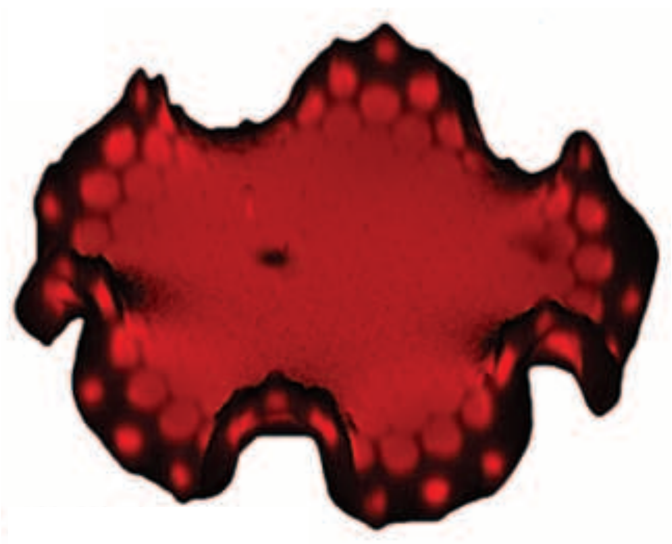
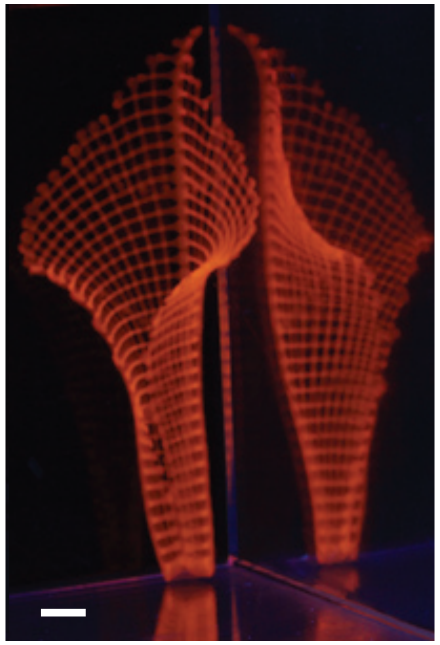
# MAE 545: Lecture 8 (3/7)

# Shapes of growing and swelling sheets





# Shapes of flowers and leaves

**saddles**

**wrinkled edges**

**helices**

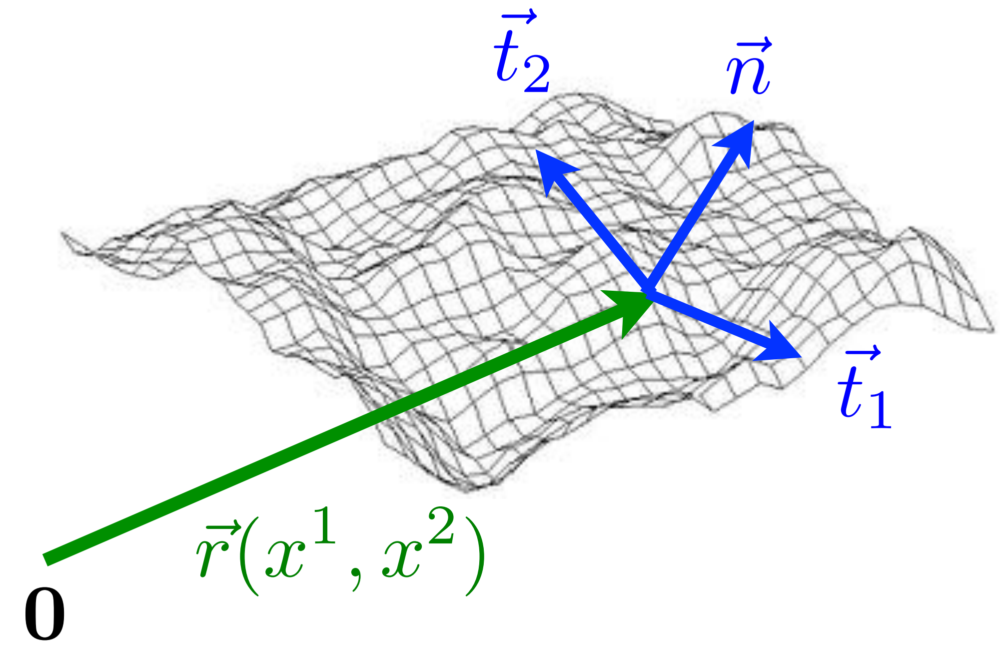
# Metric tensor for measuring distances on surfaces

$x^1, x^2$ **parameters describing position along the surface**

$\vec{r}(x^1, x^2)$ **function describing shape of the surface**

$\vec{t}_i = \dfrac{\partial \vec{r}}{\partial x^i}$ **local tangent vectors to the surface**

$\vec{n} = \dfrac{\vec{t}_1 \times \vec{t}_2}{|\vec{t}_1 \times \vec{t}_2|}$ **unit normal vector of the surface**



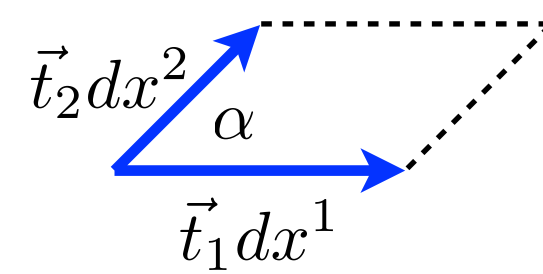

**metric tensor for measuring lengths**

$$d\ell^2 = d\vec{r}^{\,2} = \sum_{i,j} \vec{t}_i \cdot \vec{t}_j dx^i dx^j = \sum_{i,j} g_{ij} dx^i dx^j$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} \vec{t}_1 \cdot \vec{t}_1, & \vec{t}_1 \cdot \vec{t}_2 \\ \vec{t}_2 \cdot \vec{t}_1 & \vec{t}_2 \cdot \vec{t}_2 \end{pmatrix}$$

$$g = \det(g_{ij}) = |\vec{t}_1 \times \vec{t}_2|^2$$

**area element**

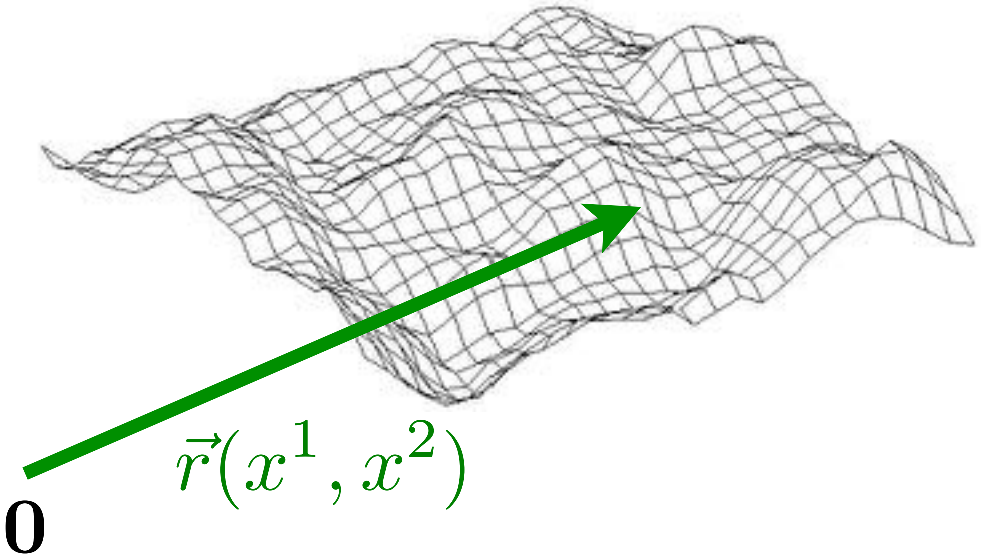



$$dA = |\vec{t}_1||\vec{t}_2| \sin\alpha \, dx^1 dx^2$$

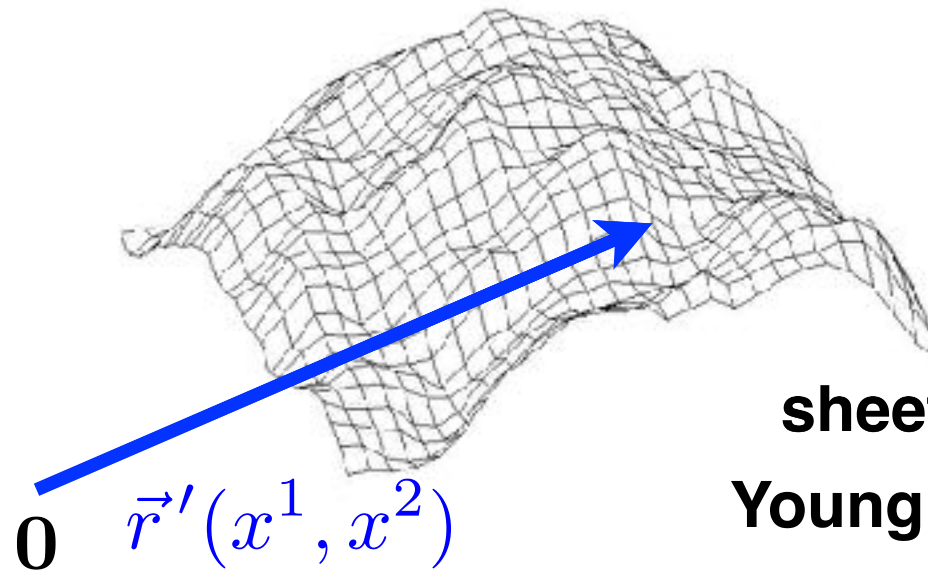
$$dA = \sqrt{g}\, dx^1 dx^2$$

# Strain tensor and energy of shell deformations

## undeformed shell




## deformed shell




**sheet thickness** $d$

**Young's modulus** $E$

**Poisson's ratio** $\nu$

$$g_{ij} = \frac{\partial \vec{r}}{\partial x^i} \cdot \frac{\partial \vec{r}}{\partial x^j}$$

$$d\ell^2 = \sum_{i,j} g_{ij} dx^i dx^j$$

$$g'_{ij} = \frac{\partial \vec{r}\,'}{\partial x^i} \cdot \frac{\partial \vec{r}\,'}{\partial x^j}$$

$$d\ell'^2 = \sum_{i,j} g'_{ij} dx^i dx^j$$

**strain tensor**

$$u_{ij} = \frac{1}{2} \sum_k (g^{-1})_{ik} \left(g'_{kj} - g_{kj}\right)$$

**inverse metric tensor**

$$\sum_k (g^{-1})_{ik} g_{kj} = \sum_k g_{ik} (g^{-1})_{kj} = \delta_{ij}$$

**Energy cost for stretching, compressing and shearing**

$$U = \int (\sqrt{g} dx^1 dx^2)\ \frac{1}{2} \left[ \lambda \left( \sum_i u_{ii} \right)^2 + 2\mu \sum_{i,j} u_{ij} u_{ji} \right]$$

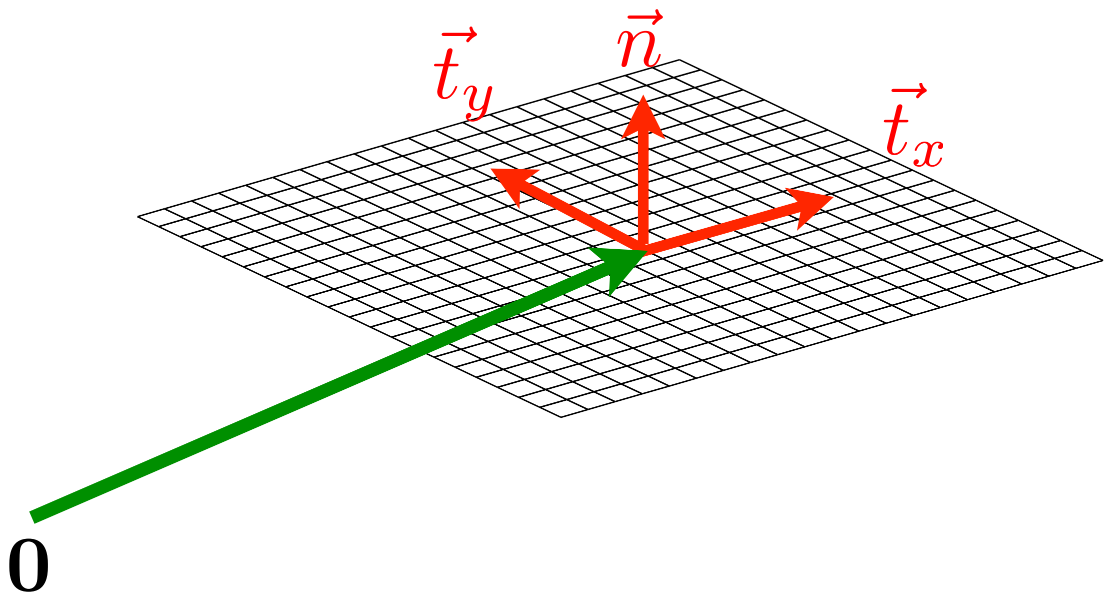
$$g = \det(g_{ij})$$

**Lame constants**

$$\lambda = \frac{E \nu d}{(1 - \nu^2)} \quad \mu = \frac{E d}{2(1 + \nu)}$$

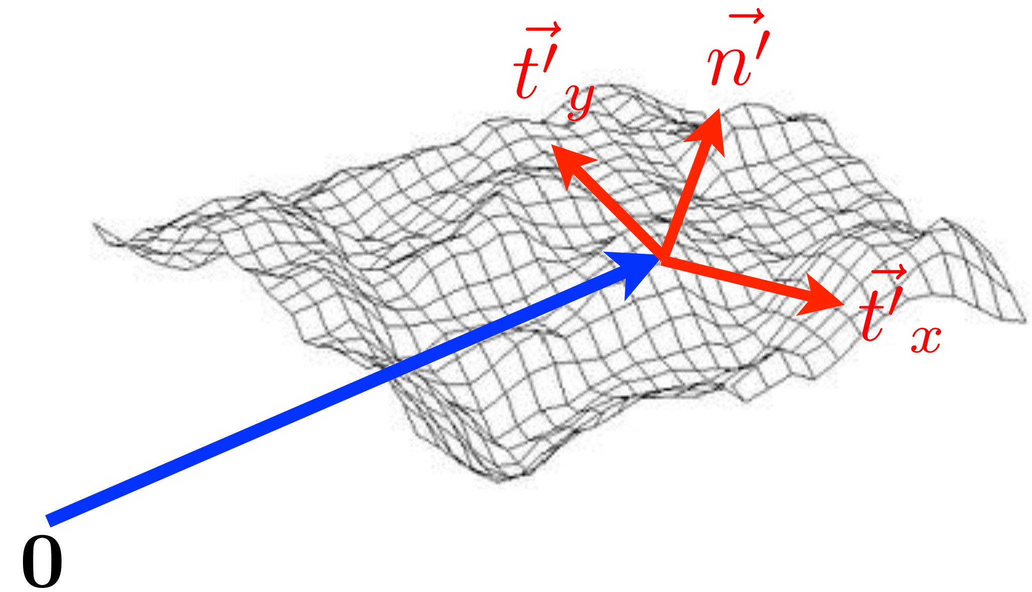
# Strain tensor for deformation of flat plates

## undeformed plate




$$\vec{r}(x,y) = x\vec{e}_x + y\vec{e}_y$$

## deformed plate




$$\vec{r'}(x,y) = \vec{r}(x,y) + u_x(x,y)\vec{e}_x + u_y(x,y)\vec{e}_y + h(x,y)\vec{e}_z$$

$$i, j, k \in \{x, y\}$$

**local tangents**

$$\vec{t}_i = \partial_i \vec{r} \equiv \frac{\partial \vec{r}}{\partial i} = \vec{e}_i$$

**metric tensor**

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \delta_{ij} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

**local tangents**

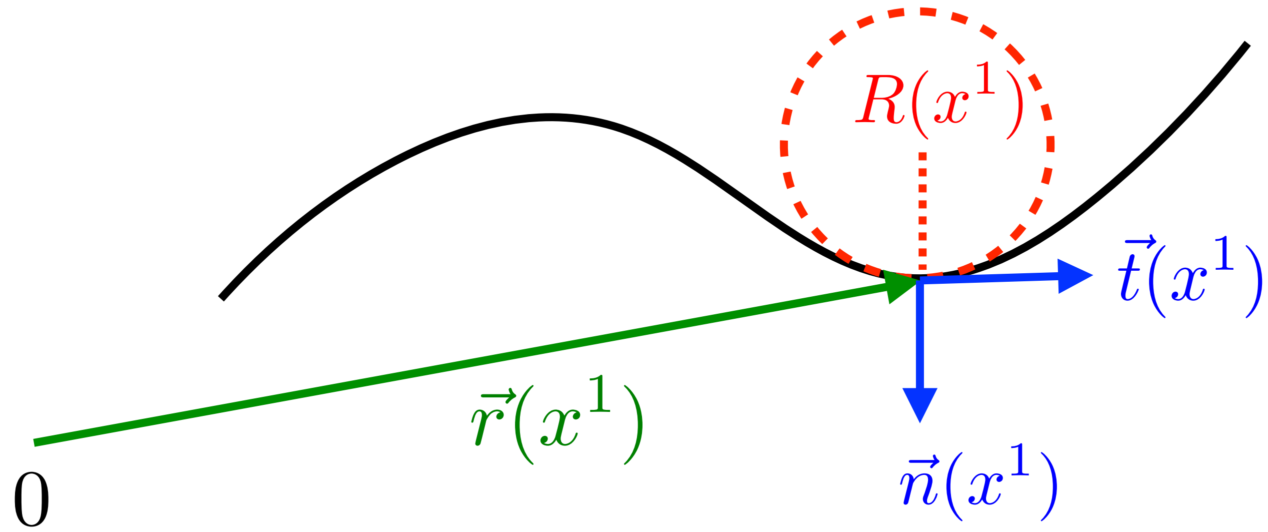
$$\vec{t'}_i = \partial_i \vec{r'} = \vec{e}_i + \sum_k (\partial_i u_k)\vec{e}_k + (\partial_i h)\vec{e}_z$$

