W = L_{\mathrm{out}} : L_{\mathrm{in}} : W$$

Note: growth with constant width (v_W=0) produces helices

Growth of spiral structures

Assume the following spiral profiles of the outer and inner layers:

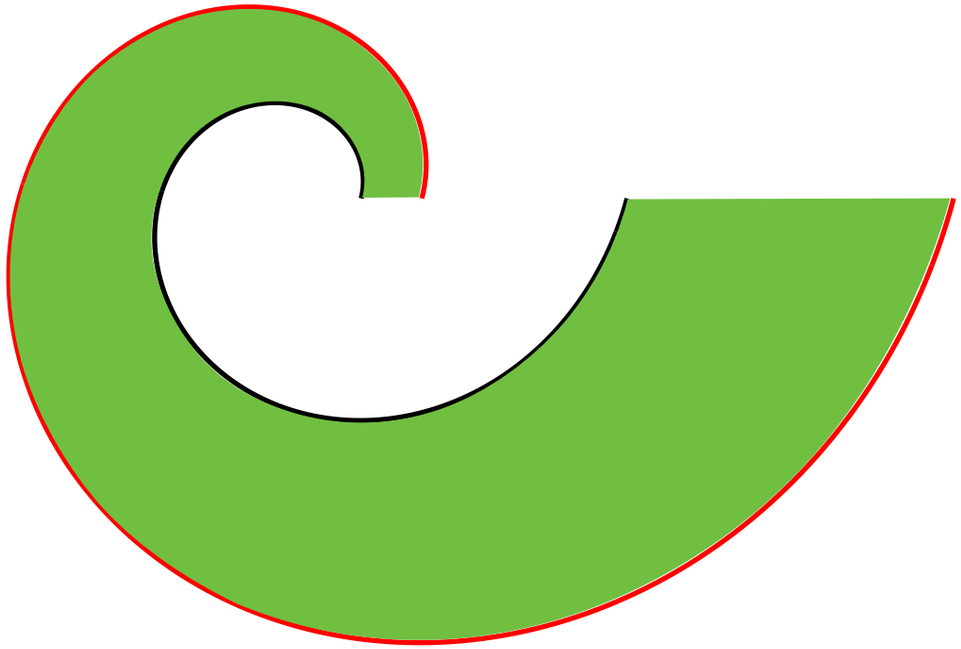
$$r_{\rm out}(\theta) = e^{\theta \cot\alpha}$$

