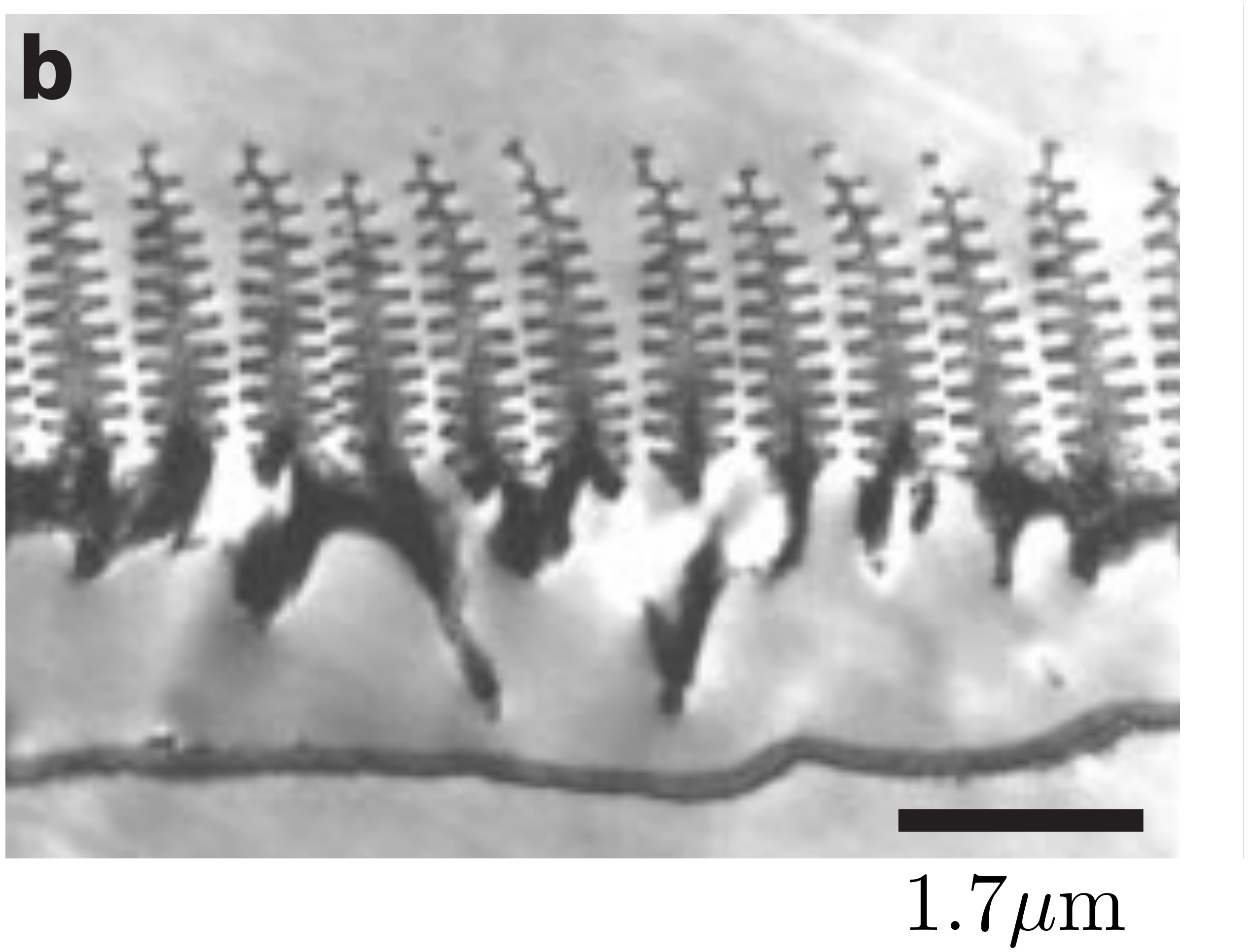
MAE 545: Lecture 2 (2/8)

Structural colors

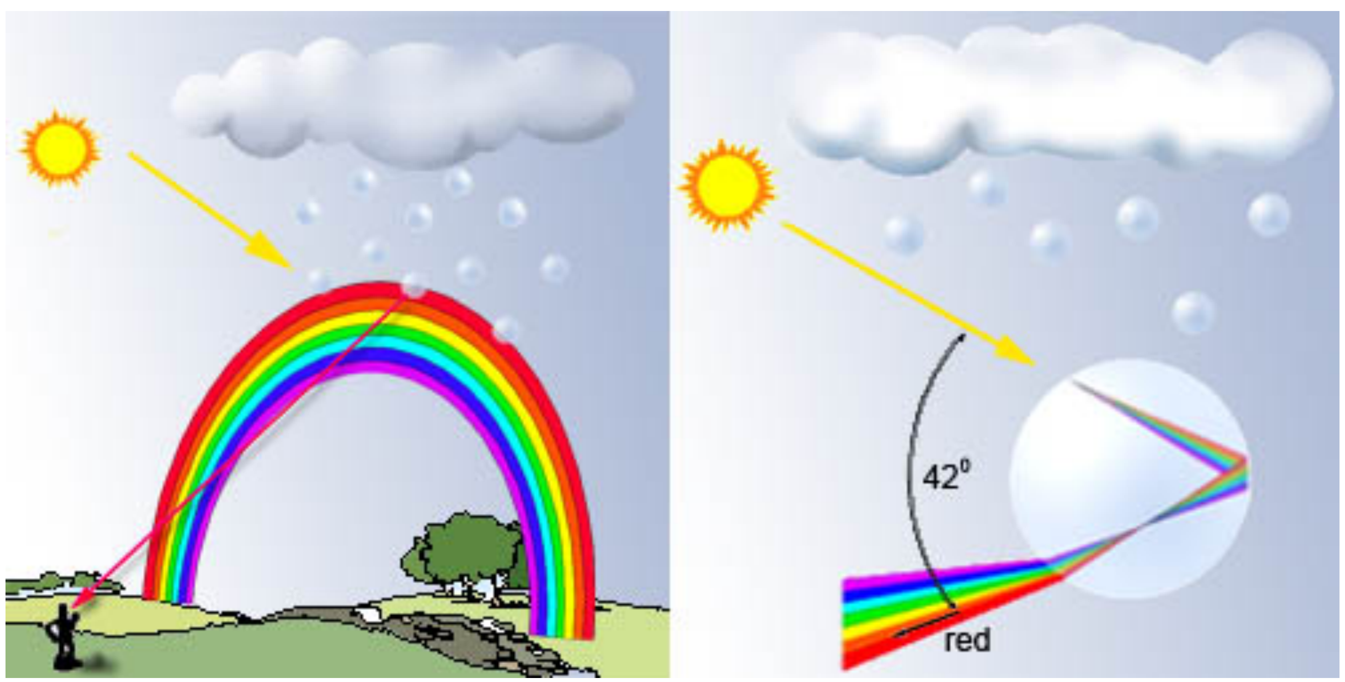


Structural color

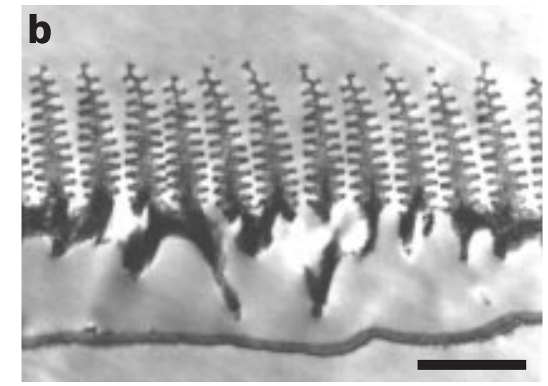
Structural colors of animals and plants appear due to the selective reflection of ambient light on structural features underneath the surface.

White light coming from the the sun consists of all colors.