**strain tensor**

$$u_{ij} = \frac{1}{2}\left(g'_{ij} - \delta_{ij}\right)$$

$$2u_{ij} = (\partial_i u_j + \partial_j u_i) + \sum_k \partial_i u_k \partial_j u_k + \partial_i h \partial_j h$$

# Curvature of curves




$x^1$ **parameter describing position along the curve**

$\vec{r}(x^1)$ **function describing shape of the curve**

$$\vec{t}(x^1) = \frac{d\vec{r}(x^1)}{dx^1}$$ **local tangent to the curve**

$\vec{n}(x^1)$ **local unit normal vector to the curve**

$g = \vec{t}^{\,2}$ **metric for measuring lengths**

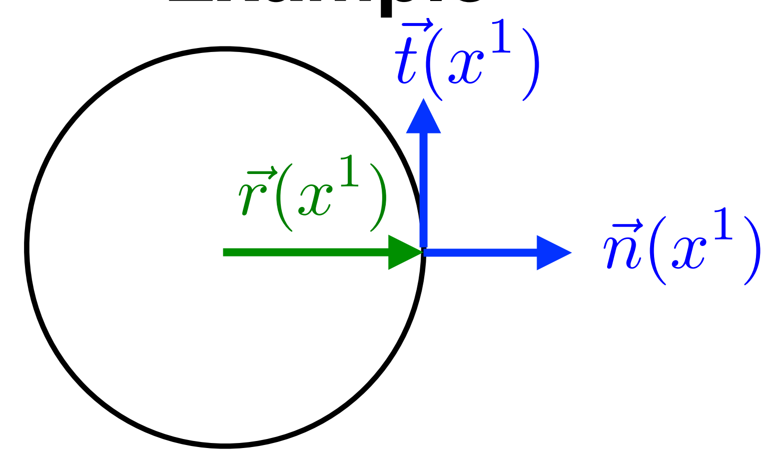
**curvature of curve**

$$\frac{1}{R} = K = \frac{1}{g}\left(\vec{n}\cdot\frac{d^2\vec{r}}{d(x^1)^2}\right)$$

**in natural parametrization with *g*=1**

$$\left|\frac{1}{R}\right| = \left|\frac{d^2\vec{r}}{d\left(x^1\right)^2}\right| = \left|\frac{d\vec{t}}{dx^1}\right|$$

## Example




$$\vec{r}(x^1) = R\big(\cos(\omega x^1), \sin(\omega x^1)\big)$$

$$\vec{n}(x^1) = \big(\cos(\omega x^1), \sin(\omega x^1)\big)$$

$$g(x^1) = R^2\omega^2$$

$$K = -\frac{1}{R}$$

# Curvature tensor for surfaces

$x^1, x^2$ parameters describing position along the surface

$\vec{r}(x^1, x^2)$ function describing shape of the surface

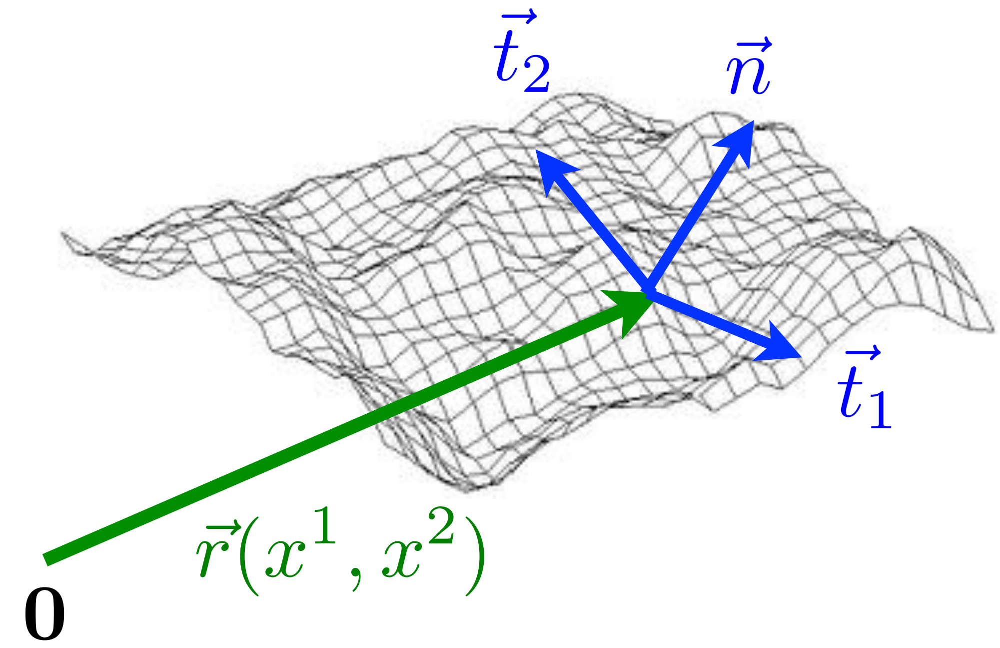
$\vec{t}_i = \dfrac{\partial \vec{r}}{\partial x^i}$ local tangent vectors to the surface

$\vec{n} = \dfrac{\vec{t}_1 \times \vec{t}_2}{|\vec{t}_1 \times \vec{t}_2|}$ unit normal vector of the surface

$g_{ij} = \vec{t}_i \cdot \vec{t}_j$ metric tensor for measuring lengths

curvature tensor for surfaces

$$K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)$$




principal curvatures correspond to the eigenvalues of curvature tensor

$$\frac{1}{R_1}, \frac{1}{R_2}$$

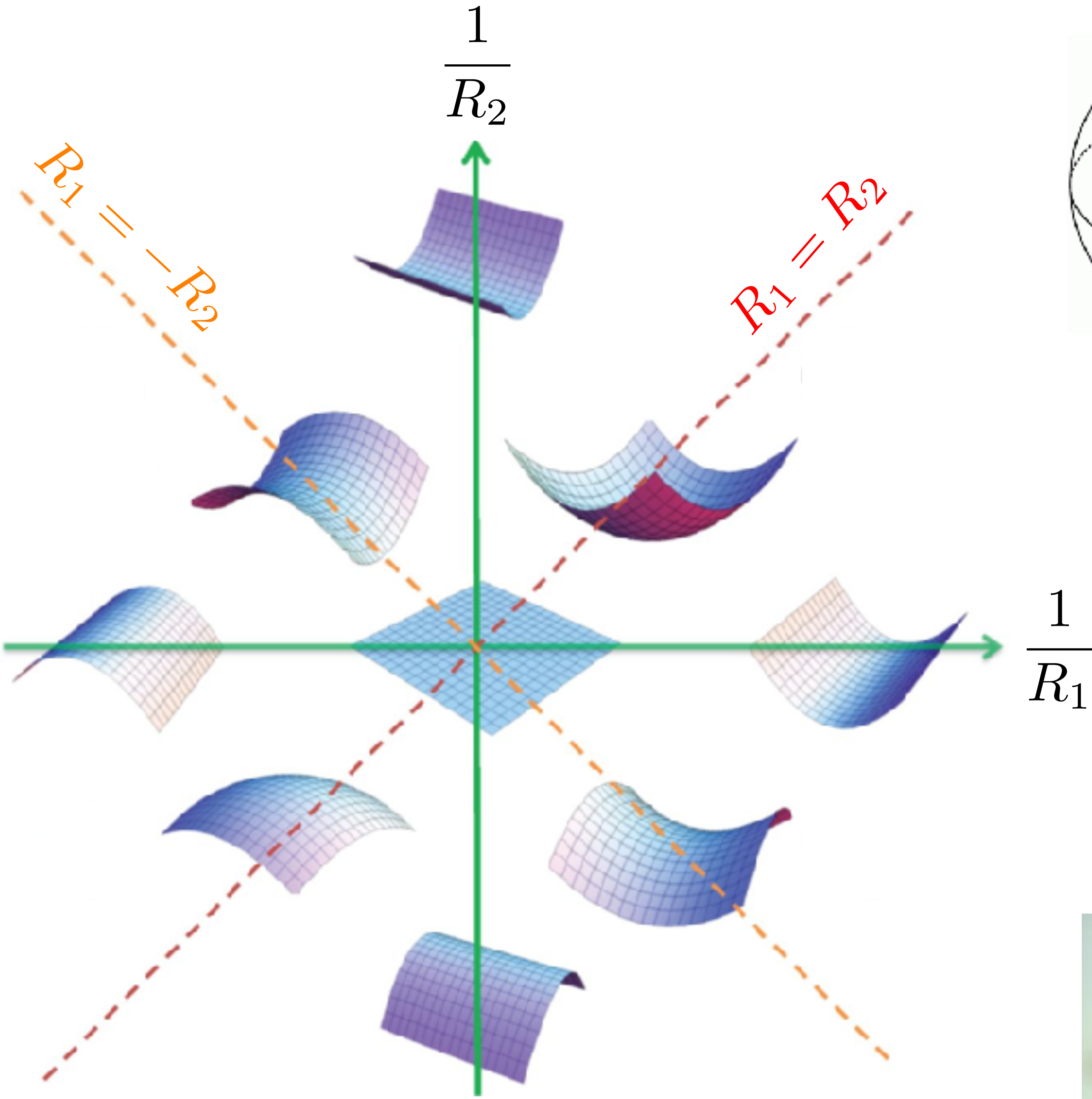
mean curvature

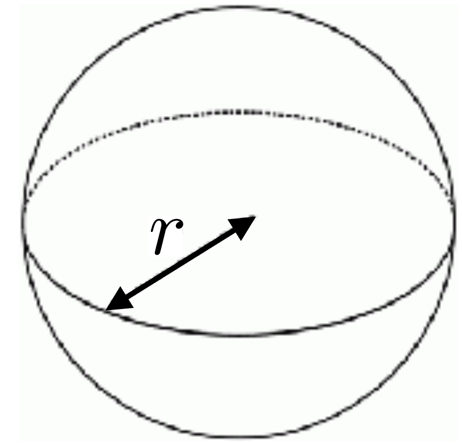
$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\sum_i K_{ii} = \frac{1}{2}\mathrm{tr}(K_{ij})$$

Gaussian curvature

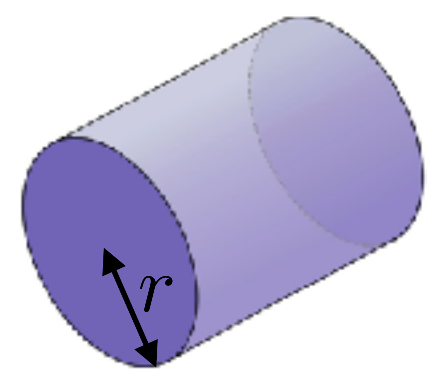
$$\frac{1}{R_1 R_2} = \det(K_{ij})$$

# Surfaces of various principal curvatures




$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{r}$$



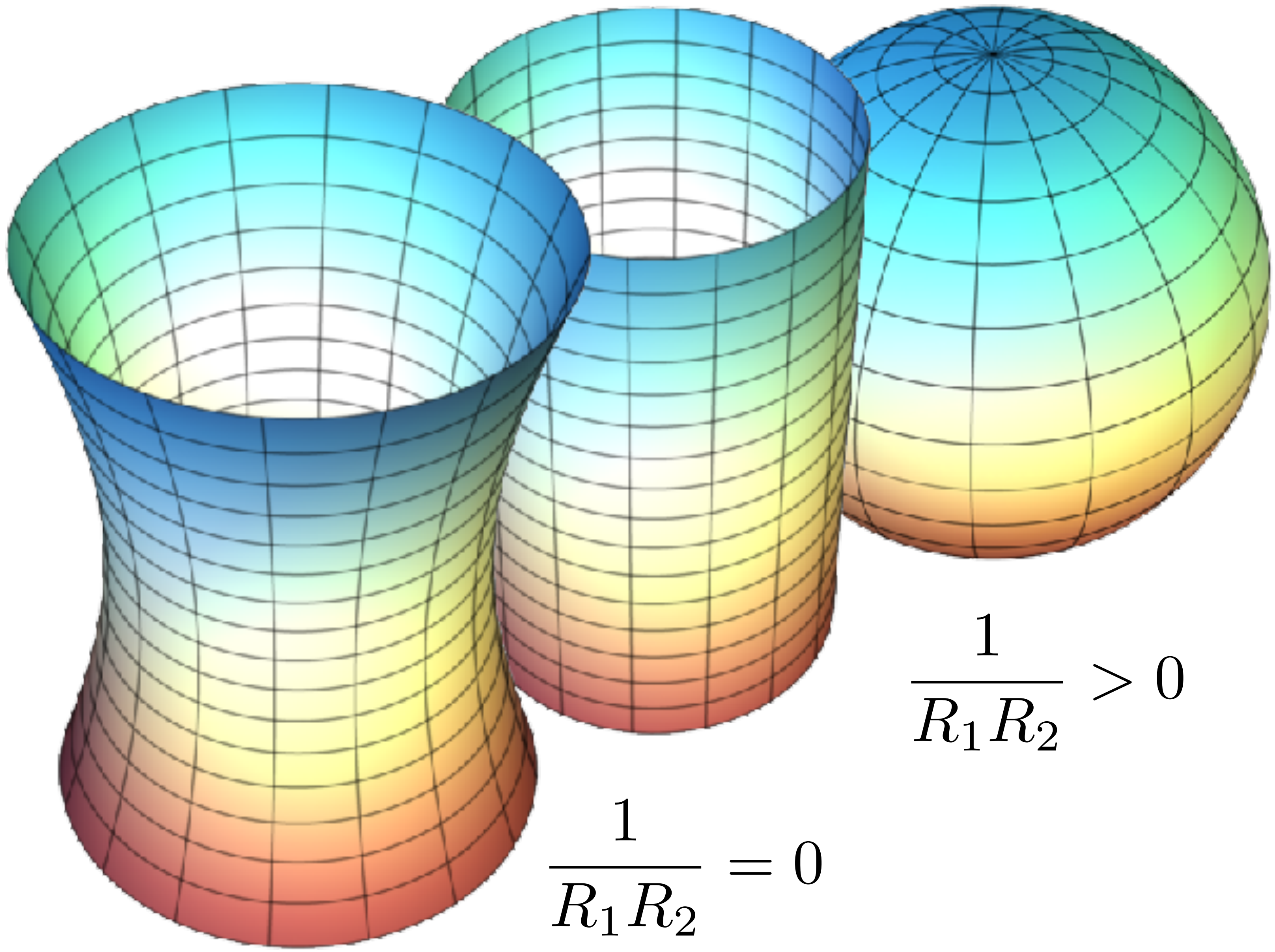

$$\frac{1}{R_1} = \frac{1}{r}$$

$$\frac{1}{R_2} = 0$$

$$\frac{1}{R_1} > 0$$

$$\frac{1}{R_2} < 0$$

# Examples for Gaussian curvature



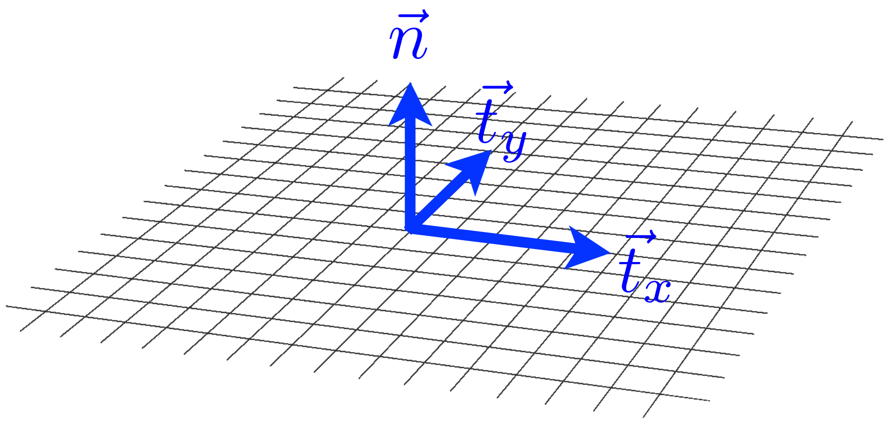
$$\frac{1}{R_1 R_2} > 0$$

$$\frac{1}{R_1 R_2} = 0$$

$$\frac{1}{R_1 R_2} < 0$$

# Examples

$$K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)$$




$$\vec{r}(x,y) = (x,y,0)$$

$$\vec{t}_x = \frac{\partial \vec{r}}{\partial x} = (1,0,0)$$

$$\vec{t}_y = \frac{\partial \vec{r}}{\partial y} = (0,1,0)$$

$$\vec{n} = \frac{\vec{t}_x \times \vec{t}_y}{|\vec{t}_x \times \vec{t}_y|} = (0,0,1)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