$$r_{\rm in}(\theta) = \lambda e^{\theta \cot\alpha}$$

$\lambda < 1$

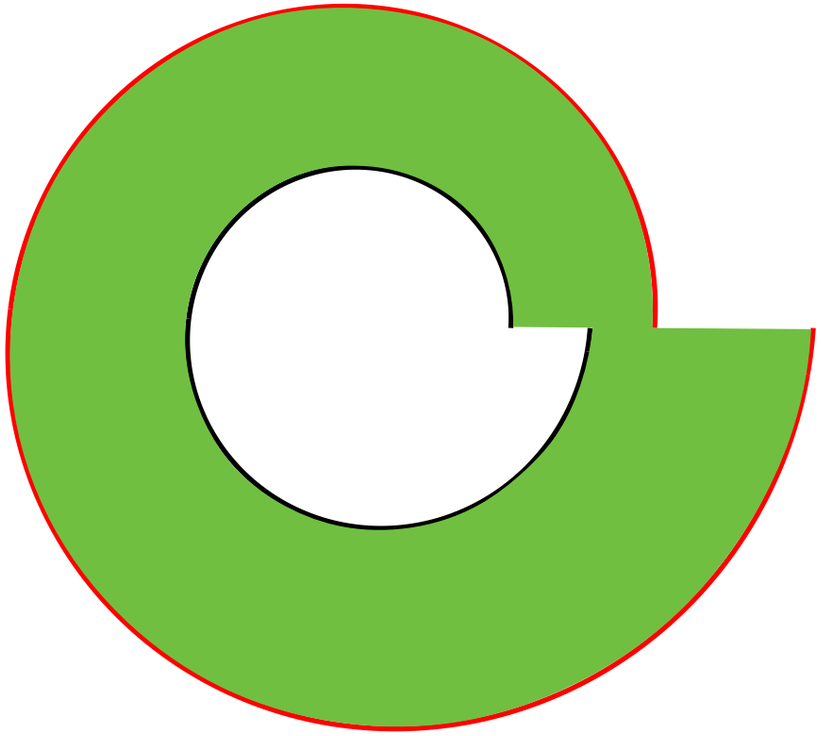
$\lambda e^{2\pi \cot\alpha} > 1$

$\lambda = 0.5,\ \alpha = 75^\circ$



$\lambda e^{2\pi \cot\alpha} < 1$

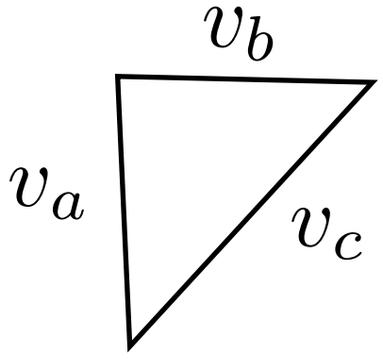
$\lambda = 0.5,\ \alpha = 86^\circ$



In some shells the inner layer does not grow at all

3D spirals

3D spiral of ram's horns is due to the triangular cross-section of the horn, where each side grows with a different velocity.



Shells of mollusks are often conical

Phyllotaxis

Phyllotaxis is classification of leaves on a plant stem

maize

distichous pattern

leaves alternating every 180^0

Coleus sp.

decussate pattern

pairs of leaves at 90^0

Veronicastrum virginicum

whorled pattern

3 or more leaves originating from the same node (180^0)