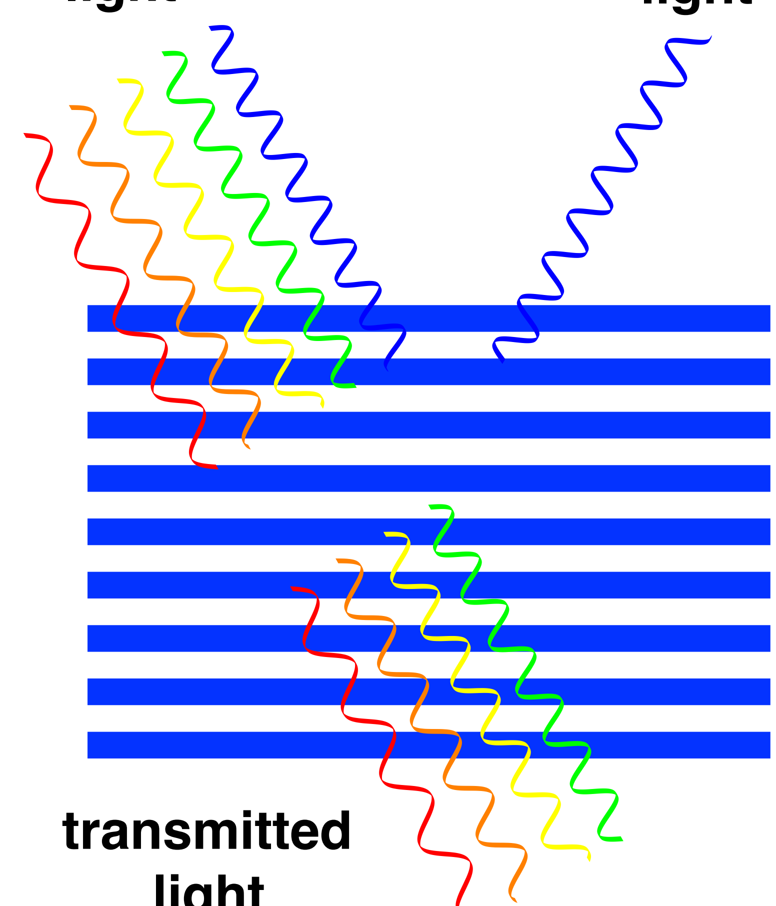
rainbow



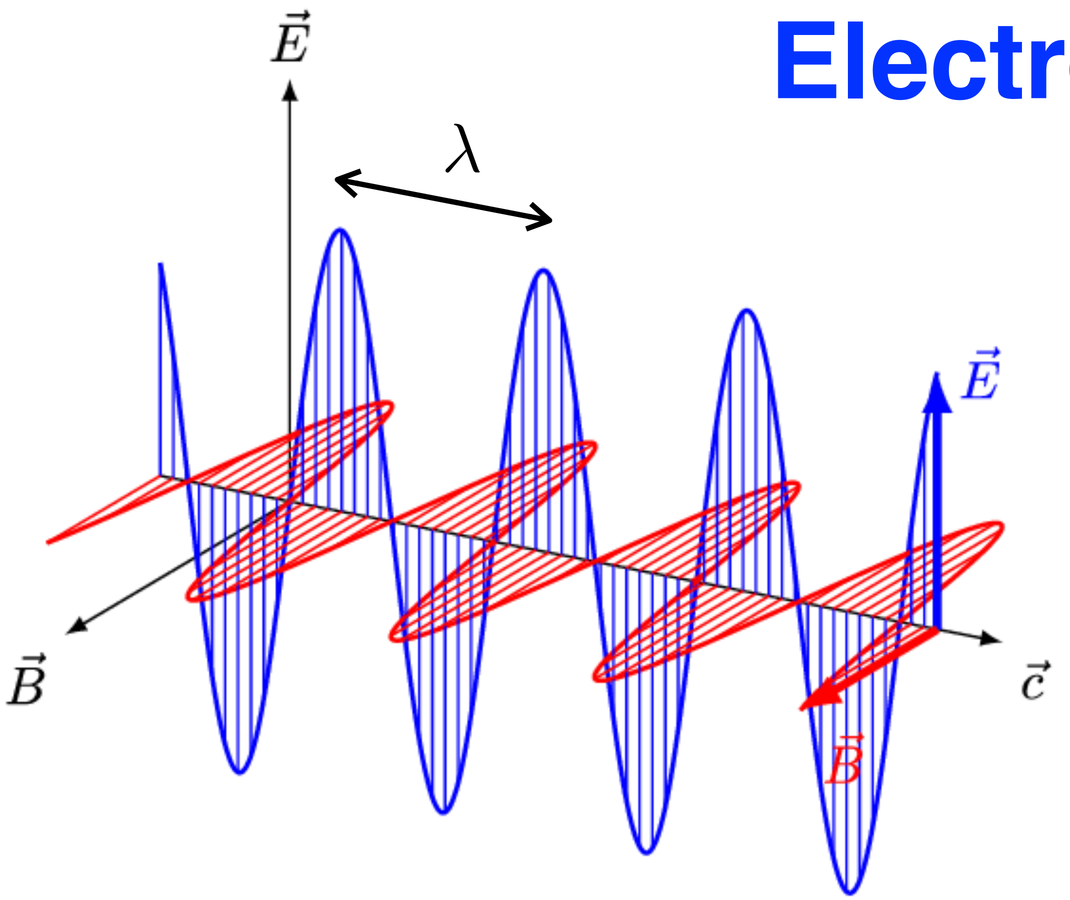
structural color



$1.7\mu m$



Electromagnetic waves



electric field

magnetic field

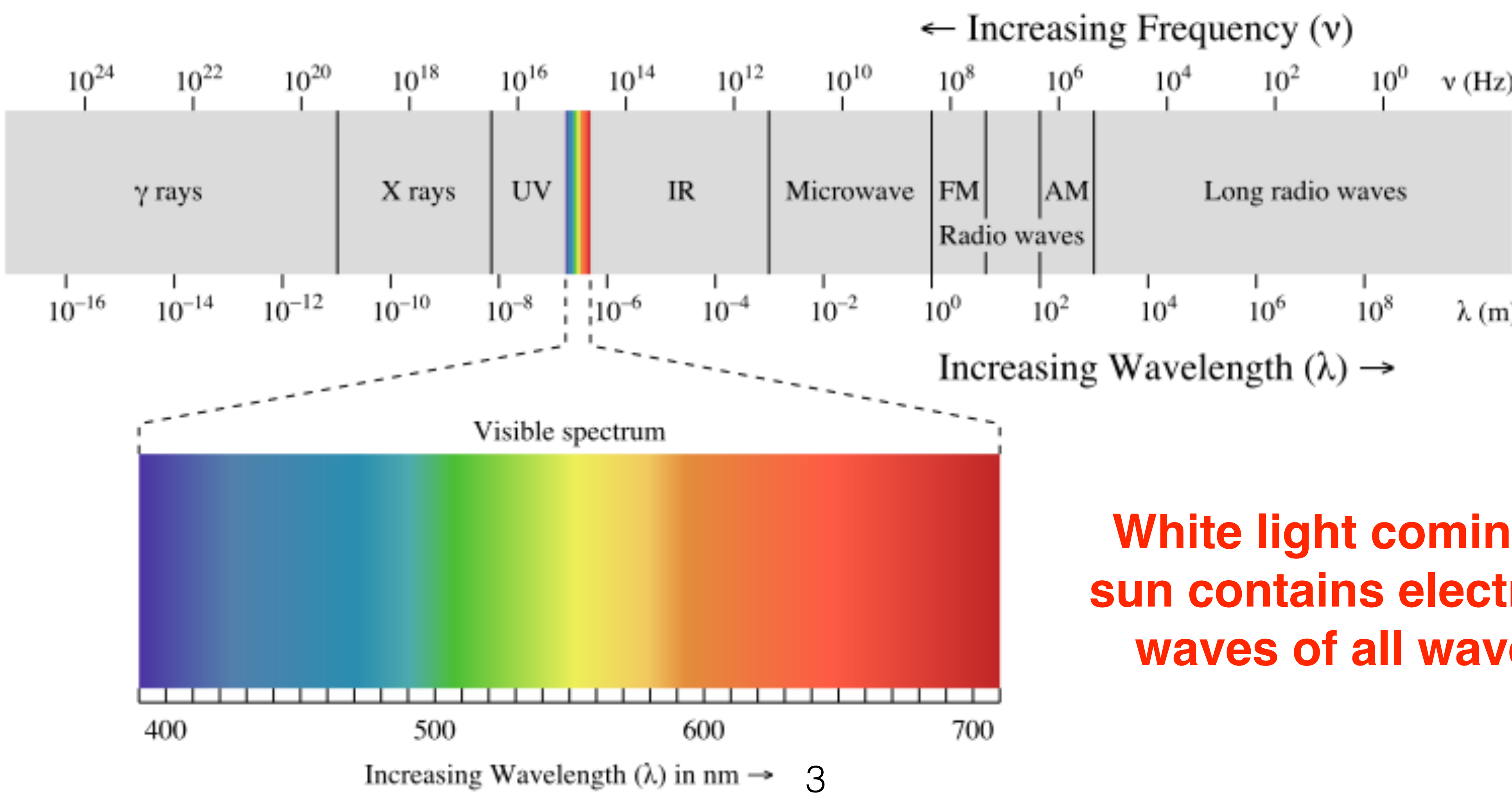
$$c^2 \vec{B}_0 = \vec{c} \times \vec{E}_0$$