$$K_{ij} = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix}$$

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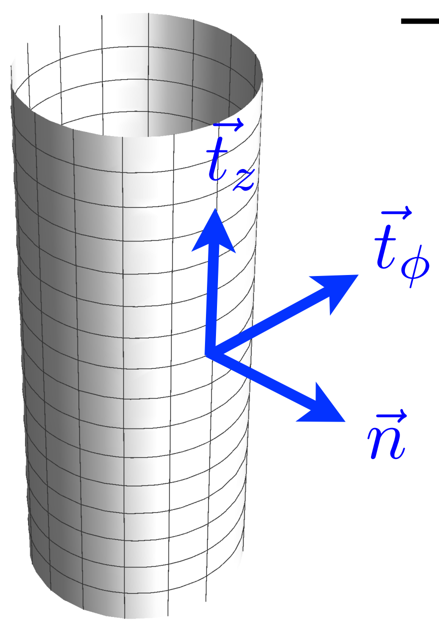


$$\vec{r}(\phi, z) = (R\cos\phi, R\sin\phi, z)$$

$$\vec{t}_\phi = \frac{\partial \vec{r}}{\partial \phi} = R(-\sin\phi, \cos\phi, 0)$$

$$\vec{t}_z = \frac{\partial \vec{r}}{\partial z} = (0,0,1)$$

$$\vec{n} = \frac{\vec{t}_\phi \times \vec{t}_z}{|\vec{t}_\phi \times \vec{t}_z|} = (\cos\phi, \sin\phi, 0)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} R^2, & 0 \\ 0, & 1 \end{pmatrix}$$

$$K_{ij} = \begin{pmatrix} -\frac{1}{R}, & 0 \\ 0, & 0 \end{pmatrix}$$

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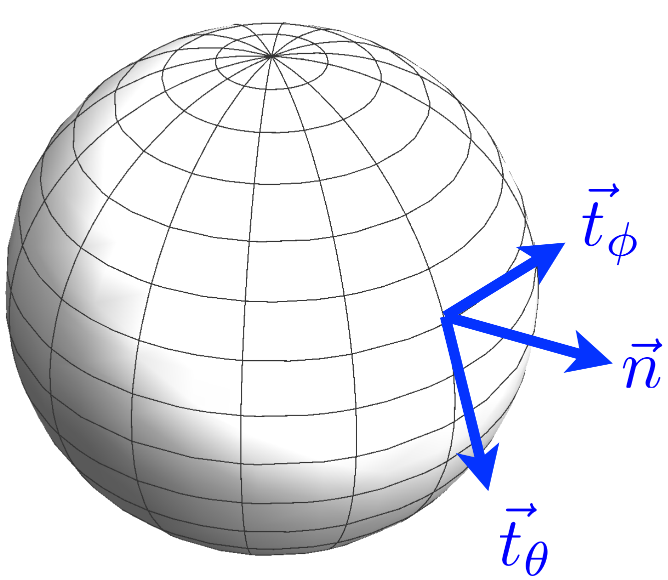



$$\vec{r}(\theta, \phi) = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{t}_\theta = \frac{\partial \vec{r}}{\partial \theta} = R(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\vec{t}_\phi = \frac{\partial \vec{r}}{\partial \phi} = R\sin\theta(-\sin\phi, \cos\phi, 0)$$

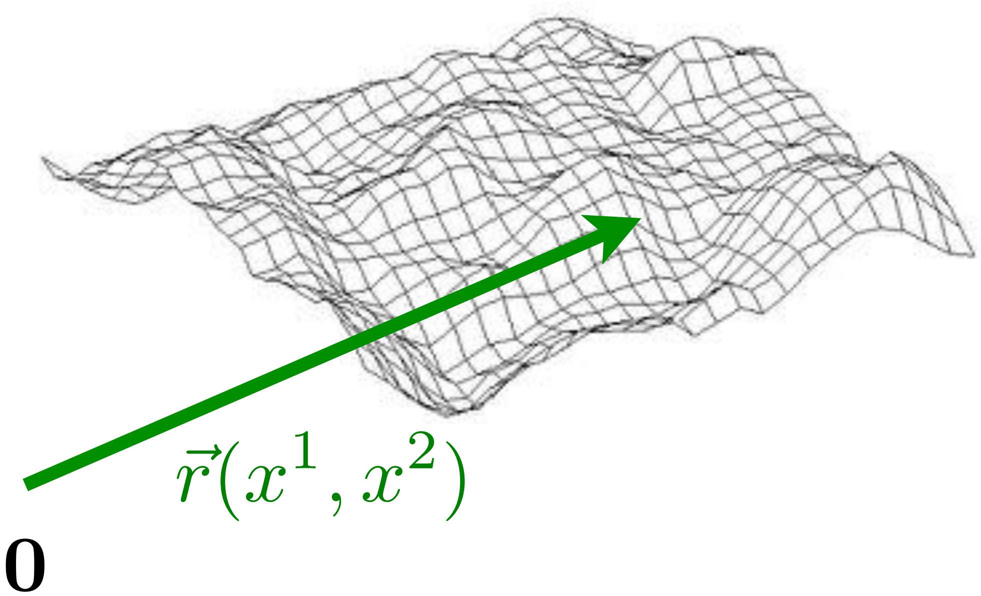
$$\vec{n} = \frac{\vec{t}_\theta \times \vec{t}_\phi}{|\vec{t}_\theta \times \vec{t}_\phi|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} R^2, & 0 \\ 0, & R^2 \sin^2\theta \end{pmatrix}$$

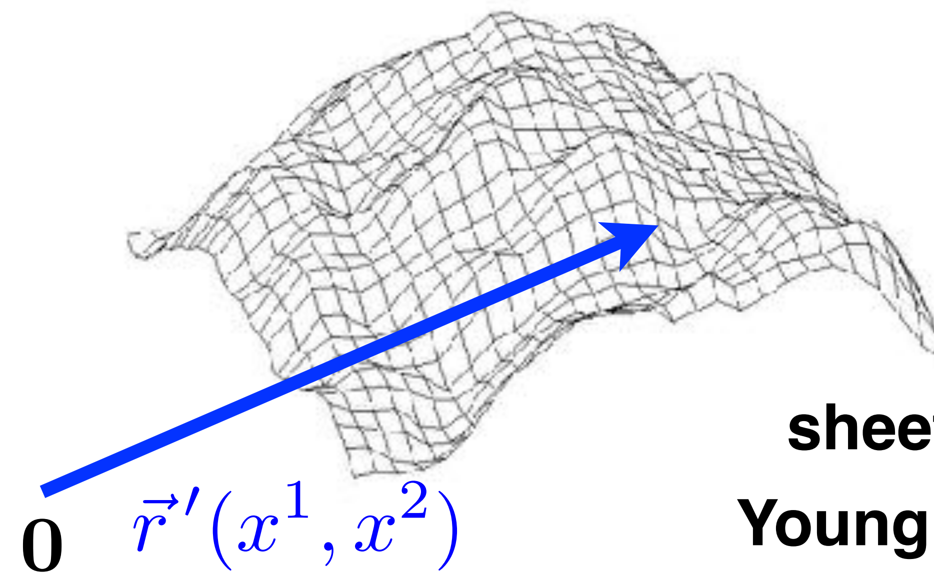
$$K_{ij} = \begin{pmatrix} -\frac{1}{R}, & 0 \\ 0, & -\frac{1}{R} \end{pmatrix}$$

# Bending energy for deformation of shells

## undeformed shell




## deformed shell




**sheet thickness** $d$

**Young's modulus** $E$

**Poisson's ratio** $\nu$

$$K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)$$

$$K'_{ij} = \sum_k \left(g'^{-1}\right)_{ik} \left(\vec{n}' \cdot \frac{\partial^2 \vec{r}\,'}{\partial x^k \partial x^j}\right)$$

## bending strain tensor

$$b_{ij} = K'_{ij} - K_{ij}$$

**(local measure of deviation from preferred curvature)**

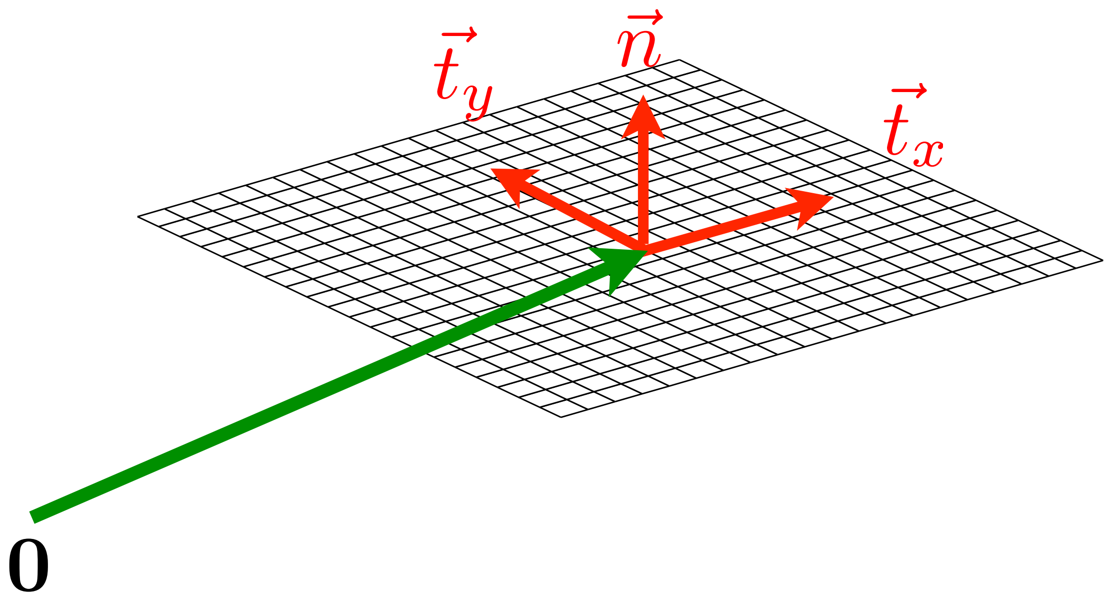
## Energy cost of bending

$$U = \int \left(\sqrt{g}\, dx^1 dx^2\right) \left[\frac{1}{2}\kappa \left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij})\right]$$

$$\kappa = \frac{Ed^3}{12(1-\nu^2)} \qquad \kappa_G = -\frac{Ed^3}{12(1+\nu)}$$

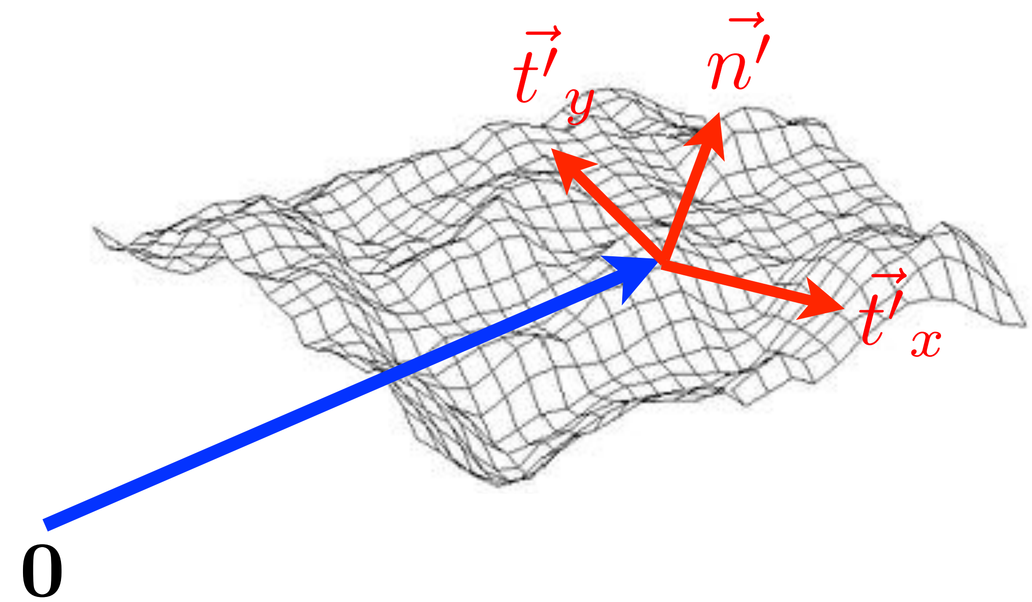
# Bending strain for deformation of flat plates

## undeformed plate




$$\vec{r}(x,y) = x\vec{e}_x + y\vec{e}_y$$

## deformed plate




$$\vec{r'}(x,y) = \vec{r}(x,y) + u_x(x,y)\vec{e}_x + u_y(x,y)\vec{e}_y + h(x,y)\vec{e}_z$$

**local normal**

$$\vec{n} = \frac{\vec{t}_x \times \vec{t}_y}{|\vec{t}_x \times \vec{t}_y|} = \vec{e}_z$$

**local normal (neglecting in-plane deformations)**

$$\vec{n'} \approx \frac{\vec{e}_z - (\partial_x h)\,\vec{e}_x - (\partial_y h)\,\vec{e}_y}{\sqrt{1 + (\partial_x h)^2 + (\partial_y h)^2}}$$

**reference curvature tensor**

$$K_{ij} = \vec{n} \cdot \partial_i \partial_j \vec{r} = 0$$

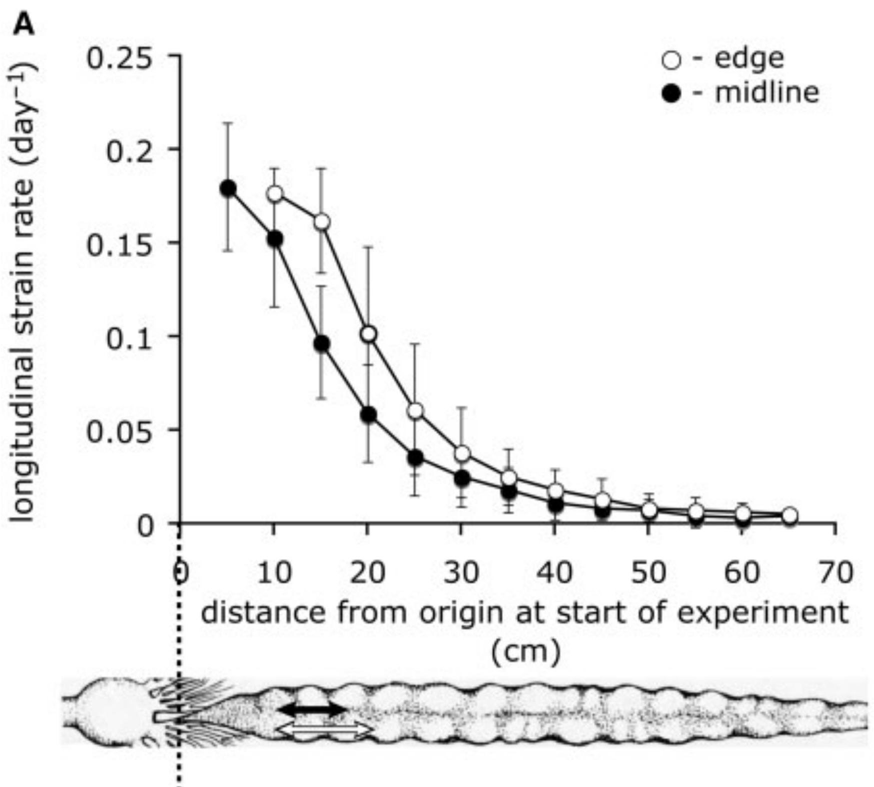
**bending strain tensor**

$$b_{ij} = K'_{ij} \approx \partial_i \partial_j h + \cdots$$

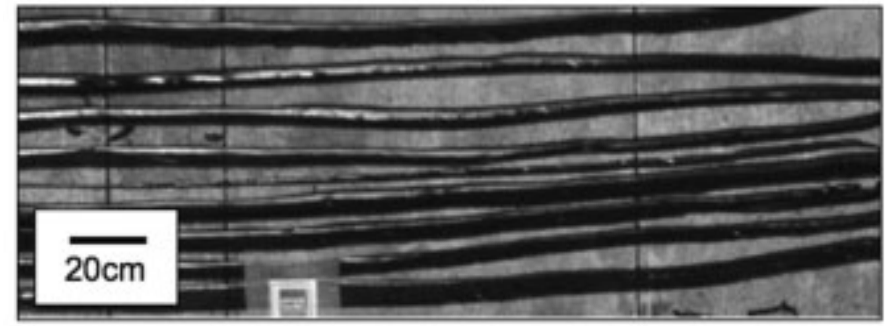
# Wrinkled and straight blades in macroalgae