sunflower

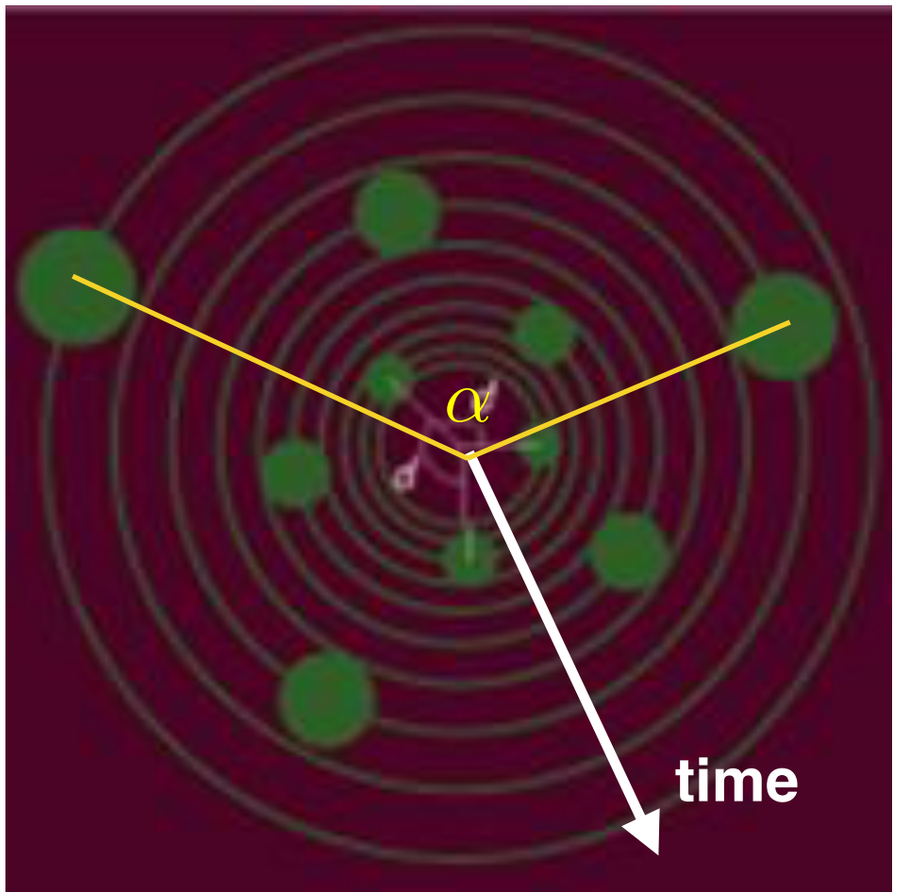
alternate (spiral) pattern

successive leaves at 137.5^0

http://www.sciteneg.com/PhiTaxis/PHYLLOTAXIS.htm

Spiral phyllotaxis

schematic description of leaves arrangement



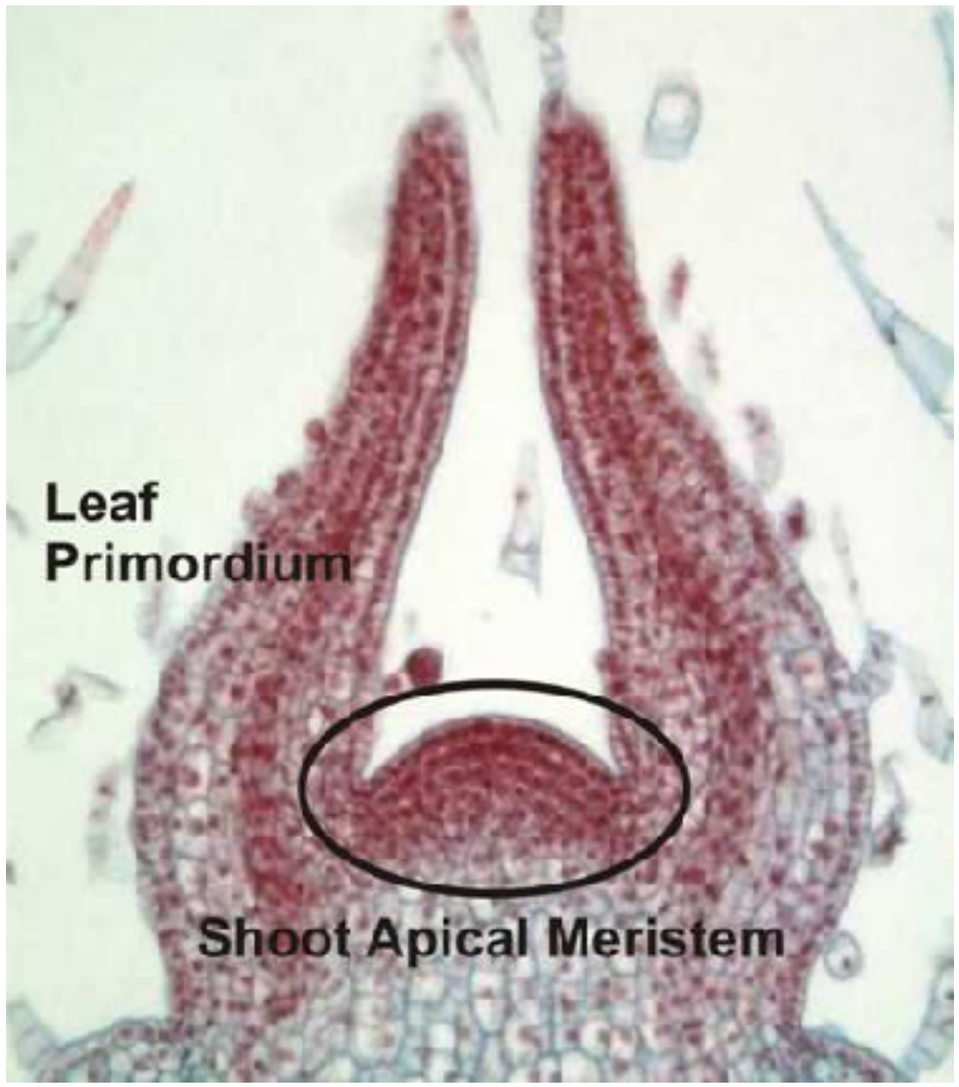
$$\alpha \approx 137.5^\circ$$

florets (petals)