speed of light

$$c_0 = \lambda \nu = 3 \times 10^8 \mathrm{m/s}$$

wavelength λ

frequency ν



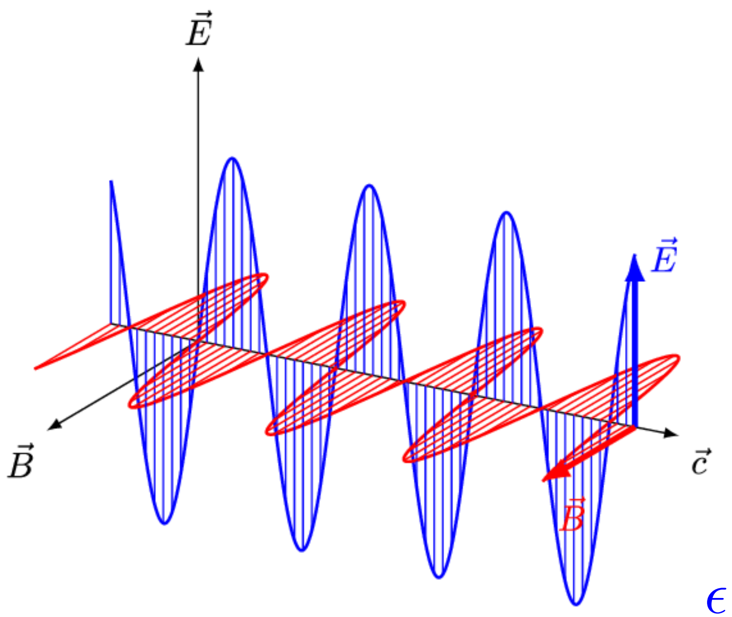
White light coming from the sun contains electromagnetic waves of all wavelengths!

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Solutions are traveling waves with velocity c.