**Slow water flow environment (v~0.5 m/s)**

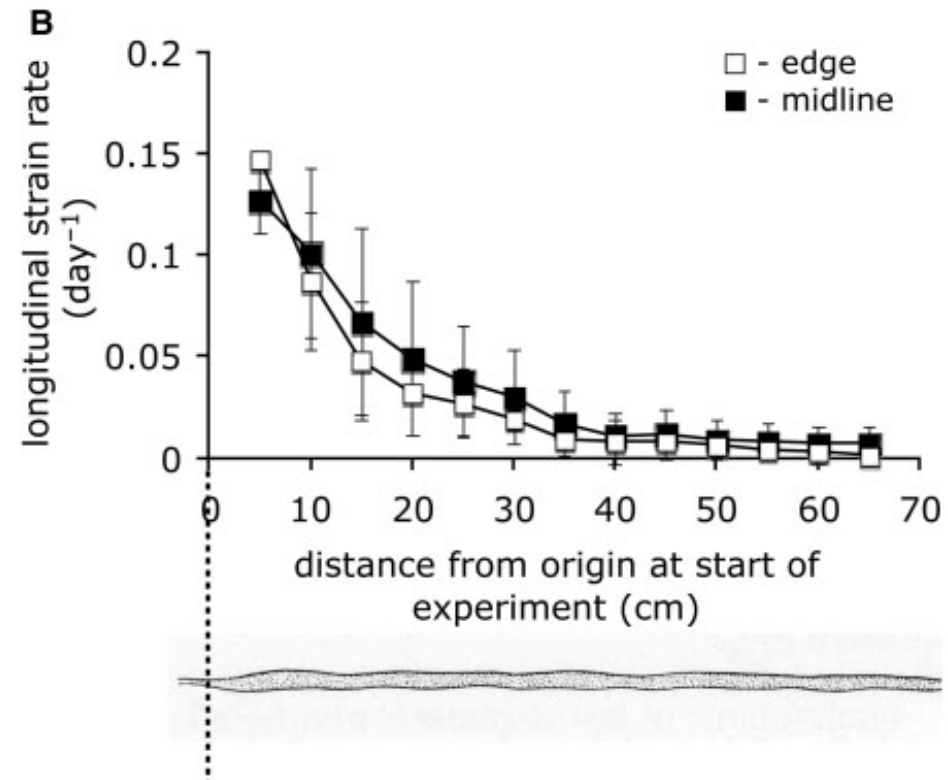

**edges of blades grow faster than the midline**




**Fast water flow environment (v~1.5 m/s)**




**edges of blades grow at the same speed as the midline**




**What is the effect of differential growth rate between the edge and the midline of the blade?**

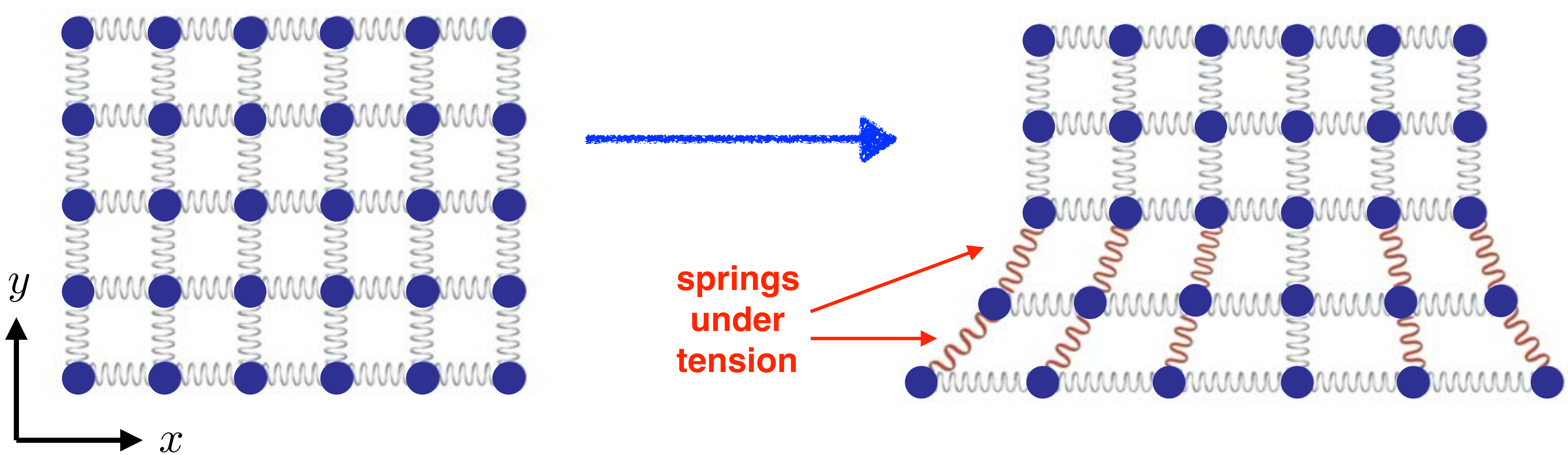


M. Koehl et al., Integ. Comp. Biol. **48**, 834 (2008)

# Differential growth produces internal stress

**before growth**

**faster growth of the bottom edge in x direction**




$$d\ell^2 = \sum_{i,j} g_{ij}\, dx^i dx^j$$

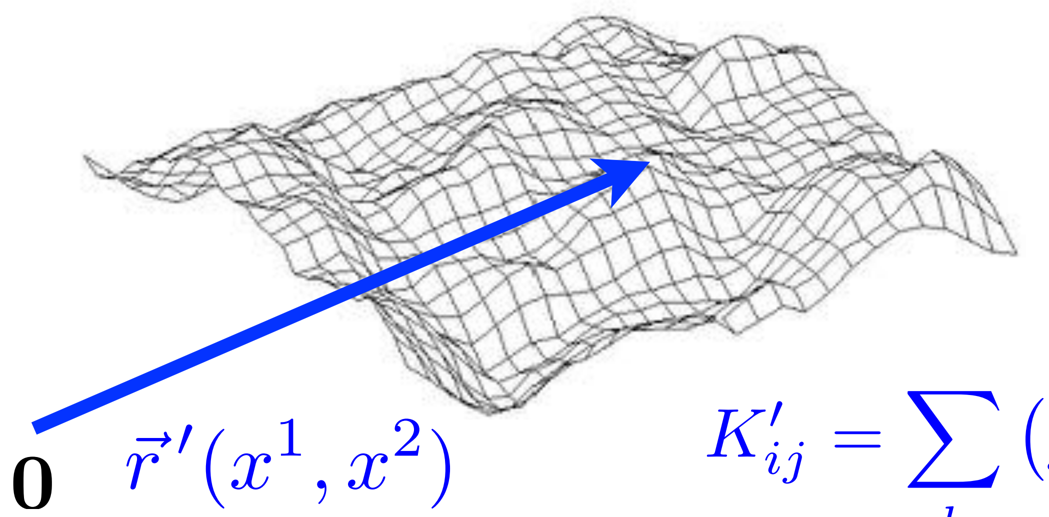
## Growth modifies the metric tensor of sheet!

$$g_{ij} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} \longrightarrow g_{ij} = \begin{pmatrix} f(y), & 0 \\ 0, & 1 \end{pmatrix}$$

**Note: If growth is different between the top and bottom of the sheet, then the curvature tensor $K_{ij}$ is modified as well!**

# Mechanics of growing sheets

**Growth defines preferred metric tensor $g_{ij}$, and preferred curvature tensor $K_{ij}$.**




$\vec{r}\,'(x^1, x^2)$

$$g'_{ij} = \frac{\partial \vec{r}\,'}{\partial x^i} \cdot \frac{\partial \vec{r}\,'}{\partial x^j}$$

$$K'_{ij} = \sum_k \left(g'^{-1}\right)_{ik} \left(\vec{n}\,' \cdot \frac{\partial^2 \vec{r}\,'}{\partial x^k \partial x^j}\right)$$

**strain tensors**

$$u_{ij} = \frac{1}{2} \sum_k \left(g^{-1}\right)_{ik} \left(g'_{kj} - g_{kj}\right)$$

$$b_{ij} = K'_{ij} - K_{ij}$$

**The equilibrium membrane shape $\vec{r}\,'(x^1, x^2)$ corresponds to the minimum of elastic energy:**

$$U = \int \left(\sqrt{g}\, dx^1 dx^2\right) \left[\frac{1}{2}\lambda \left(\sum_i u_{ii}\right)^2 + \mu \sum_{i,j} u_{ij} u_{ji} + \frac{1}{2}\kappa \left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij})\right]$$

**Growth can independently tune the metric tensor $g_{ij}$ and the curvature tensor $K_{ij}$, which may not be compatible with any surface shape that would produce zero energy cost!**

**Zero energy shape exists only when preferred metric tensor $g_{ij}$ and preferred curvature tensor $K_{ij}$ satisfy Gauss-Codazzi-Mainardi relations!**

# Mechanics of growing membranes

**One of the Gauss-Codazzi-Mainardi equations (Gauss's Theorema Egregium) relates the Gauss curvature to metric tensor**

$$\det(K'_{ij}) = \mathcal{F}(g'_{ij})$$

**The equilibrium membrane shape $\vec{r}\,'(x^1, x^2)$ corresponds to the minimum of elastic energy:**

$$U = \int (\sqrt{g}dx^1dx^2) \left[ \frac{1}{2}\lambda \left( \sum_i u_{ii} \right)^2 + \mu \sum_{i,j} u_{ij}u_{ji} + \frac{1}{2}\kappa \left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij}) \right]$$

**scaling with membrane thickness d**

$$\lambda, \mu \sim Ed$$

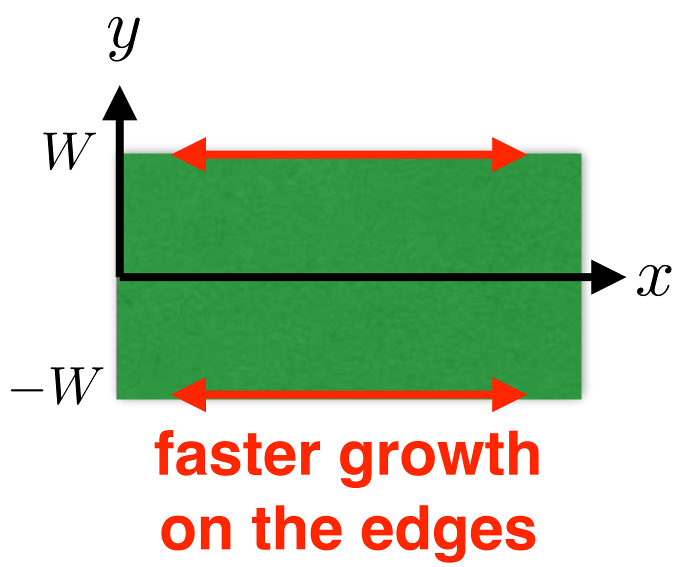
$$\kappa, \kappa_G \sim Ed^3$$

**For very thin membranes the equilibrium shape matches the preferred metric tensor to avoid stretching, compressing and shearing. This also specifies the Gauss curvature!**

$$g'_{ij} = g_{ij}$$

$$\det(K'_{ij}) = \mathcal{F}(g_{ij})$$

# Example




**Assume that differential growth in x direction produces metric tensor of the form**

$$g_{ij} = \begin{pmatrix} f(y), & 0 \\ 0, & 1 \end{pmatrix} \qquad f(y) = 1 + ce^{(|y|-W)/\lambda}$$

**For thin membranes the metric tensor wants to be matched**

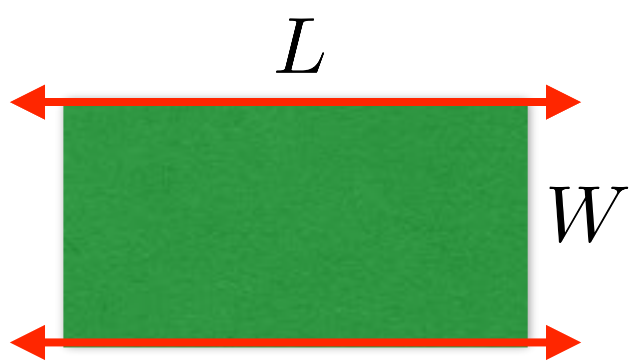
$$g'_{ij} = g_{ij}$$

**Gauss's Theorema Egregium provides Gauss curvature**

$$\det(K'_{ij}(y)) = \mathcal{F}(g_{ij}) = -\frac{1}{f}\frac{d^2 f(y)}{dy^2} = -\frac{1}{\lambda^2} \times \frac{ce^{(|y|-W)/\lambda}}{\left(1 + ce^{(|y|-W)/\lambda}\right)} < 0$$

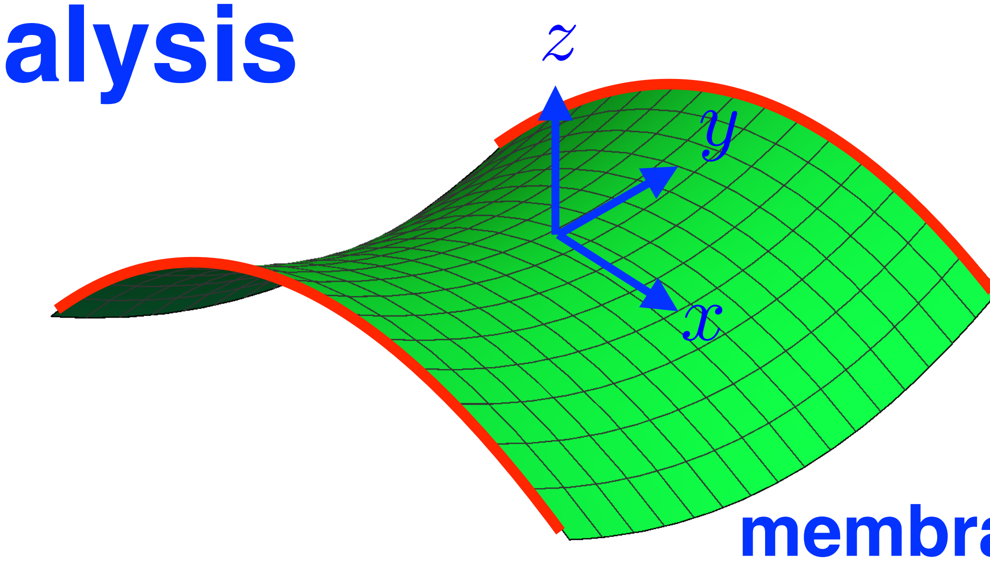
**For thin membranes faster growth on edges produces shapes that locally look like saddles!**

# Scaling analysis




faster growth increases the edge length by factor $\epsilon$

stress released by bending

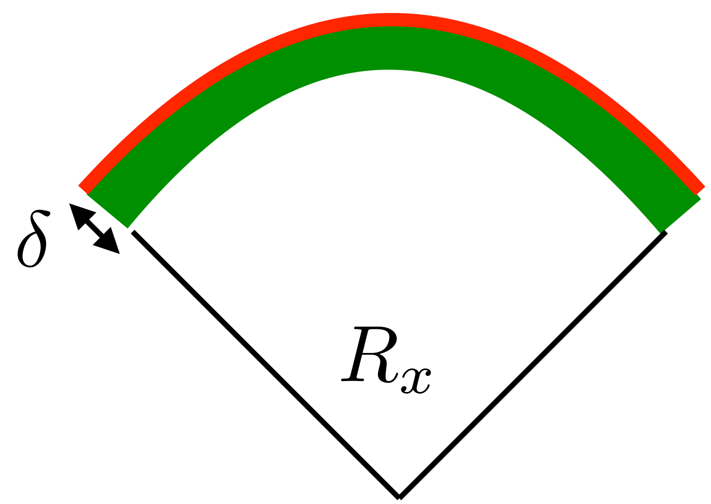



membrane thickness d

**y-z cross-section**


$$\frac{1}{R_y} \sim \frac{\delta}{W^2}$$

**projection to x-z plane**




$$\frac{L(1+\epsilon)}{L} \sim \frac{R_x + \delta}{R_x}$$

$$\frac{1}{R_x} \sim \frac{\epsilon}{\delta}$$

## Membrane bending energy

$$U_b \sim A \times \kappa \times \left( \frac{1}{R_x^2} + \frac{1}{R_y^2} + \frac{1}{R_x R_y} \right) \sim A \times E_m d^3 \times \left( \frac{\epsilon^2}{\delta^2} + \frac{\delta^2}{W^4} + \frac{\epsilon}{W^2} \right)$$

Minimize $U_b$ with respect to $\delta$ :

$$\delta \sim W\sqrt{\epsilon}$$

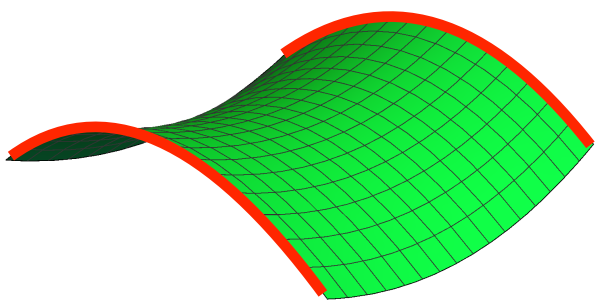
$$U_b \sim \frac{A E_m d^3 \epsilon}{W^2}$$