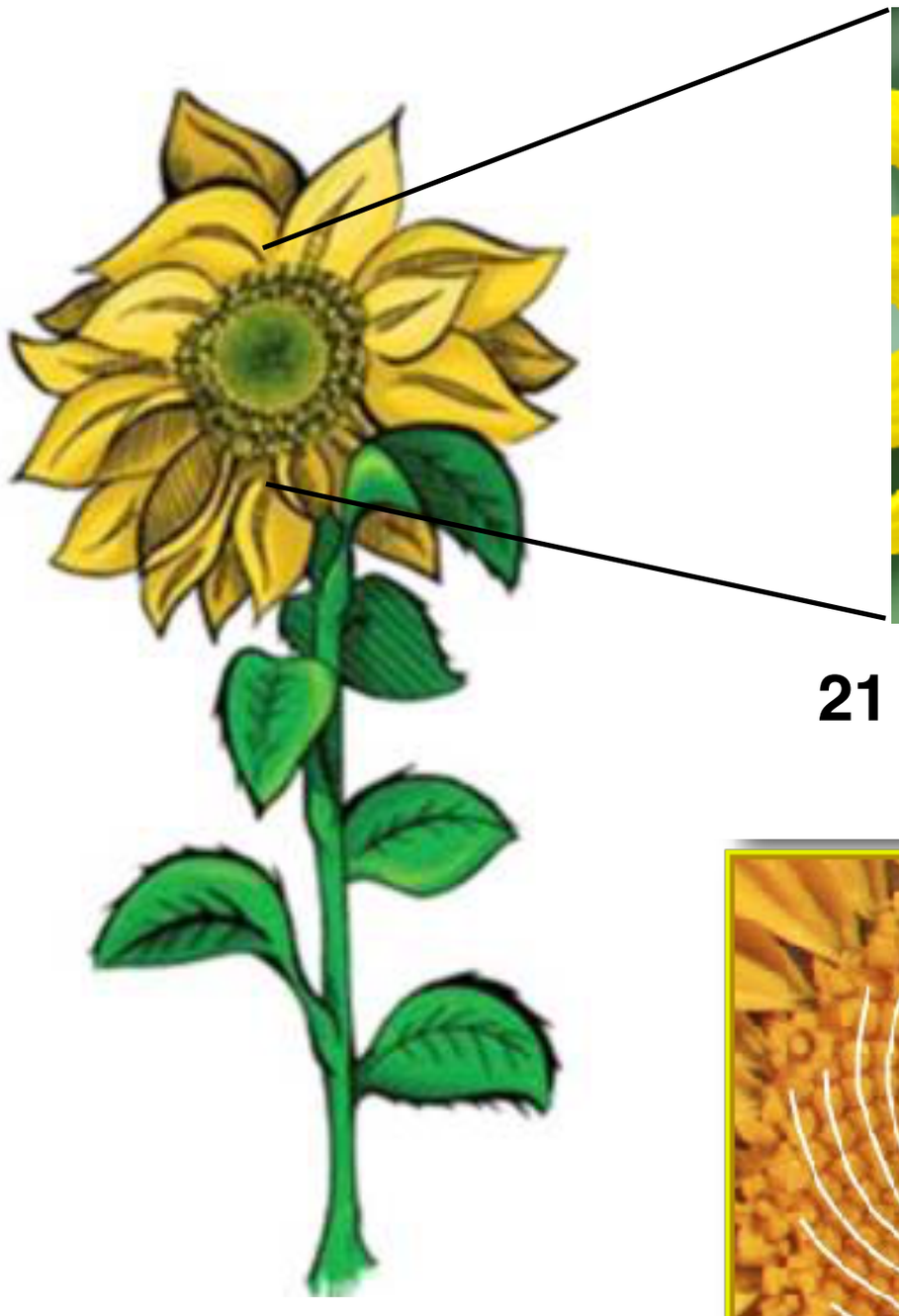
floral primordia

leaves

leaves grow from the apical meristem, which also gives rise to petals, sepals, etc.