electromagnetic waves

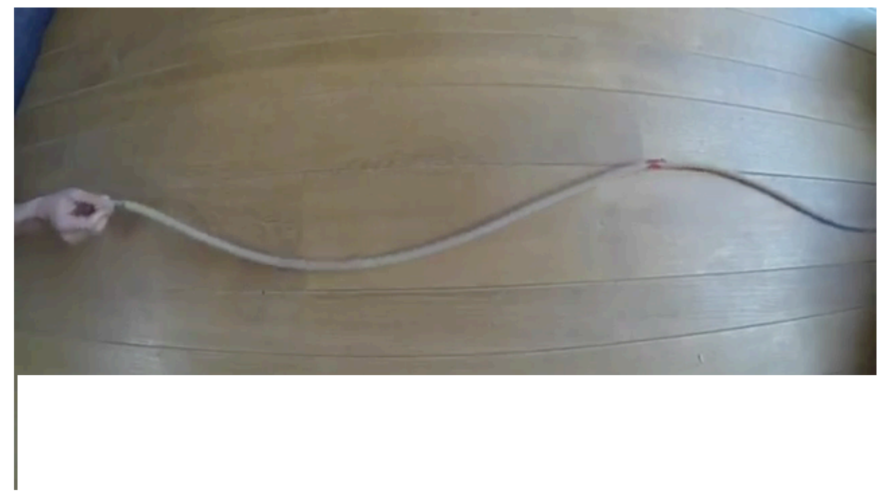


$$c = \frac{1}{\sqrt{\epsilon\mu}}$$

ϵ **permittivity**

μ **permeability**

waves in ropes under tension



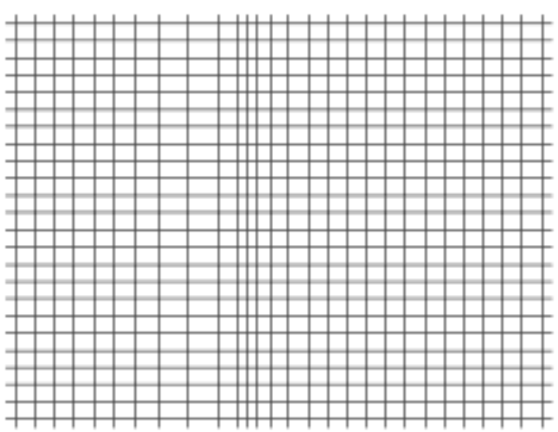
$$c = \sqrt{\frac{F}{\rho A}}$$

F **tensile force**

ρ **mass density**

A **cross-section area**

sound waves

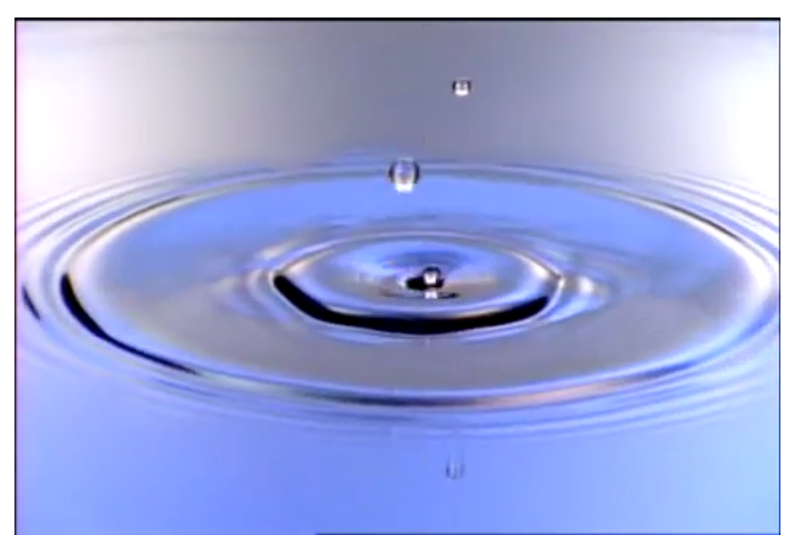


$$c = \sqrt{\frac{K}{\rho}}$$

K **bulk modulus**

ρ **mass density**

waves on liquid surfaces



shallow water

$$c = \sqrt{gh}$$

deep water

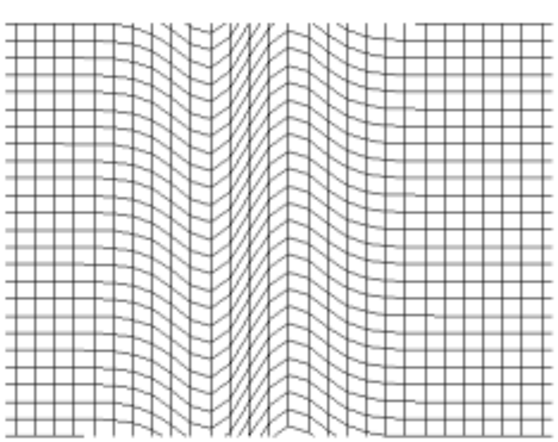
$$c = \sqrt{\frac{g\lambda}{2\pi}}$$

g **gravitational const.**

h **water depth**

λ **wavelength**

shear waves

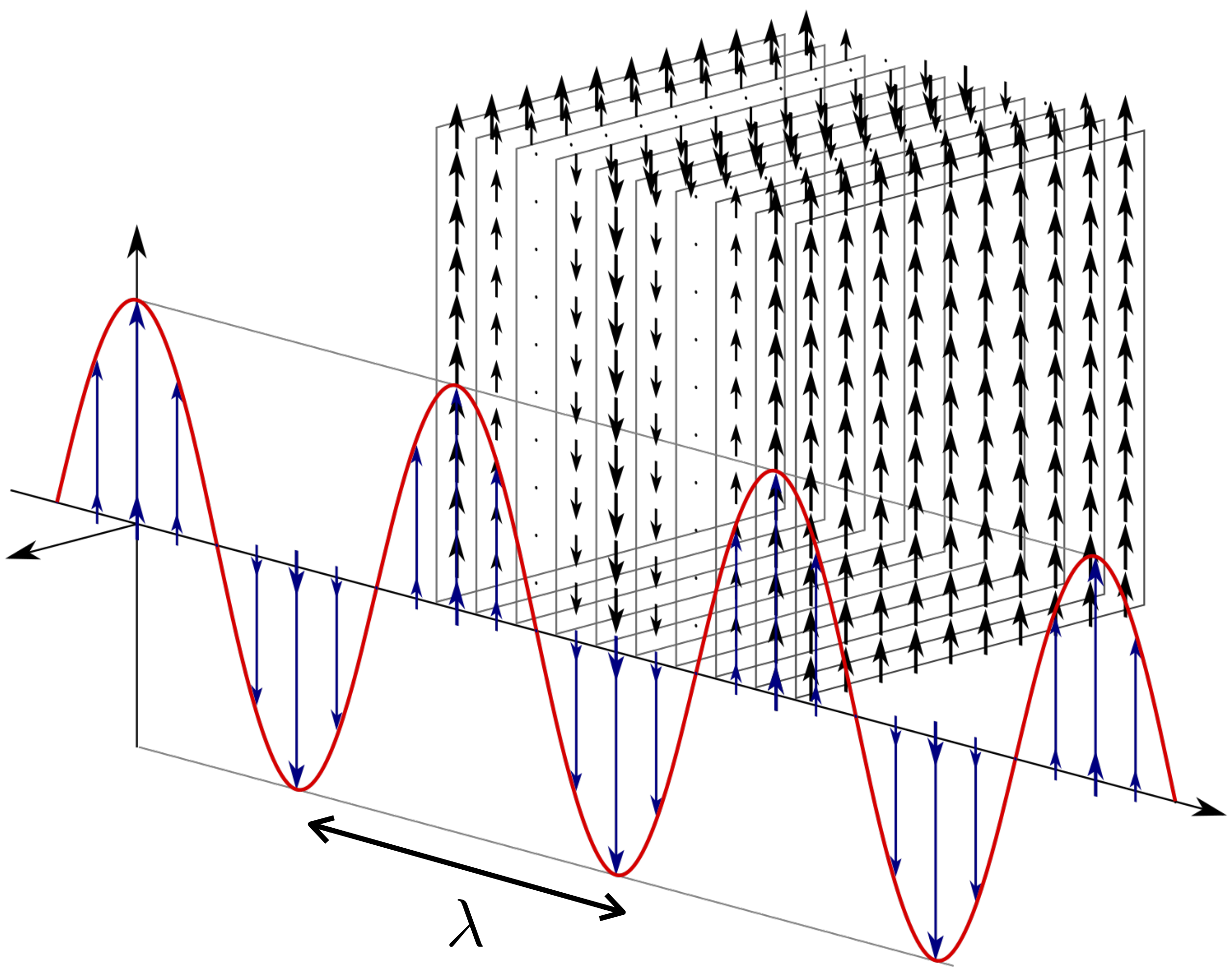


$$c = \sqrt{\frac{\mu}{\rho}}$$

μ **shear modulus**

ρ **mass density**

Plane waves



Solutions of wave equation can be described as a linear superposition of plane waves:

$$u(x,t) = \sum_{\vec{k}} A_{\vec{k}} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$k = \dfrac{2\pi}{\lambda}$ **wavevector**

$\omega = 2\pi\nu$ **angular frequency**

Plane waves travel in direction of $\vec{k}$ with velocity:

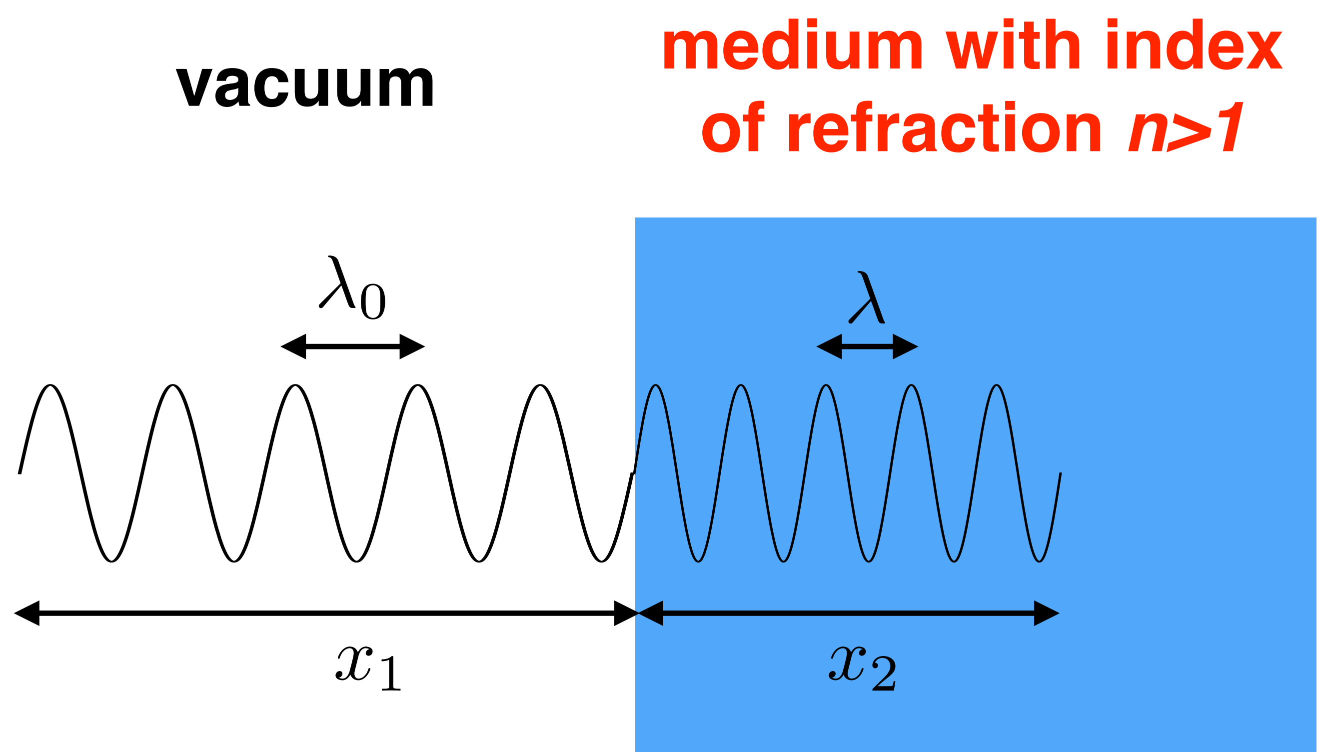
$$c = \frac{\omega}{k} = \lambda\nu$$

Planes of constant phases:

$$\vec{k}\cdot\vec{r} = \mathrm{const}$$

Note: velocity of plane waves may depend on the wavevector $c(\vec{k})$!