# Scaling analysis

**membrane compression**

$$U_c \sim A E_m d \epsilon^2$$

**membrane bending**



$$U_b \sim \frac{A E_m d^3 \epsilon}{W^2}$$

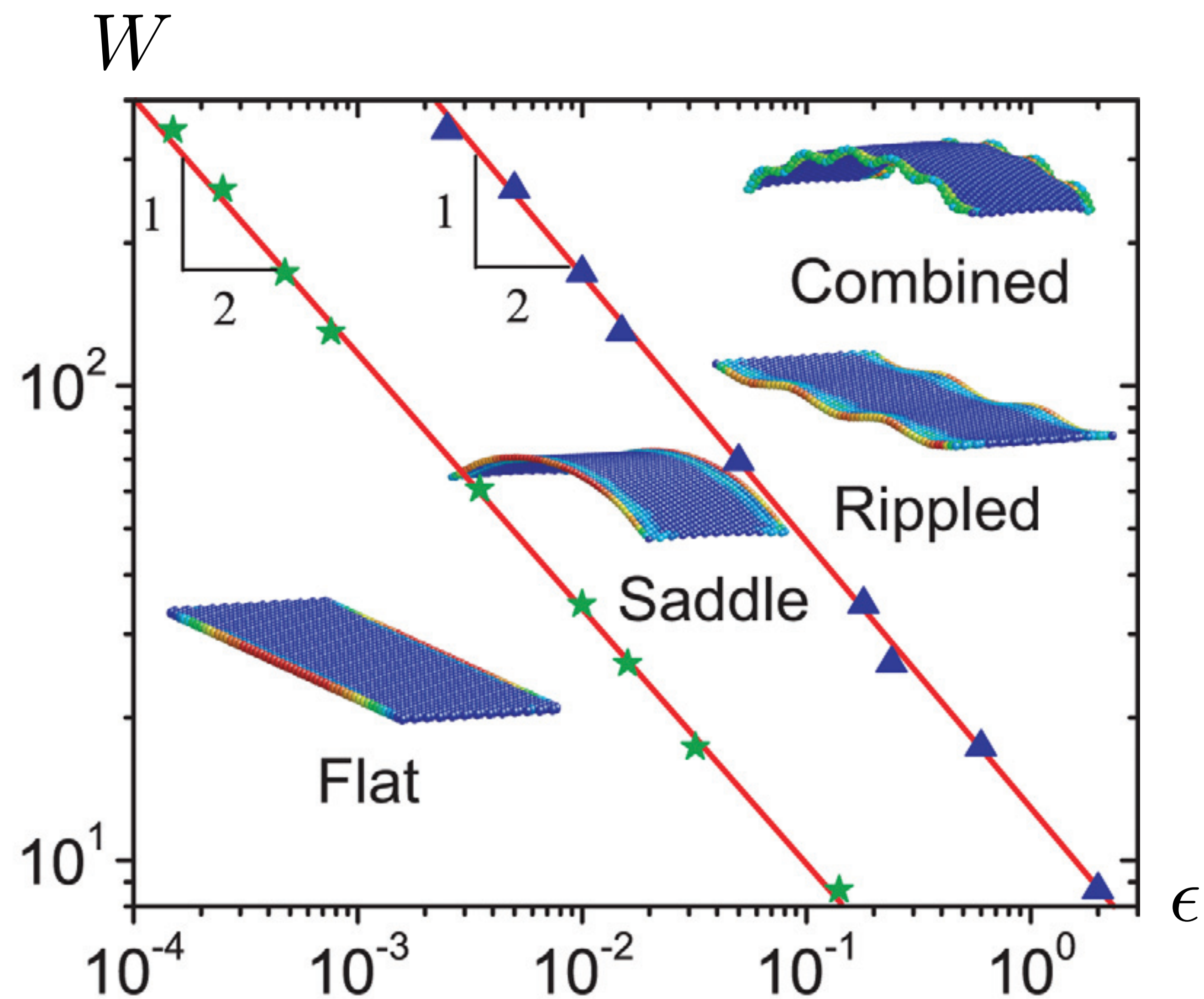
**membrane bends above the critical strain**

$$\epsilon > \epsilon_c \sim \frac{d^2}{W^2}$$

**amplitude of bending at the critical strain**

$$\delta^* \sim W\sqrt{\epsilon_c} \sim d$$

**numerical simulations**




H. Liang and L. Mahadevan, PNAS **106**, 22049 (2009)

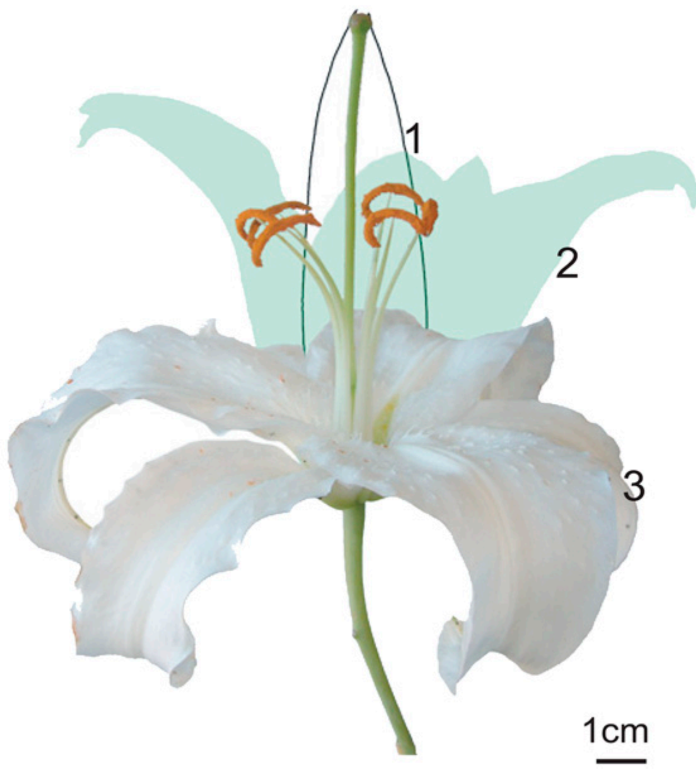
# Shapes of flowers and leaves

**Faster growth of the edge is consistent with observed saddles and edge wrinkles, which indeed correspond to the negative Gauss curvature!**

**saddles**

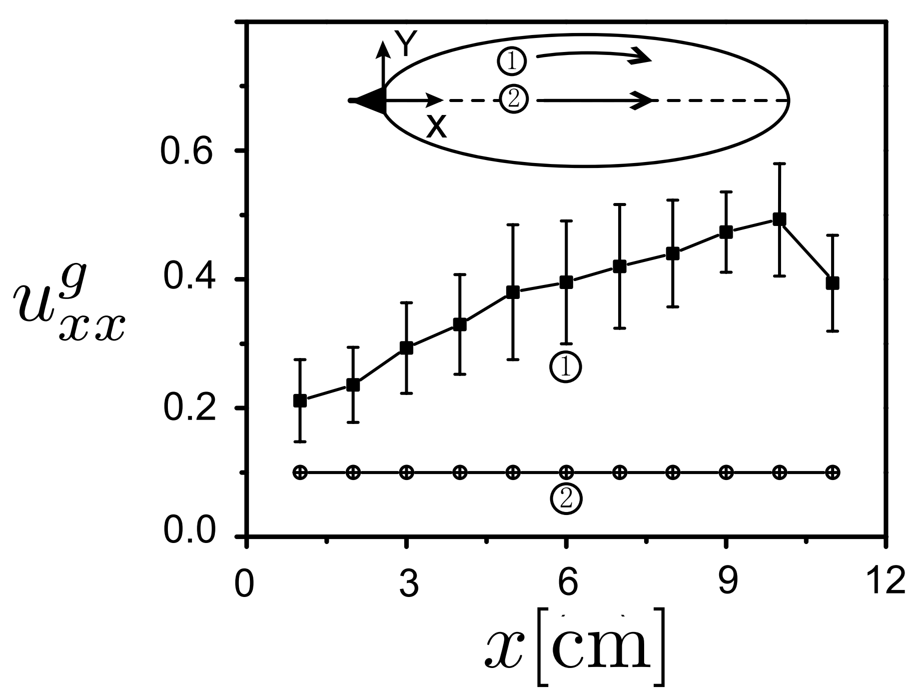
**wrinkled edges (+saddles)**

# Growth of a blooming lily




in lab blooming takes 4.5 days under constant fluorescent light (1 frame/min)

faster growth of the edge




H. Liang and L. Mahadevan, PNAS **108**, 5516 (2011)

# How flowers open in the morning and close in the evening?

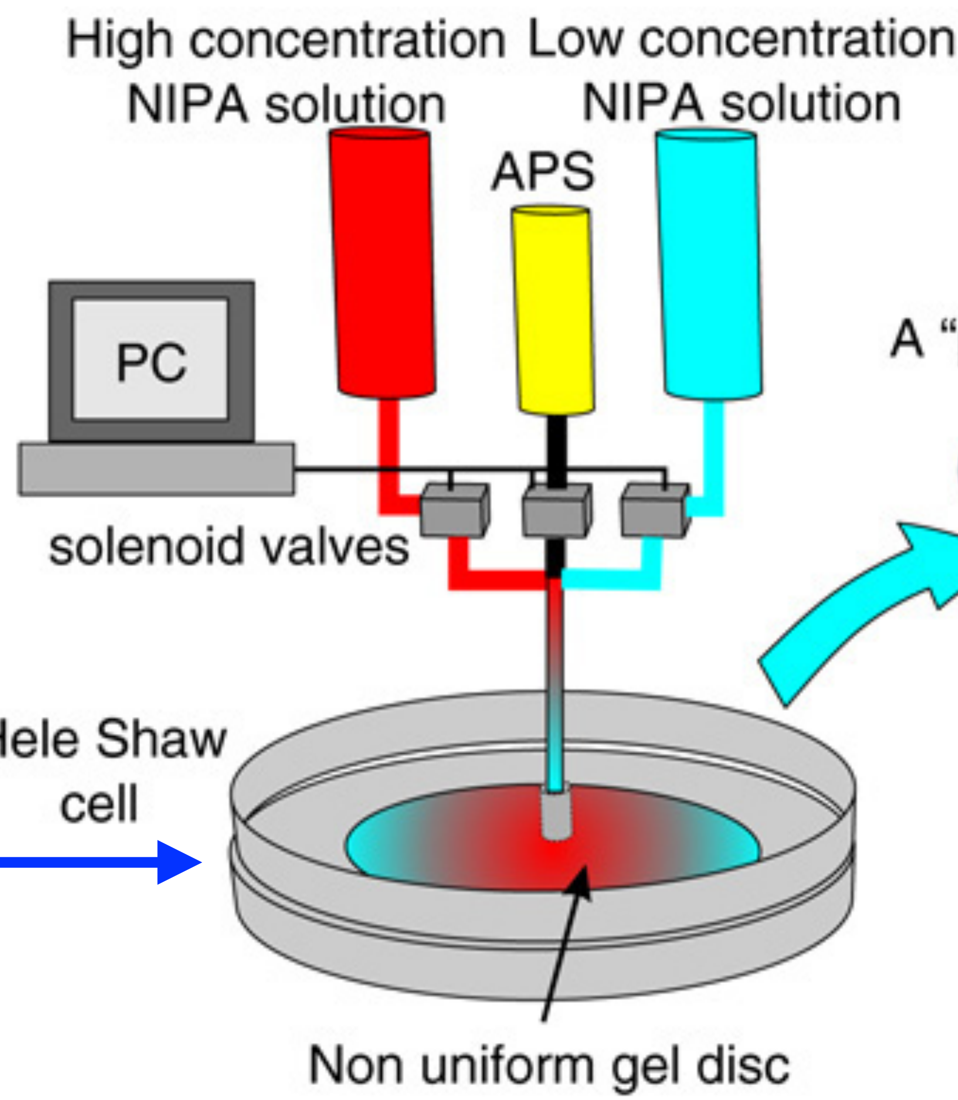
https://vimeo.com/98276732

When temperature increases in the morning, flowers regulate their growth pattern to grow more new cells on the inside of flower leaves. This results in curling of leaves and opening of flowers.

When temperature drops in the evening, flowers regulate their growth pattern to grow more new cells on the outside of flower leaves. This results in straightening of leaves and closing of flowers.

# Shaping of gel membranes by differential shrinking

Computer software controls valves to inject a predefined time depend concentration of NIPA polymers in water solution.

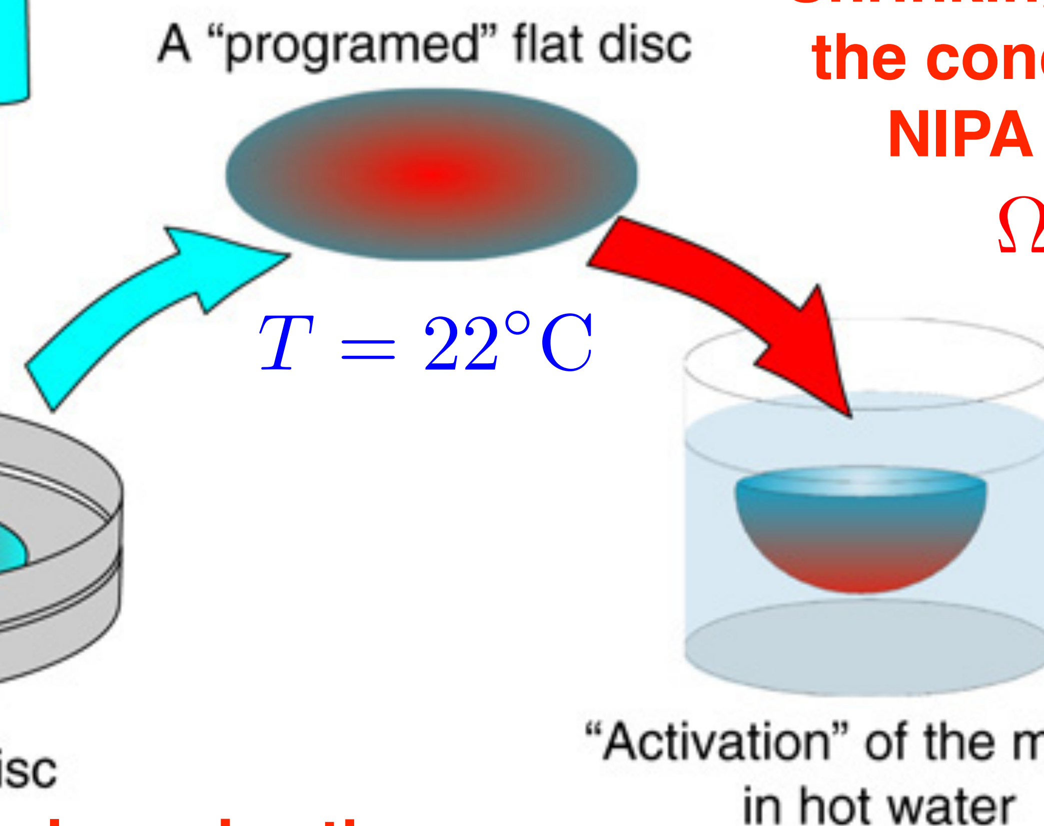



**thickness 0.25 or 0.5 mm**

Active cross-linkers (APS) polymerize the polymer solution within one minute, before polymers get a chance to diffuse around.

**Frozen NIPA concentration profile**

$C(r)$




$T = 22^{\circ}\mathrm{C}$

$T = 45^{\circ}\mathrm{C}$

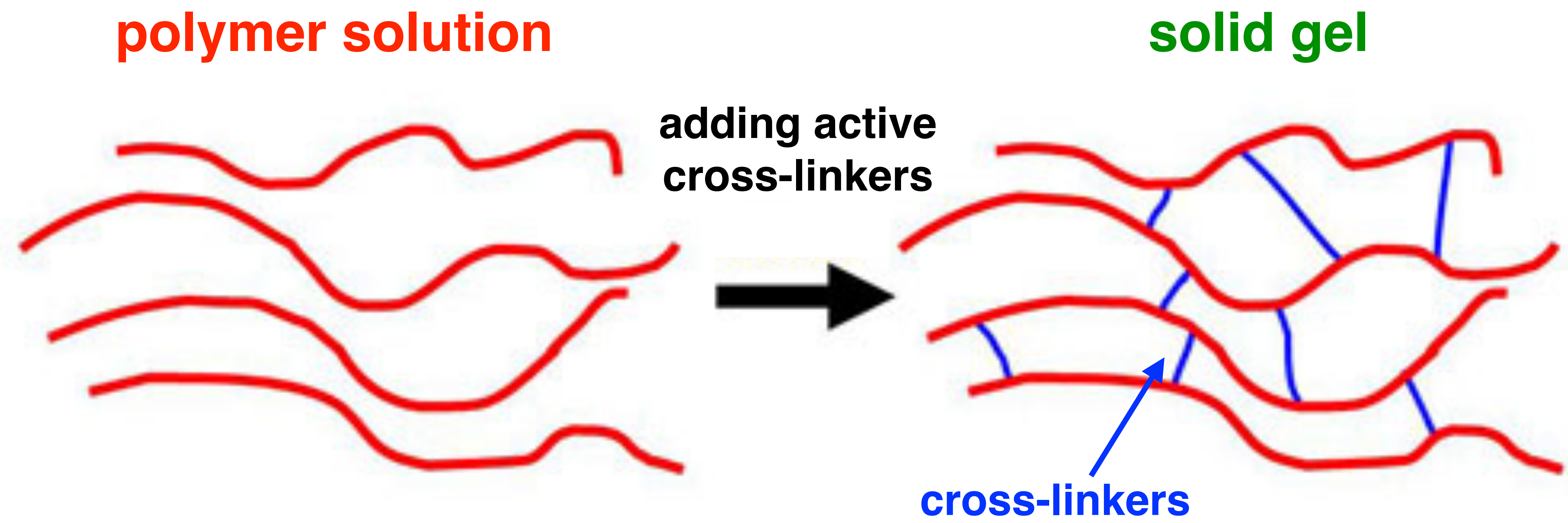
At higher temperatures gel becomes hydrophobic and expels some water. Shrinking depends on the concentration of NIPA polymers.