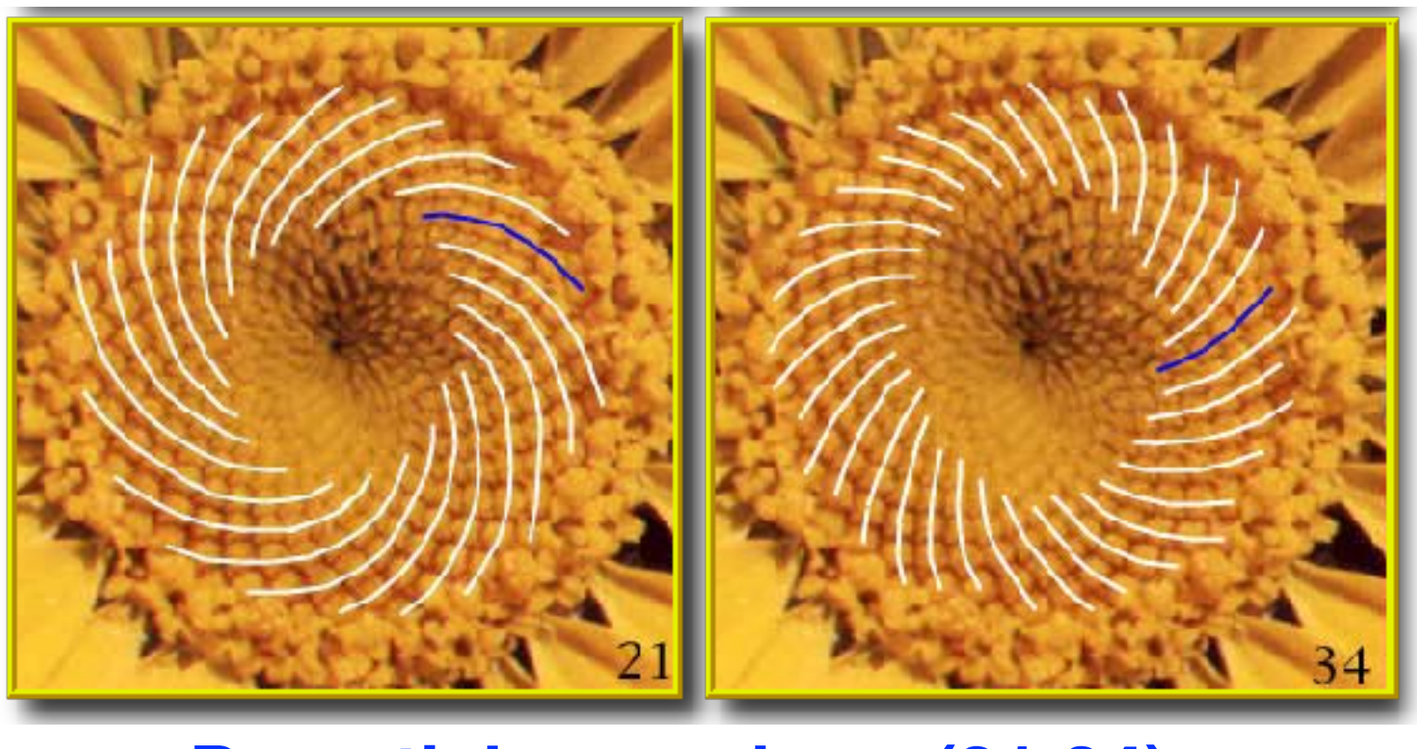


Parastichy numbers



21 left-handed spirals

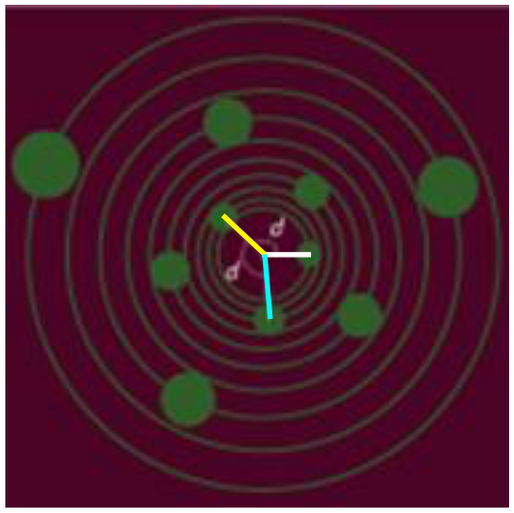
34 right-handed spirals



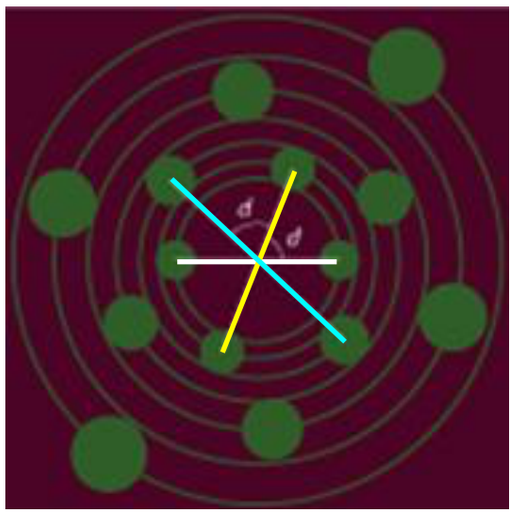
Parastichy numbers (21,34)

Parastichy numbers

spiral phyllotaxis

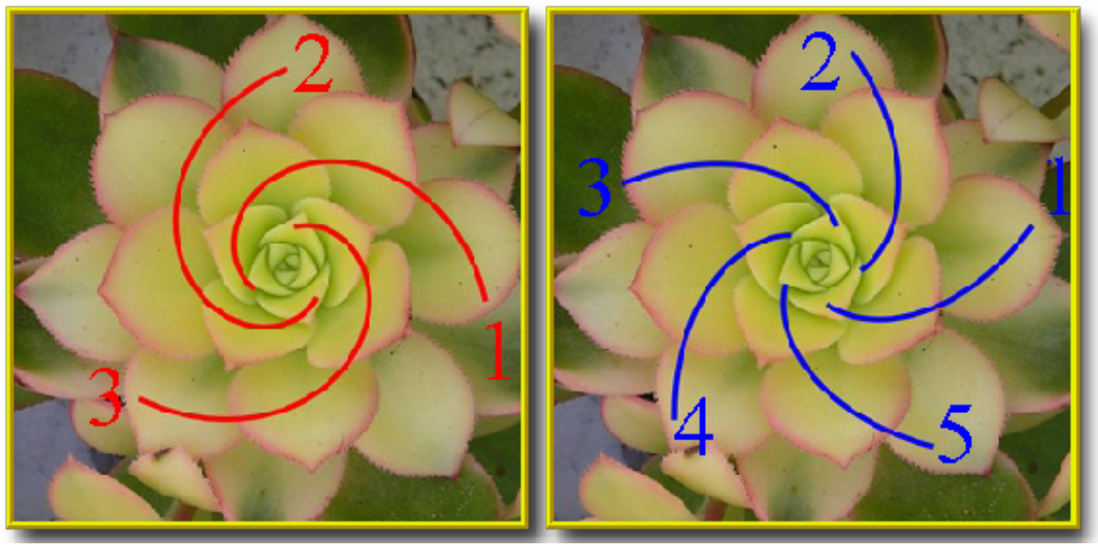


multijugate phyllotaxis



(e.g. 2 new leaves are added at the same time)

succulent plant (3,5)