Propagation of light in medium



speed of light	$c_0 = 3 \times 10^8 \mathrm{m/s}$	$c = c_0/n$
frequency	ν_0	$\nu = \nu_0$
wavelength	λ_0	$\lambda = \lambda_0/n$
	$c_0 = \nu_0 \lambda_0$	$c = \nu\lambda$

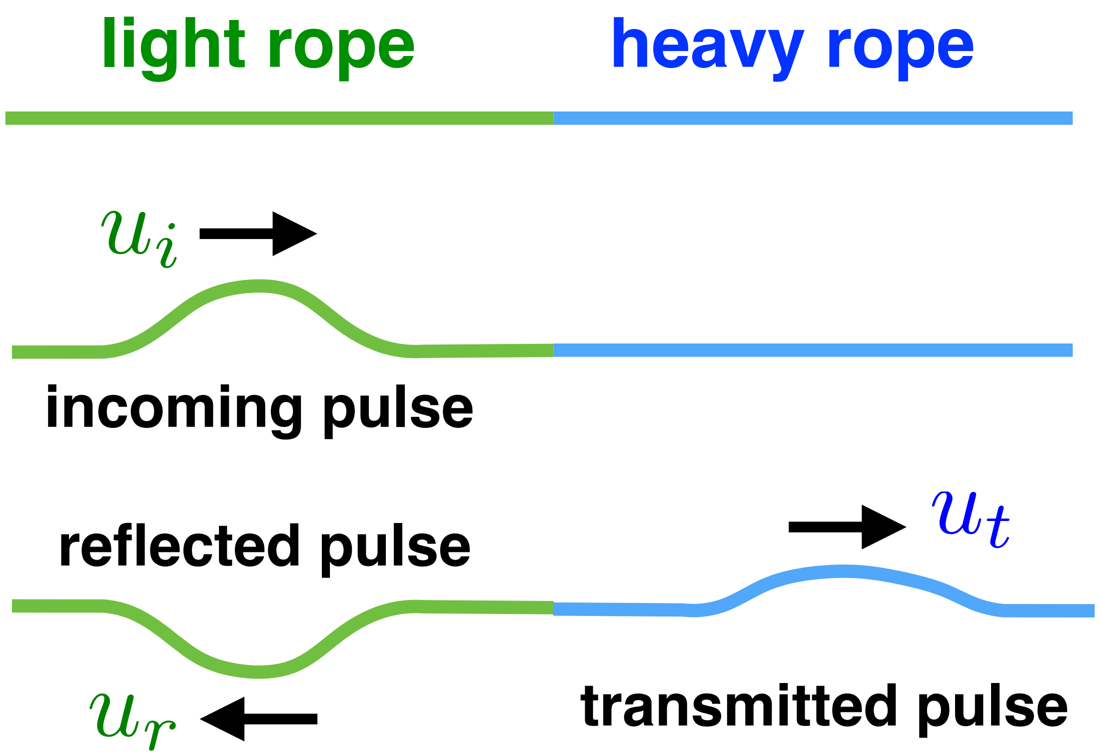
---

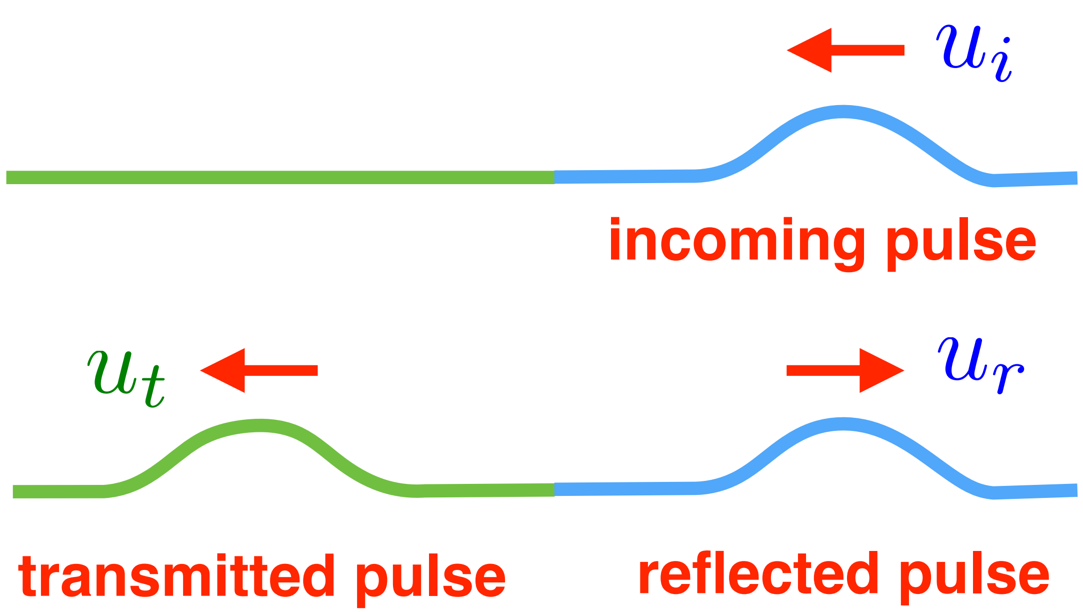
total number of cycles

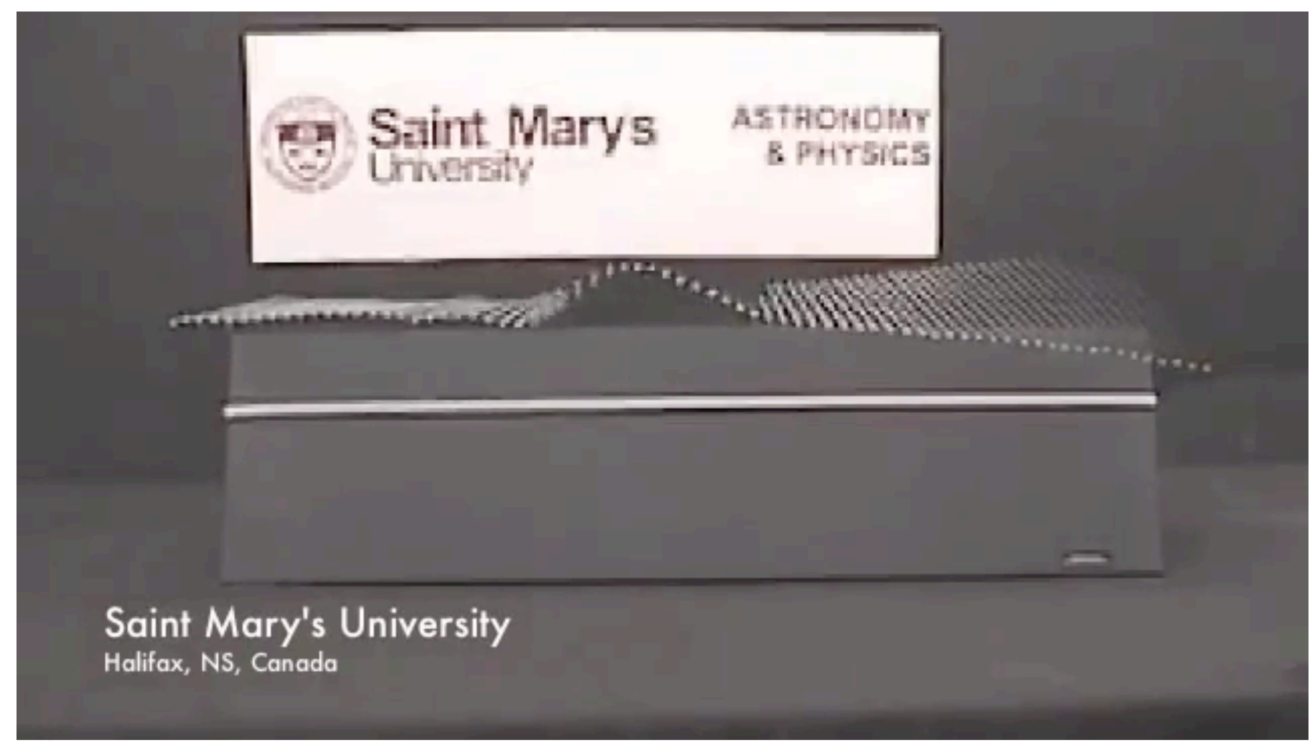
$$\frac{x_1}{\lambda_0} + \frac{x_2}{\lambda} = \frac{x_1 + n x_2}{\lambda_0}$$

Optical path length is geometric distance multiplied by the index of refraction!