$\Omega(C(r))$



E. Efrati et al., Physica D **235**, 29 (2007)

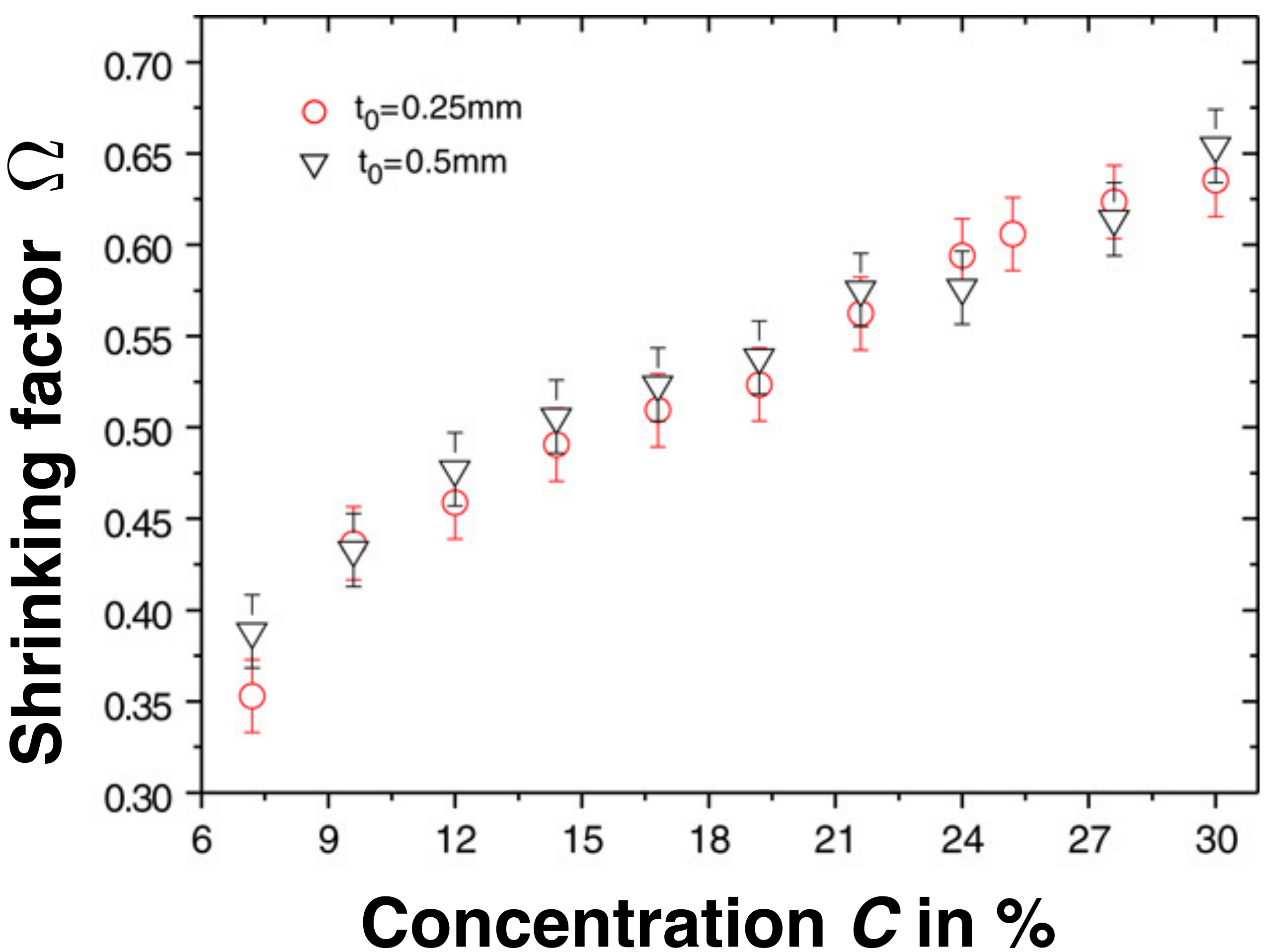
# Cross-linking of polymers result in a solid gel



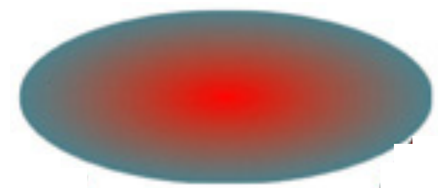

**Note: some cross-linkers can be chemically activated by UV light exposure. Duration of UV light exposure controls the degree of cross-linking and therefore the Young's modulus *E* for gels.**

# Shaping of gel membranes by differential shrinking

## Shrinking of gels at T=45°C




$T = 22°\mathrm{C}$

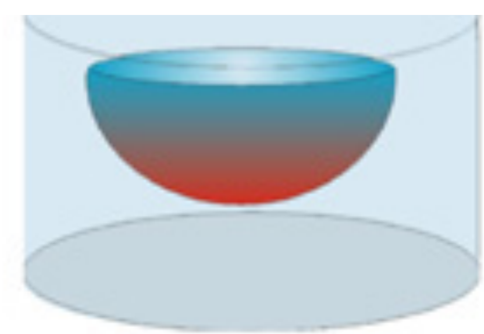


$C(r)$

$$g_{ij} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

**Shrinking modifies metric**

$T = 45°\mathrm{C}$



$$g_{ij} = \begin{pmatrix} \Omega(r), & 0 \\ 0, & \Omega(r) \end{pmatrix}$$

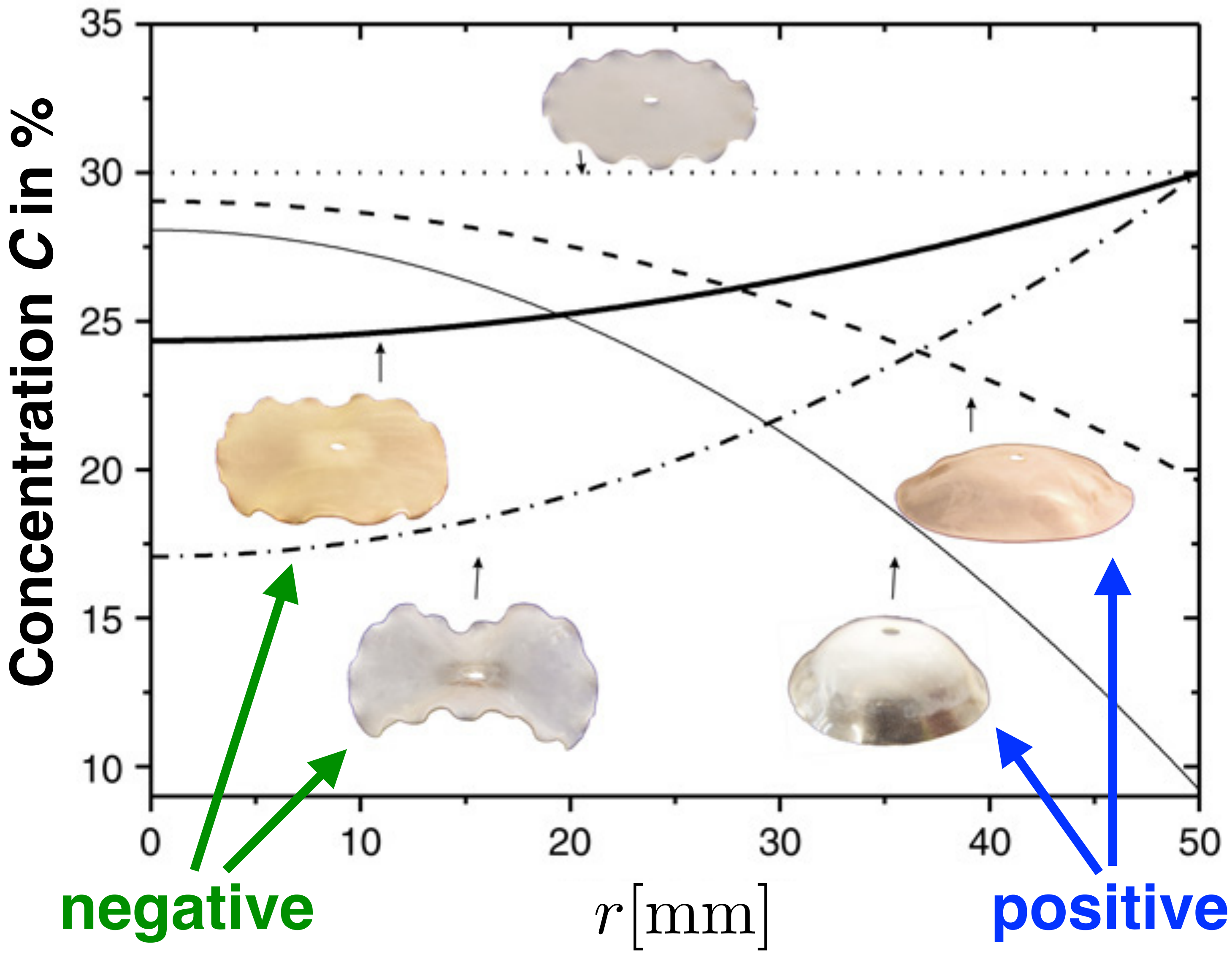
**locally isotropic shrinking**

**For thin membranes the target Gauss curvature is**

$$\det(K'_{ij}(r)) = -\frac{\nabla^2(\ln \Omega(r))}{2\Omega(r)}$$

E. Efrati et al., Physica D **235**, 29 (2007)

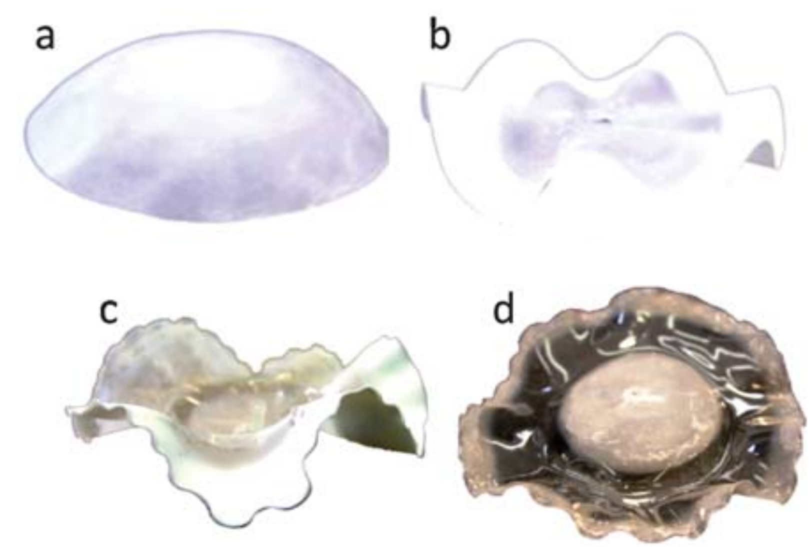
# Shaping of gel membranes by differential shrinking



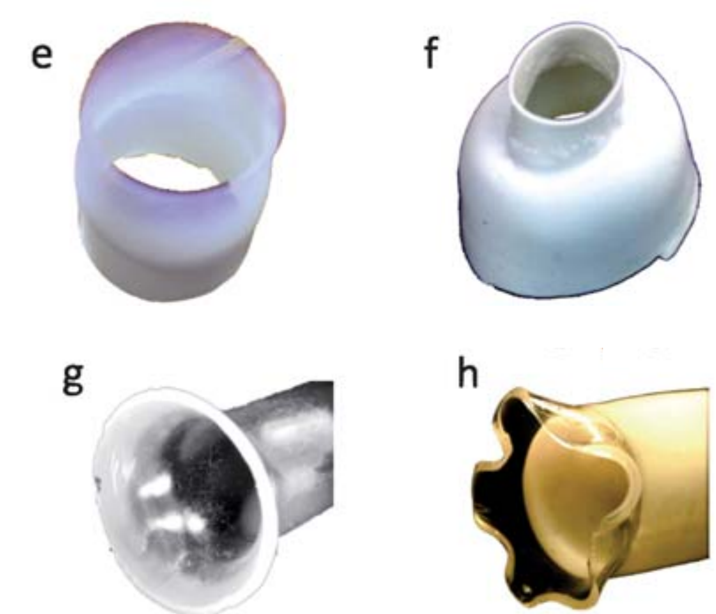

**negative Gauss curvature**

**positive Gauss curvature**

## Shrinking of sheets




## Shrinking of tubes




E. Efrati et al., Physica D 235, 29 (2007)

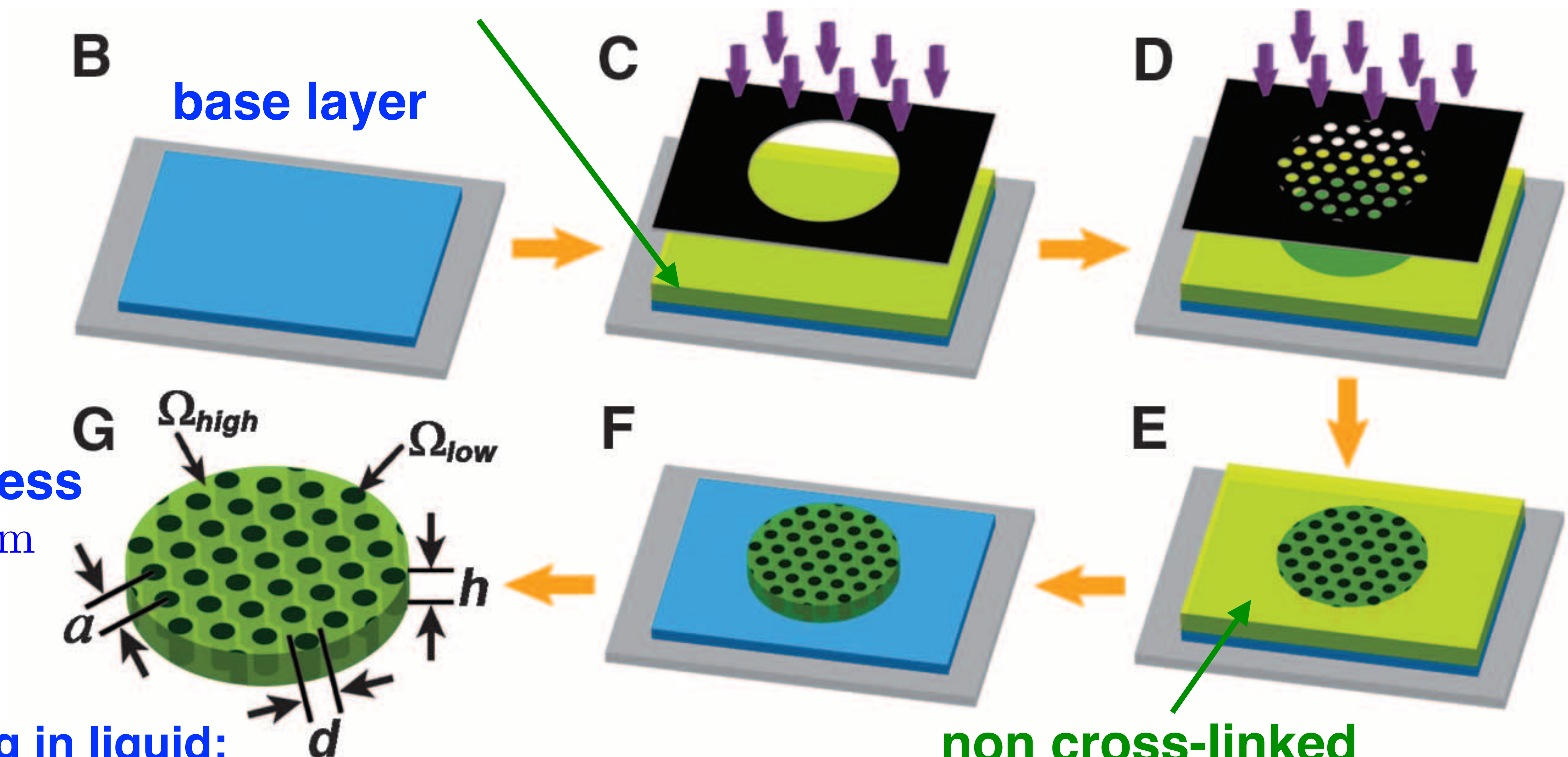


E. Sharon and E. Efrati, <u>Soft Matter</u> **6**, 5693 (2010)

# Shaping of gel membrane properties by lithography

thin film of polymer solution with premixed inactive cross-linkers

UV light activates cross-linkers. Time of UV light exposure determines the degree of polymer cross-linking.