Gymnocalycium (10,16)=2(5,8)

Parastichy numbers

aonium (2,3)

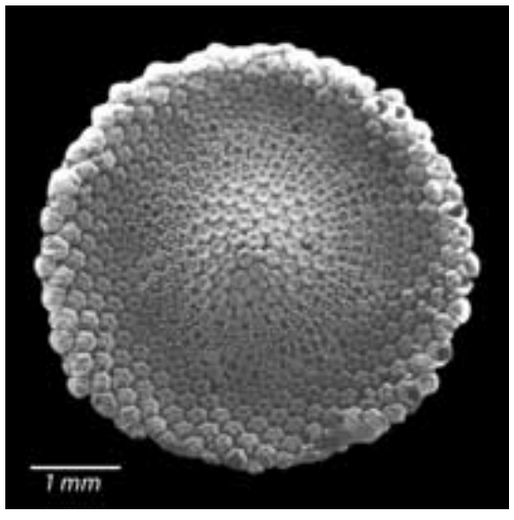
succulent plant (3,5)

aloe (5,8)

pine cone (8,13)

sunflower (21,34)

artichoke (34,55)



Parastichy numbers very often correspond to successive Fibonacci numbers!

Fibonacci numbers

$$F_1 = 1$$
$$F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

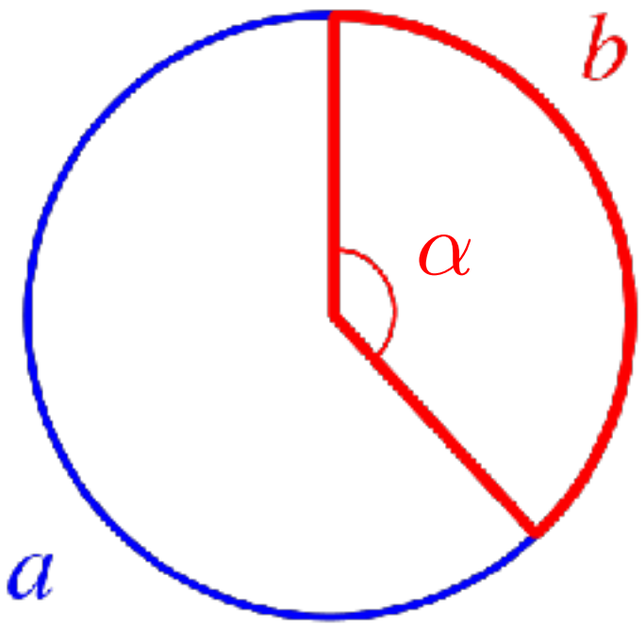
Golden ratio $\varphi = \dfrac{1+\sqrt{5}}{2}$

$$F_n = \frac{1}{\sqrt{5}}\left[\varphi^n - (1-\varphi)^n\right]$$

Sequence of Fibonacci numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

Golden angle



divide perimeter in golden ratio

$$\frac{a+b}{a} = \frac{a}{b} \longrightarrow \frac{a}{b} = \varphi$$

$$\alpha = 360^\circ \frac{b}{(a+b)} = \frac{360^\circ}{\varphi^2} \approx 137.5^\circ$$

In spiral phyllotaxis successive leaves grow at approximately Golden angle!