Reflection of waves







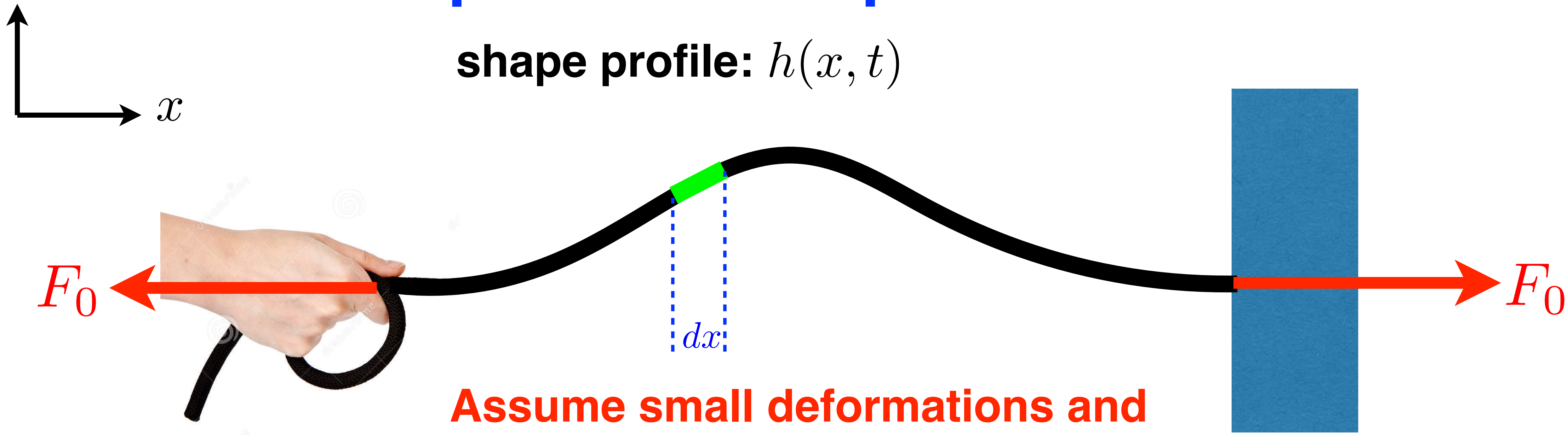
amplitude of reflected pulse

$$\frac{u_r}{u_i} = \frac{c_2 - c_1}{c_1 + c_2}$$

amplitude of transmitted pulse

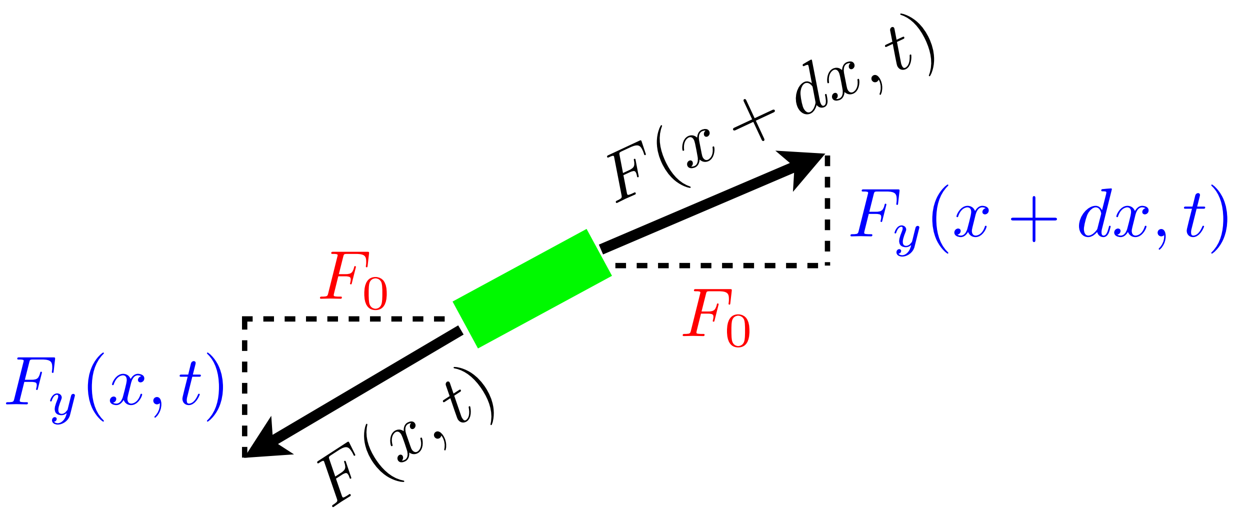
$$\frac{u_t}{u_i} = \frac{2c_2}{c_1 + c_2}$$

Wave equation for rope under tension



Assume small deformations and ignore movement in *x* direction!

Forces acting on a small rope element:



Forces act only in direction of the rope:

$$\frac{F_y(x,t)}{F_0} = \frac{\partial h(x,t)}{\partial x}$$

Second Newton's law for a small rope element:

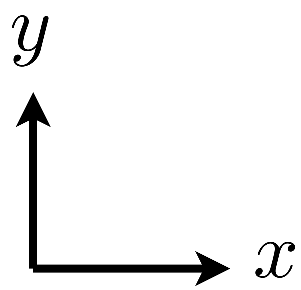
$$\rho A dx \frac{\partial^2 h}{\partial t^2} = F_y(x+dx,t) - F_y(x,t)$$

$$\rho A \frac{\partial^2 h}{\partial t^2} = \frac{\partial F_y}{\partial x} = F_0 \frac{\partial^2 h}{\partial x^2}$$

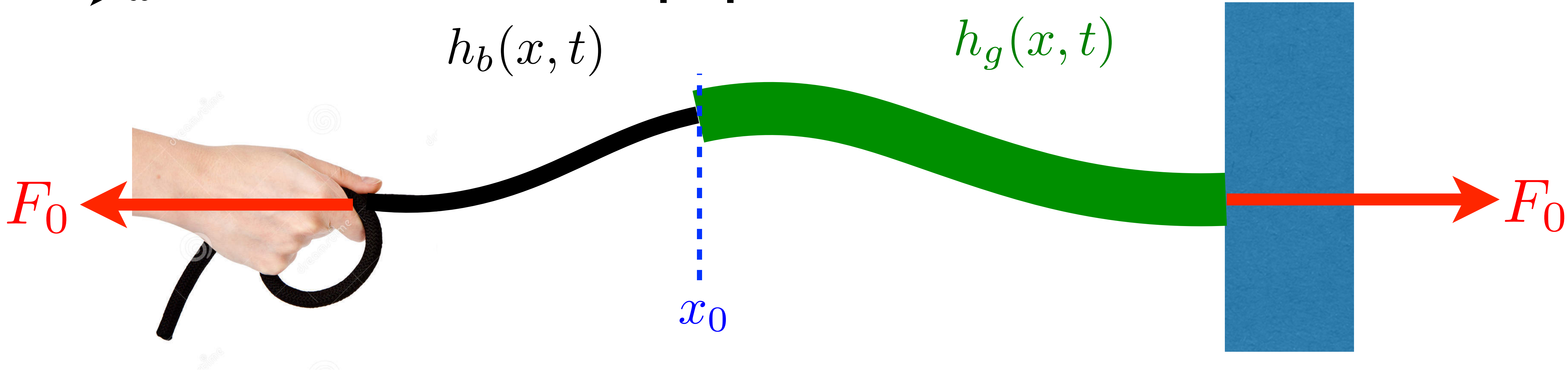
Wave equation:

$$\frac{\partial^2 h}{\partial t^2} = \frac{F_0}{\rho A} \frac{\partial^2 h}{\partial x^2} \equiv c^2 \frac{\partial^2 h}{\partial x^2}$$