thickness $7\text{-}17\mu\text{m}$

swelling in liquid:

more cross-linked regions swell less $\Omega_{\text{low}}$

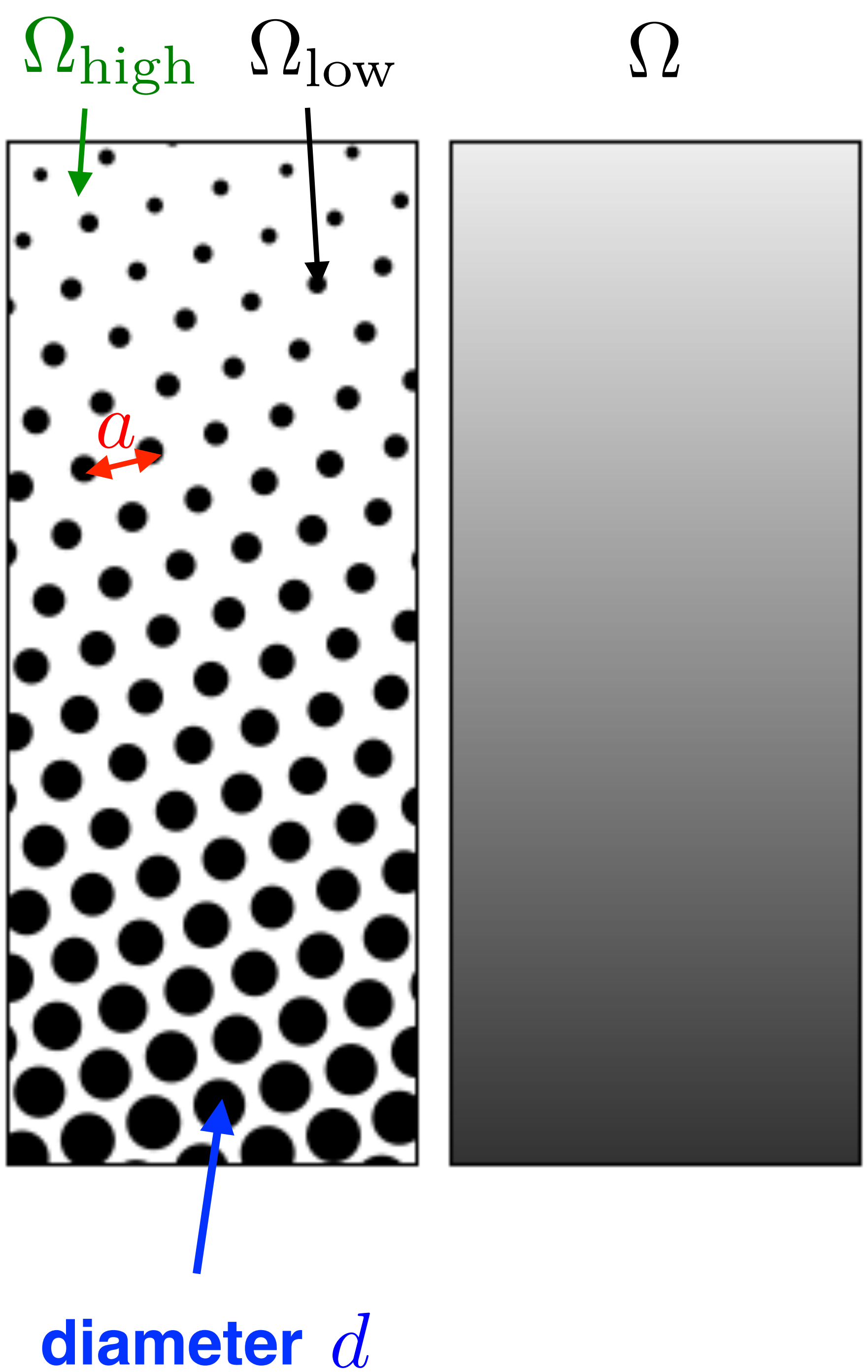
less cross-linked regions swell more $\Omega_{\text{high}}$

non cross-linked polymer solution is washed away



J. Kim et al., Science **335**, 1201 (2012)

# Halftoning

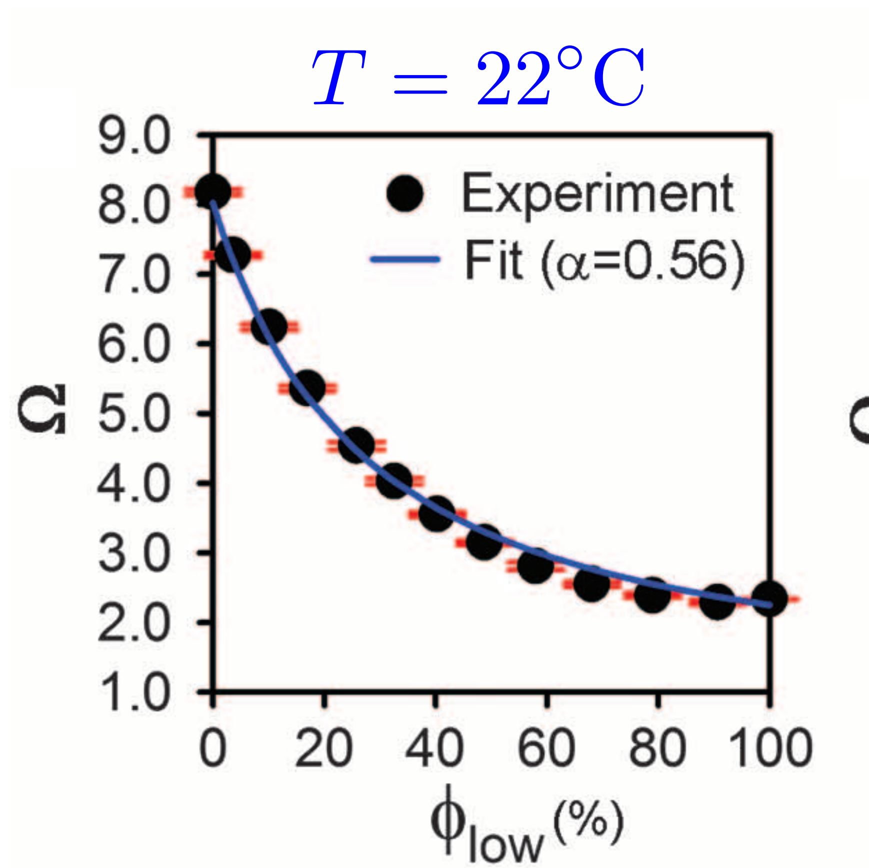



**local area fraction of the low swelling regions**

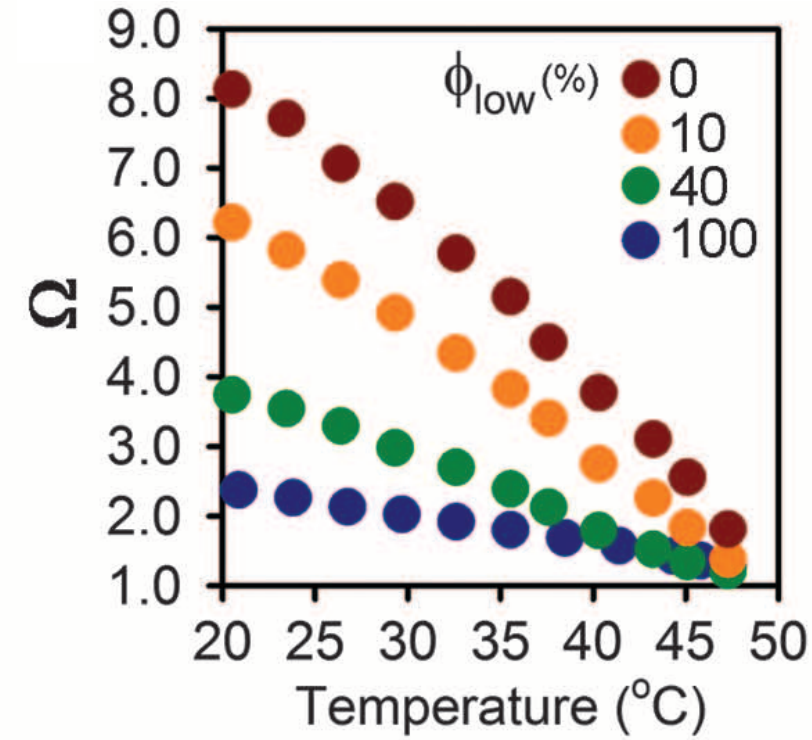
$$\phi_{\mathrm{low}} = \frac{\Delta A_{\mathrm{low}}}{\Delta A_{\mathrm{low}} + \Delta A_{\mathrm{high}}} = \frac{\pi}{2\sqrt{3}} \left(\frac{d}{a}\right)^2$$

**Effective swelling $\Omega$ can be estimated from local force balance as**

$$\frac{\phi_{\mathrm{low}} + \alpha(1-\phi_{\mathrm{low}})}{\Omega^{1/2}} = \frac{\phi_{\mathrm{low}}}{\Omega_{\mathrm{low}}^{1/2}} + \frac{\alpha(1-\phi_{\mathrm{low}})}{\Omega_{\mathrm{high}}^{1/2}}$$



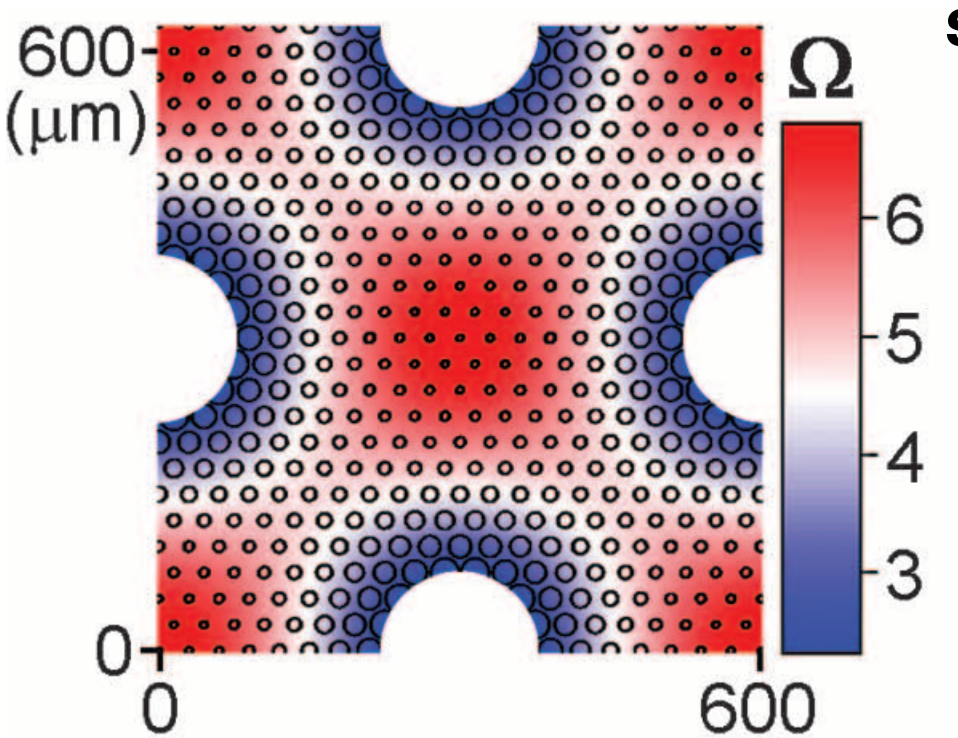

**swelling depends on *T***



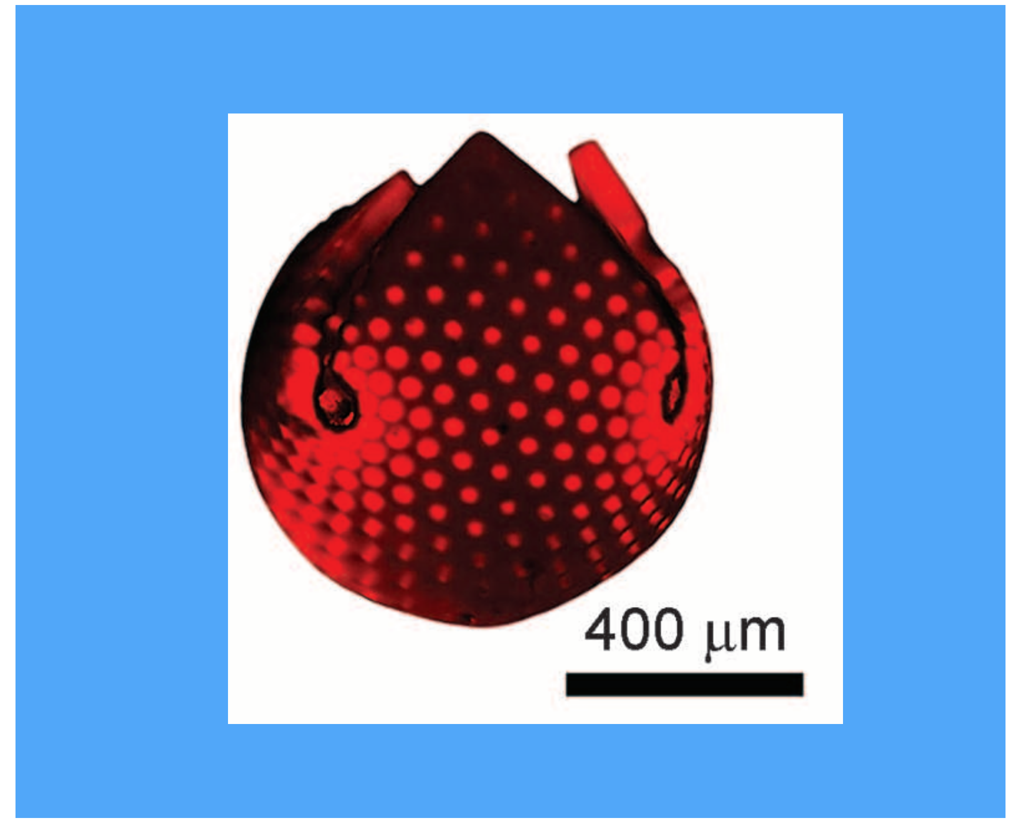


J. Kim et al., Science **335**, 1201 (2012)

# Shaping of gel membrane properties by halftone lithography




**Differential swelling in liquid deforms square membrane to a closed sphere**




**metric tensor**

$$g_{ij} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

**locally isotropic swelling**

$$g_{ij} = \begin{pmatrix} \Omega(x,y), & 0 \\ 0, & \Omega(x,y) \end{pmatrix}$$

**For thin membranes the target Gauss curvature is**

$$\det(K'_{ij}(x,y)) = -\frac{\nabla^2(\ln\Omega(x,y))}{2\Omega(x,y)}$$

**Inverse problem can be solved with conformal maps.**



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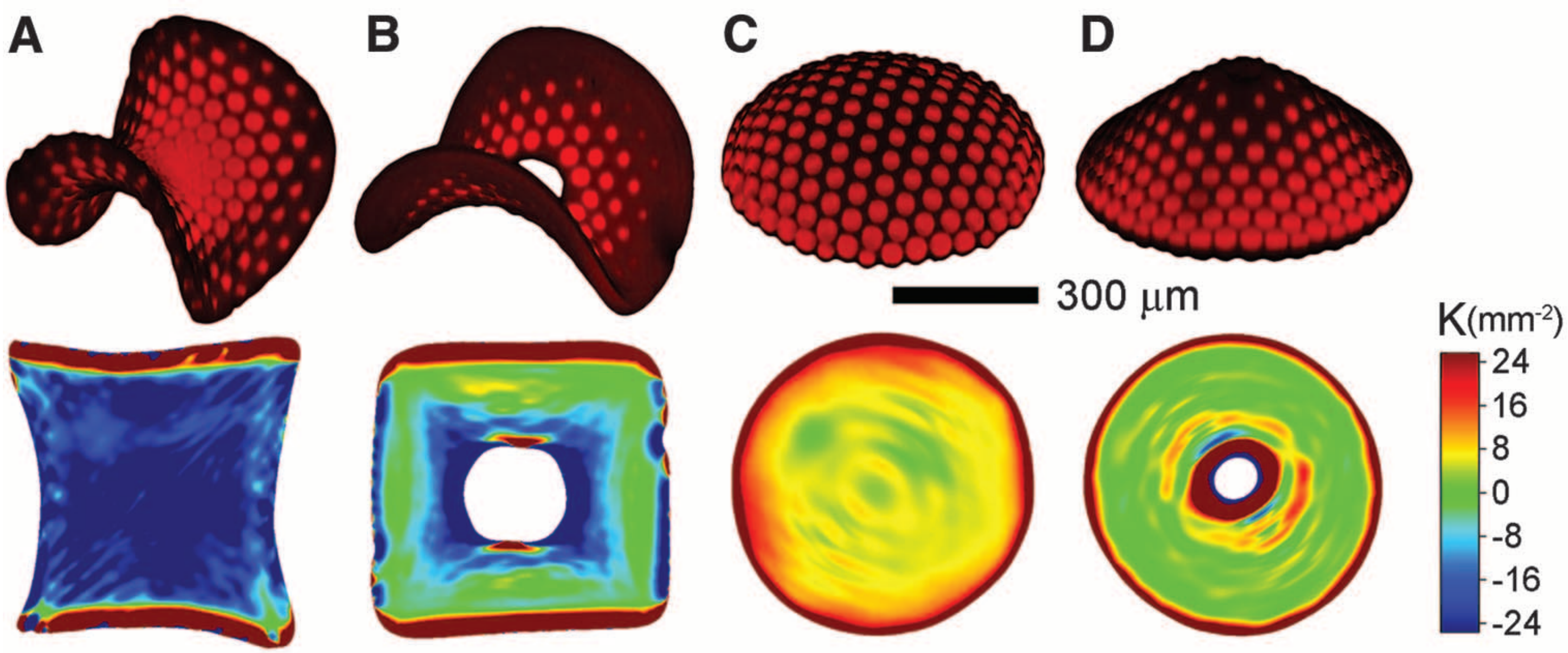
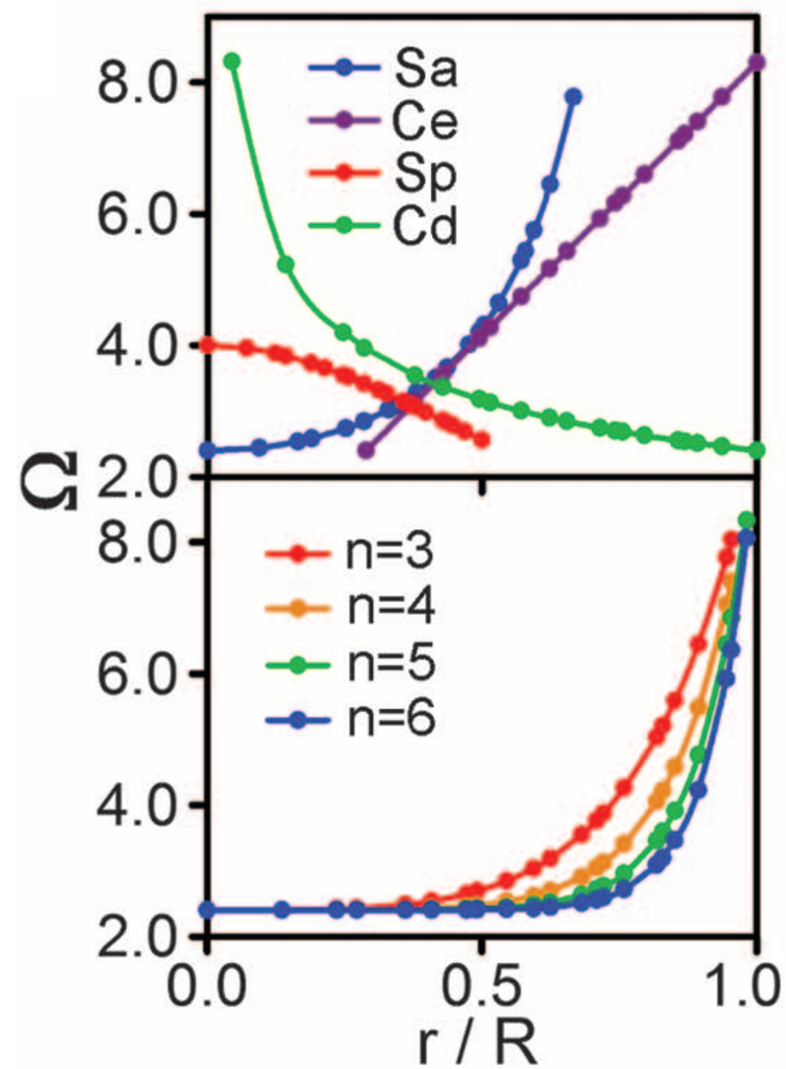
# Shaping of gel membrane properties by halftone lithography