Non-Fibonacci parastichy numbers

Statistics for pine trees in Norway

95% Fibonacci numbers
4% Lucas numbers
1% not properly formed

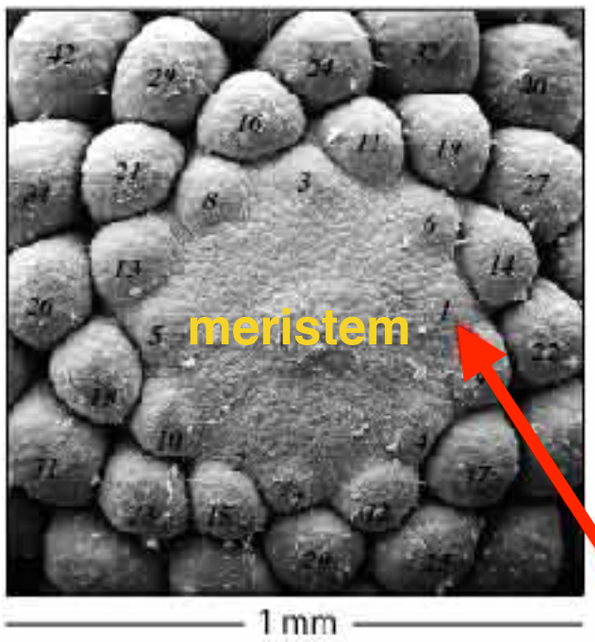
Lucas numbers

$$L_1 = 1$$
$$L_2 = 3$$
$$L_n = L_{n-1} + L_{n-2}$$

Sequence of Lucas numbers
1, 3, 4, 7, 11, 18, 29, 47, 76

Spiral phyllotaxis

Norway spruce

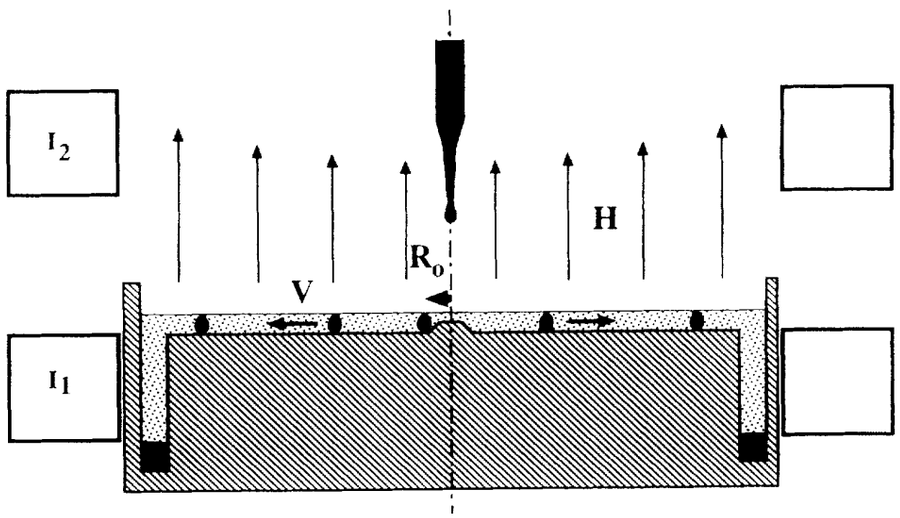


New primordia start growing at the site where plant hormone auxin is depleted.

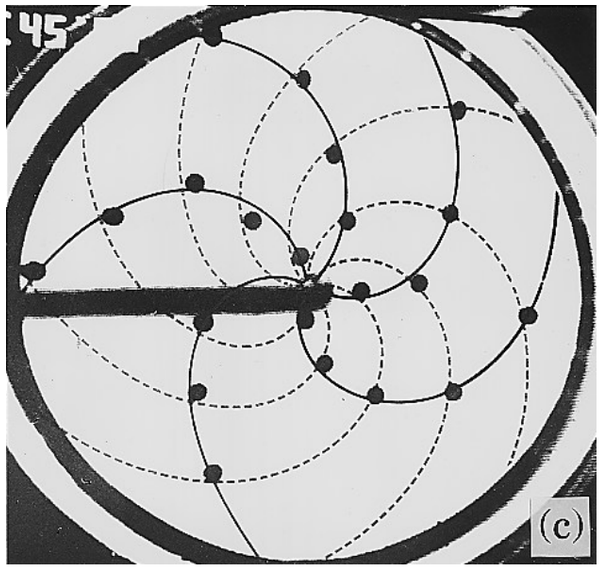
Auxin hormones are released by growing primordia. New primordium wants to be as far apart as possible from the existing primordia.

new primordial

Mechanical analog with magnetic repelling particles



magnetic field drives particles away from the center

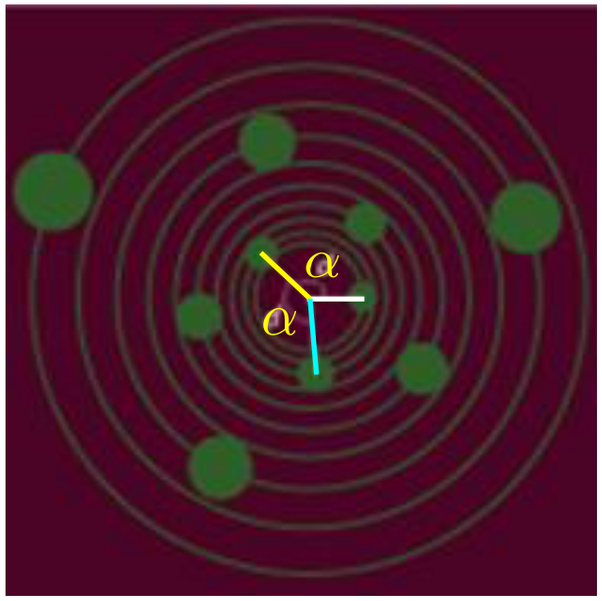


Parastichy numbers (5,8)

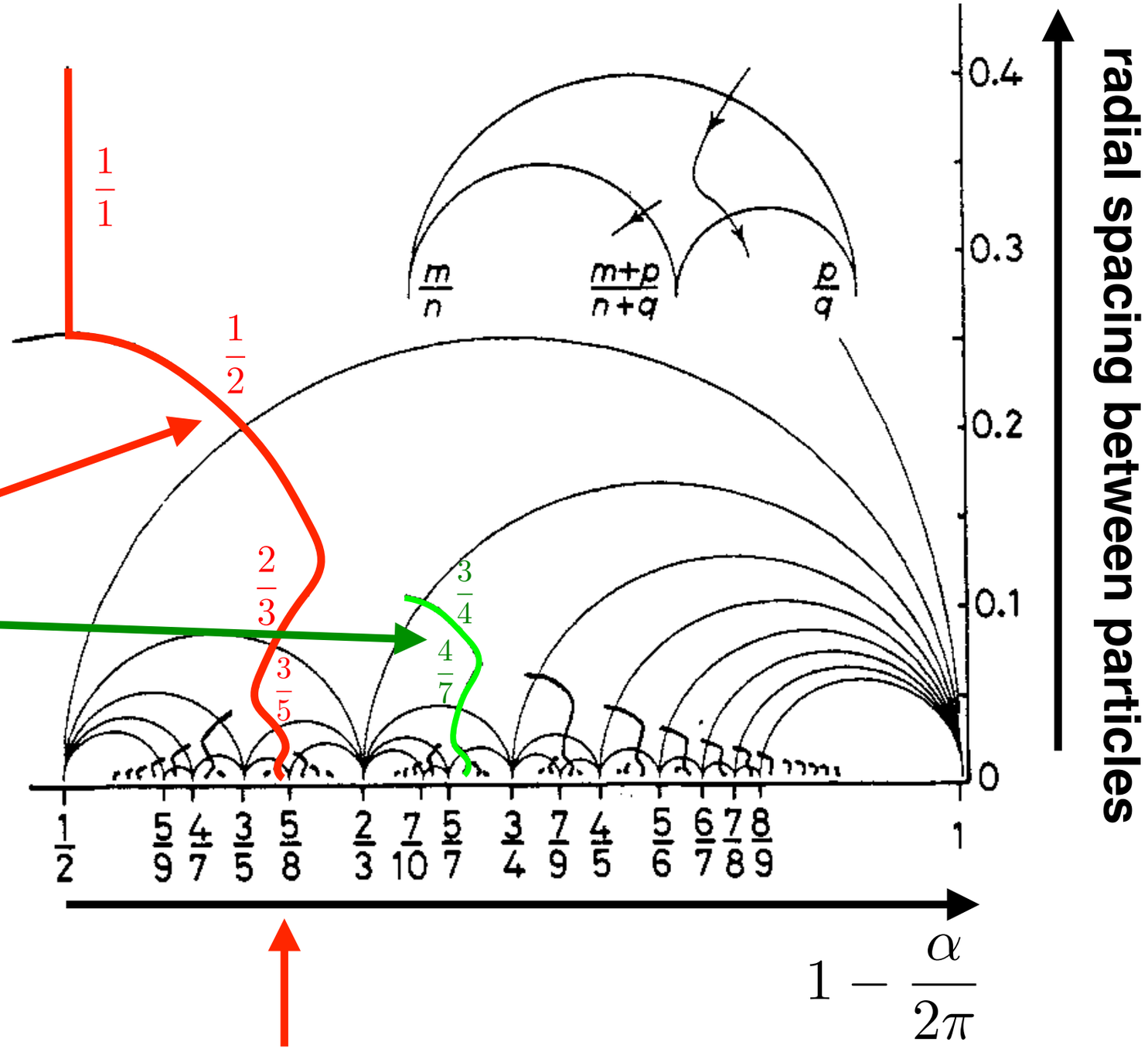
particles repel via magnetic dipol interactions

 S. Douady and Y. Couder, PRL 68, 2098 (1992)

Energy minimization between repelling particles



Local energy minima for repelling particles



Fibonacci numbers

Lucas numbers

As the plant is growing it is gradually reducing the time delay between formation of new primordia. The spiral patterns then go sequentially through all the Fibonacci parastichies.

Occasional excursions to the neighbor local minima produce Lucas parastichy numbers.

golden angle

L. Levitov, PRL 66, 224 (1991)

L. Levitov, EPL 14, 533 (1991)

Further reading