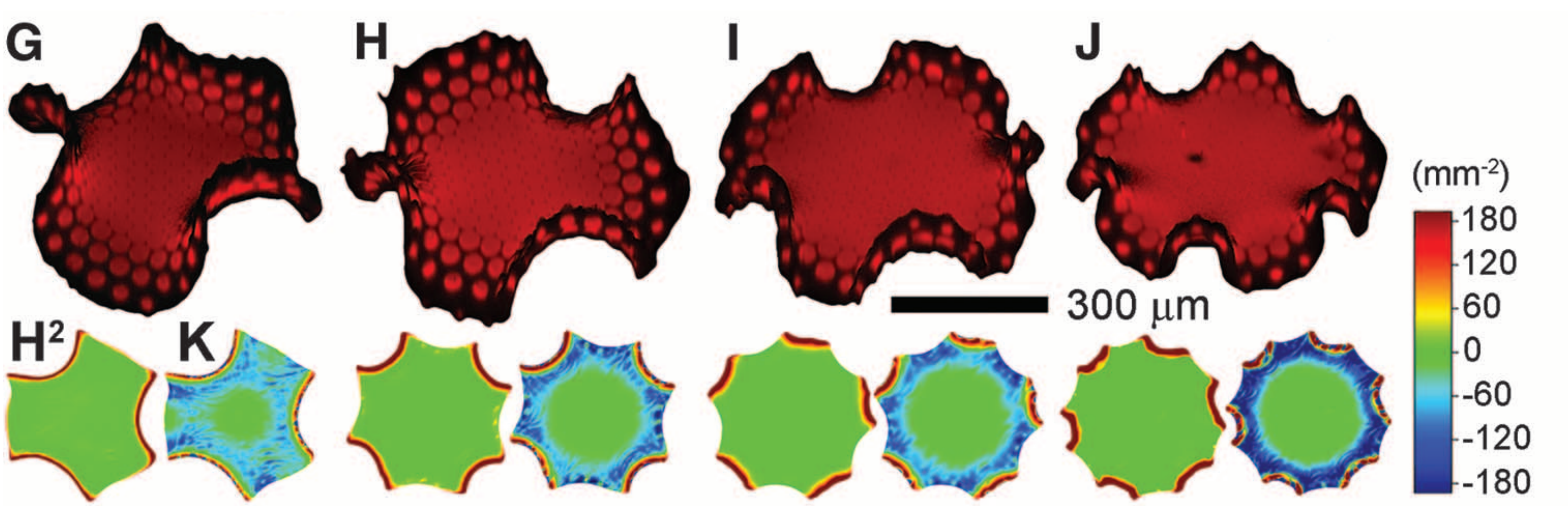
saddle (Sa) | cone with excess angle (Ce) | spherical cap (Sp) | cone with deficit angle (Cd)




swelling profiles




## Enneper's minimal surfaces (H=0)




**H - mean curvature**

**K - Gauss curvature**



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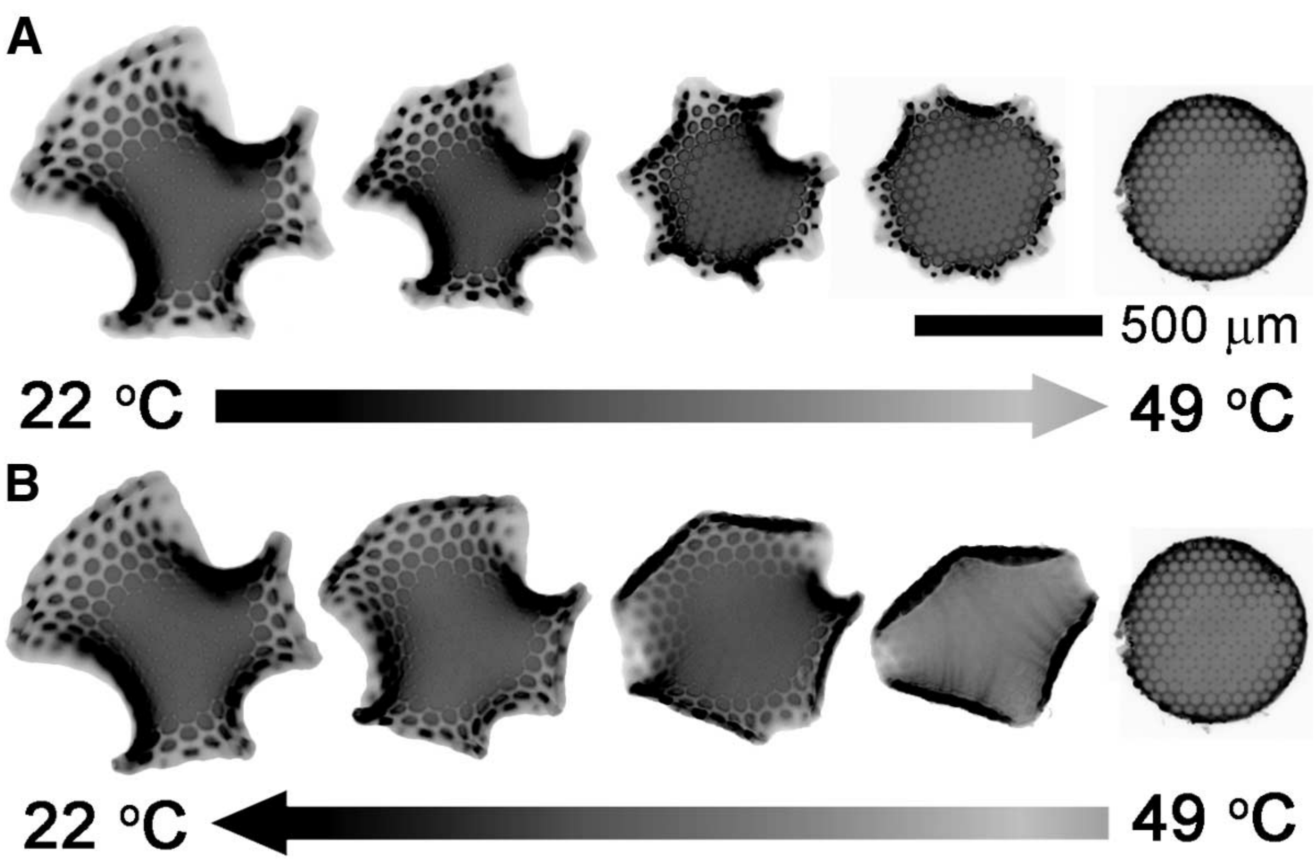
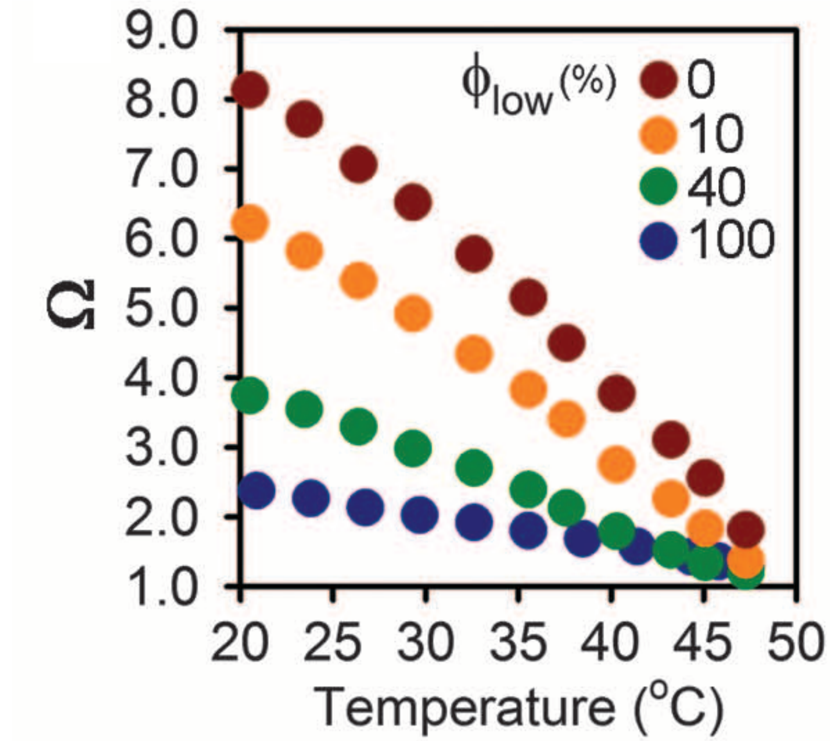
# Temperature controls swelling and thus the deformed shape




**swelling depends on *T***




**Note different intermediate shapes!**
**By slowly varying the temperature**
**we stay in a local energy minimum!**



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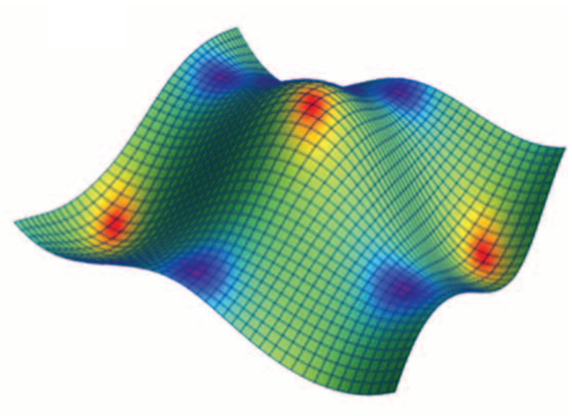
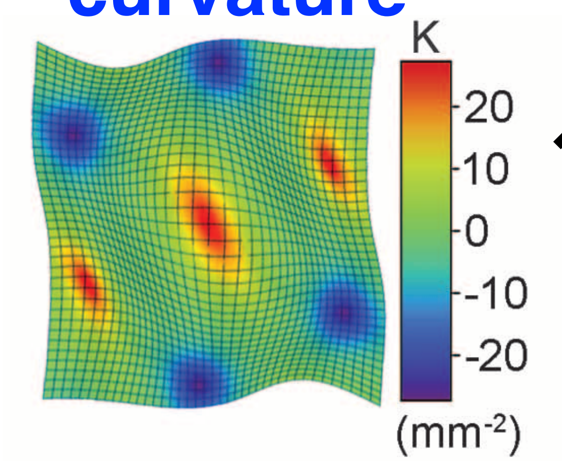
# Gaussian curvature does not uniquely specify the shape!

**target shape**

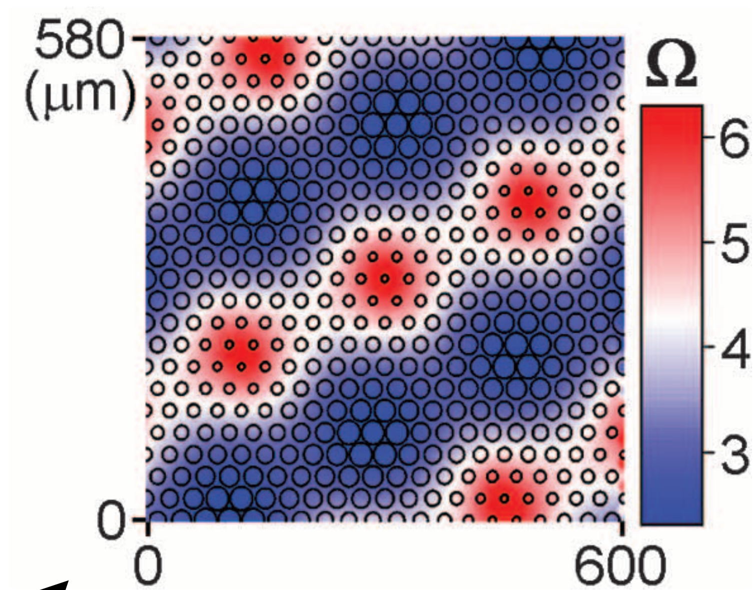


**target Gauss curvature**

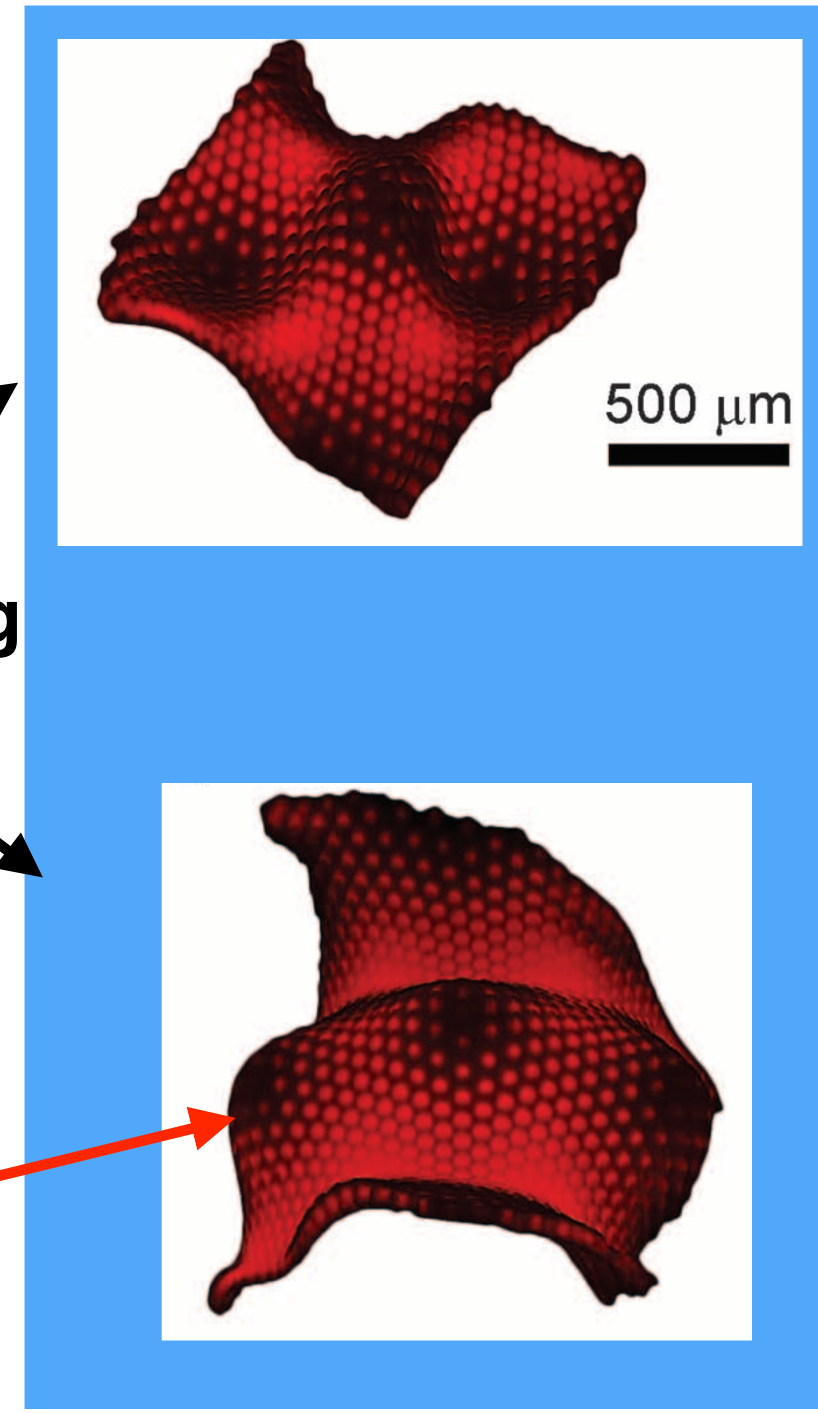



**conformal map**

**swelling pattern**




**swelling**




**this bump buckled on the wrong side**



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