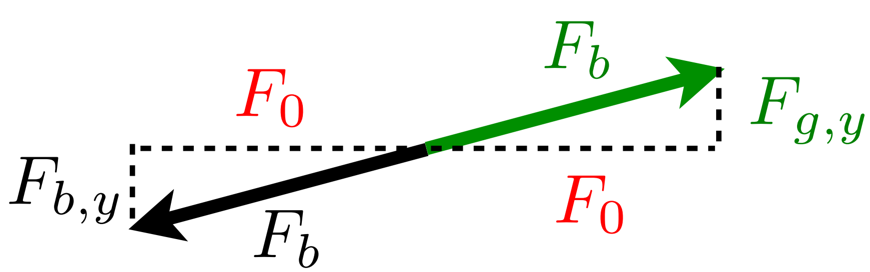
Boundary conditions between connected ropes



shape profiles:



Forces acting on the massless point, where ropes are connected:



Newton's law for this massless point:

$$F_{g,y} - F_{b,y} = ma = 0$$

→

Continuity: ropes are connected

$$h_b(x_0,t) = h_g(x_0,t)$$

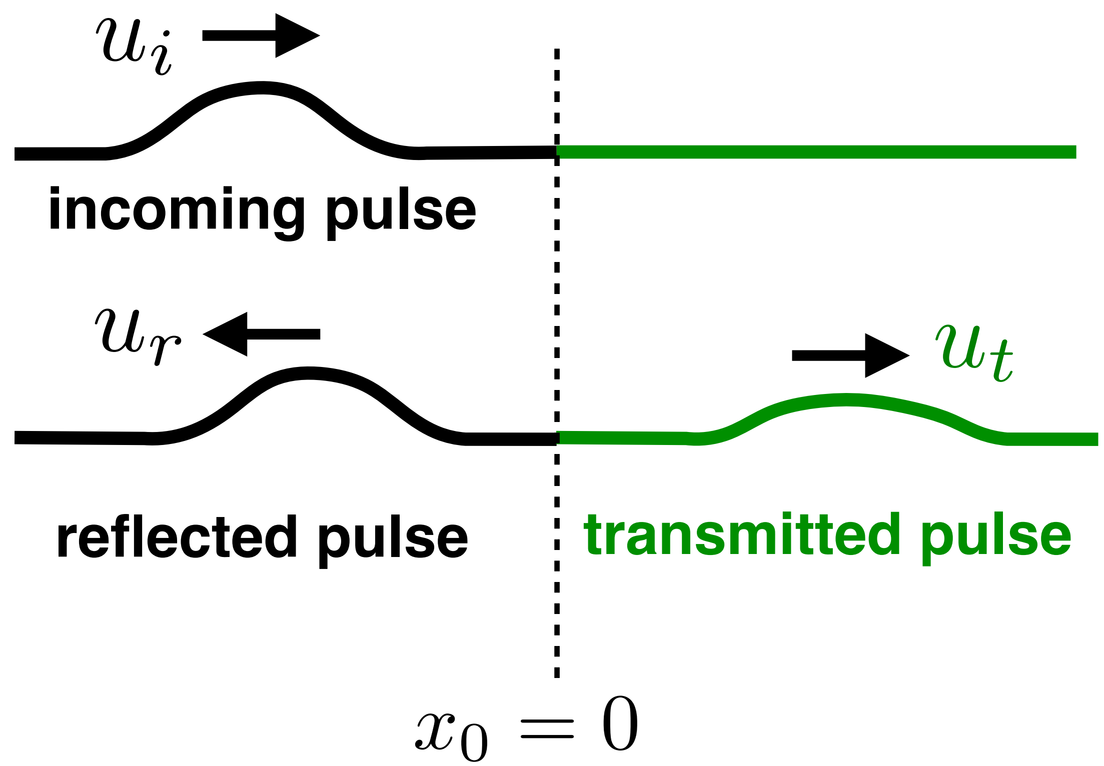
Force balance:

$$\frac{\partial h_b}{\partial x}(x_0,t) = \frac{\partial h_g}{\partial x}(x_0,t)$$

Reflection of waves on ropes

wave speed in black rope

$$c_1 = \frac{\omega}{k_1}$$



wave speed in green rope

$$c_2 = \frac{\omega}{k_2}$$

Solutions of wave equations can be expanded in Fourier series:

incoming pulse, reflected pulse

$$u_b(x,t) = \sum_{\omega} \left(A_\omega e^{i(k_1 x - \omega t)} + B_\omega e^{i(-k_1 x - \omega t)} \right)$$

transmitted pulse

$$u_g(x,t) = \sum_{\omega} \left(C_\omega e^{i(k_2 x - \omega t)} \right)$$

boundary conditions:

$$u_b(0,t) = u_g(0,t)$$

$$\frac{\partial u_b}{\partial x}(0,t) = \frac{\partial u_g}{\partial x}(0,t)$$

$$\rightarrow$$

$$A_\omega + B_\omega = C_\omega$$

$$ik_1(A_\omega - B_\omega) = ik_2 C_\omega$$

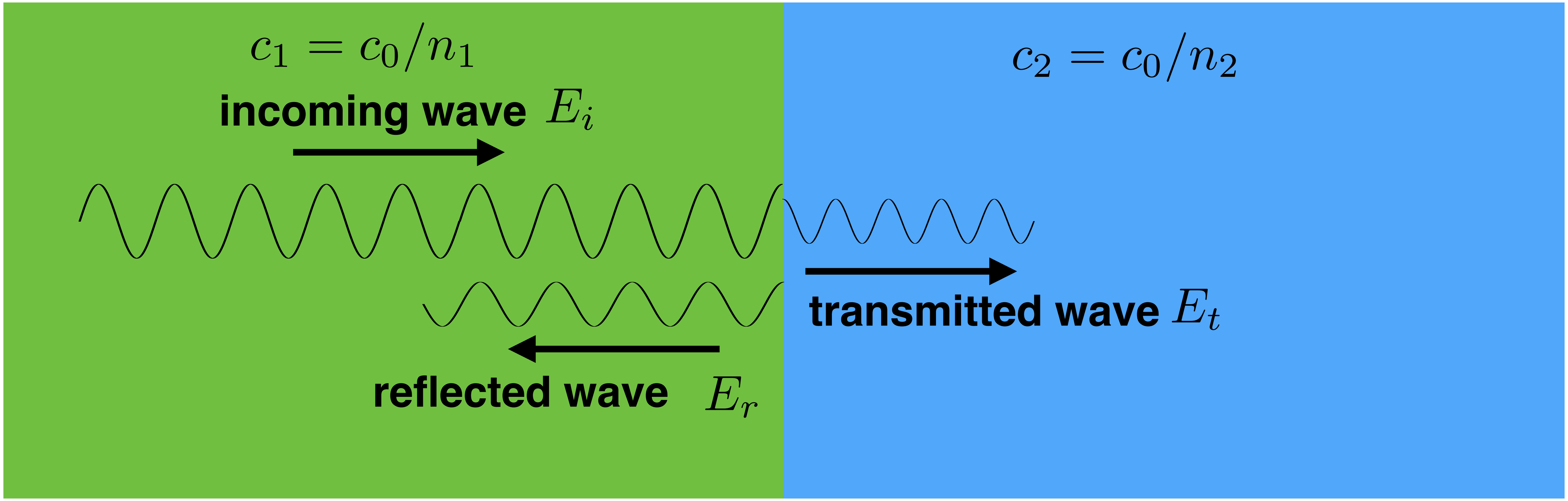
$$\rightarrow$$

amplitudes of reflected and transmitted waves:

$$B_\omega = A_\omega \, \frac{(c_2 - c_1)}{(c_1 + c_2)}$$

$$C_\omega = A_\omega \, \frac{2c_2}{(c_1 + c_2)}$$

Reflection of light at the interface between two media



boundary conditions for incident waves normal to the interface:

$$E_1 = E_2 \qquad H_1 = H_2 \rightarrow \frac{\partial E_1}{\partial x} = \frac{\partial E_2}{\partial x}$$

amplitude of reflected electric field

$$r \equiv \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2}$$

amplitude of transmitted electric field

$$t \equiv \frac{E_t}{E_i} = \frac{2n_1}{n_1 + n_2}$$

energy density of electromagnetic waves

$$\propto n|E|^2$$

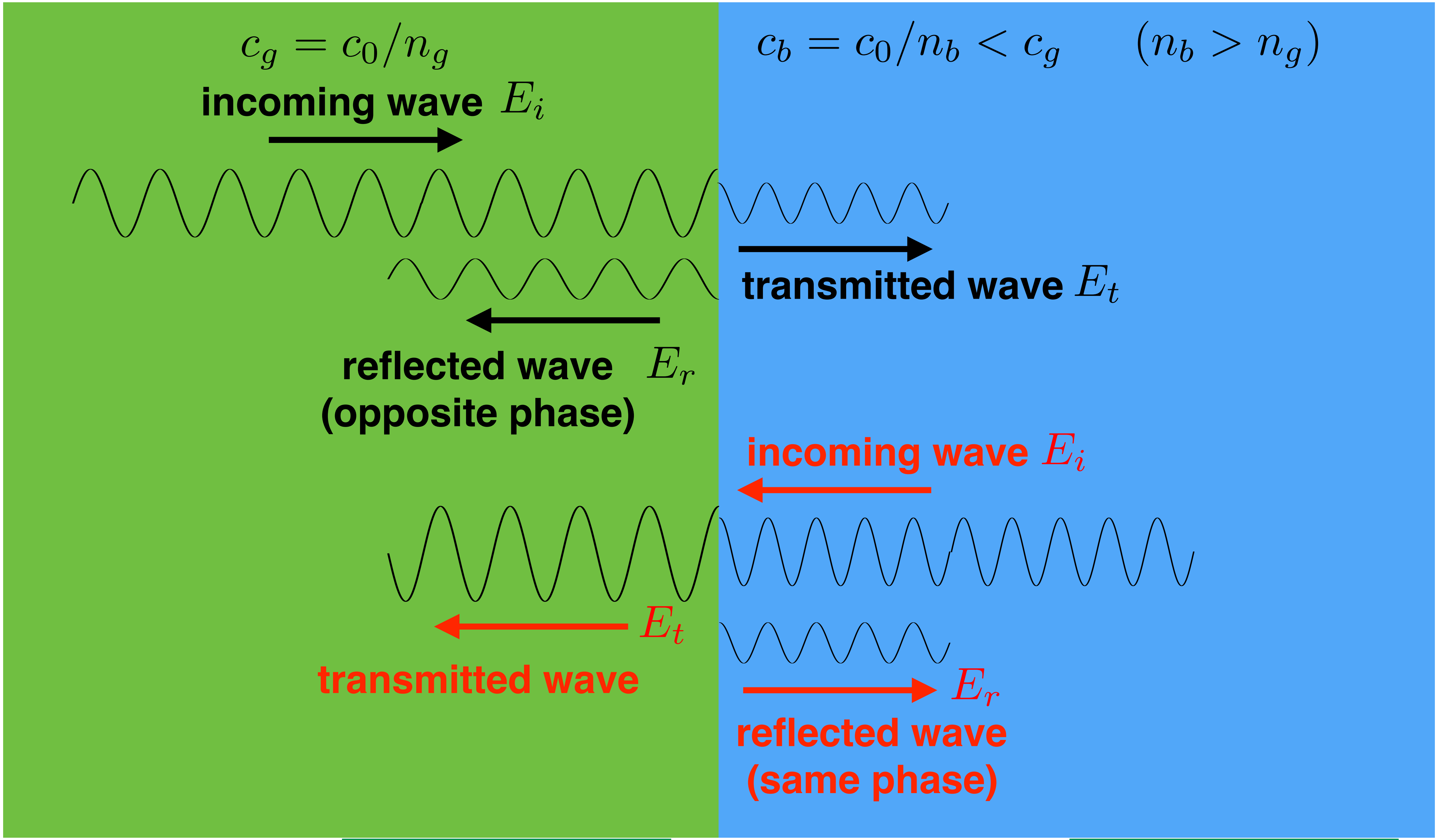
reflectance

$$R \equiv \frac{n_1|E_r|^2}{n_1|E_i|^2} = |r|^2$$

transmittance

$$T \equiv \frac{n_2|E_t|^2}{n_1|E_i|^2} = |t|^2 \frac{n_2}{n_1} = 1 - R$$

Reflection of light at the interface between two media



amplitude of reflected electric field

$$\frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2}$$

amplitude of transmitted electric field

$$\frac{E_t}{E_i} = \frac{2n_1}{n_1 + n_2}$$

Interference

constructive interference

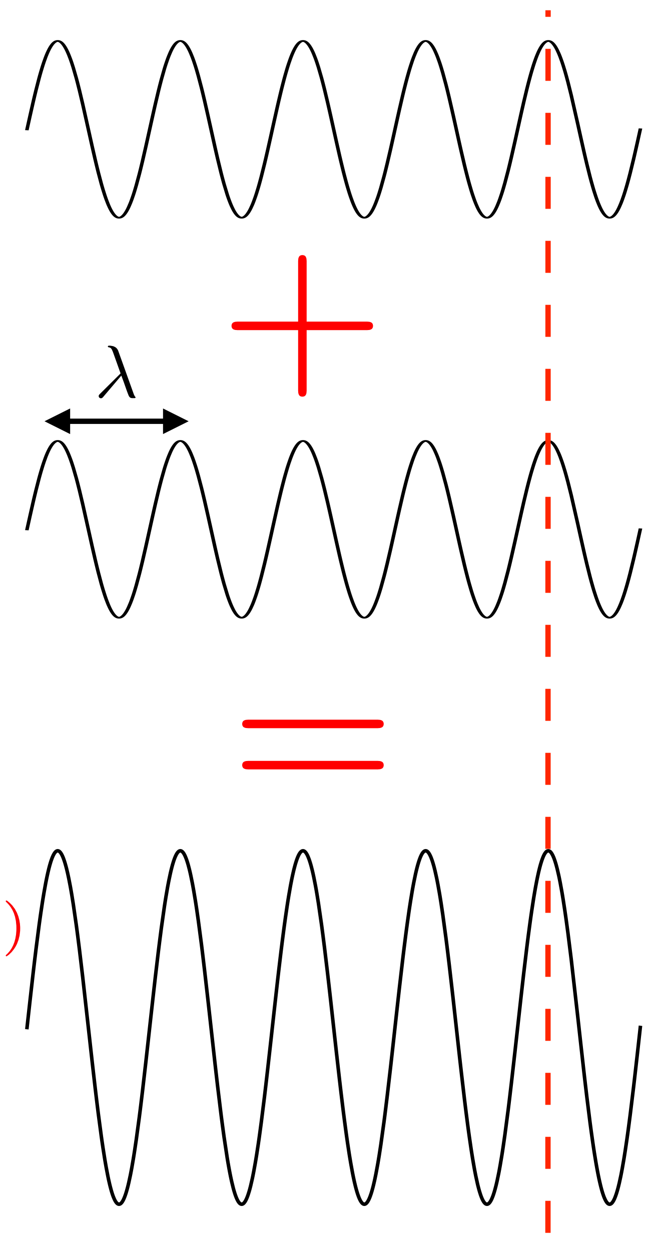
$|A|e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

+

$|B|e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

=

$(|A|+|B|)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$



Constructive interference occurs when the two waves are in phase: waves offset by $m\lambda$ **,**

$$m = 0, \pm 1, \pm 2, \ldots$$

$$e^{ikm\lambda} = e^{i2\pi m} = +1$$

destructive interference

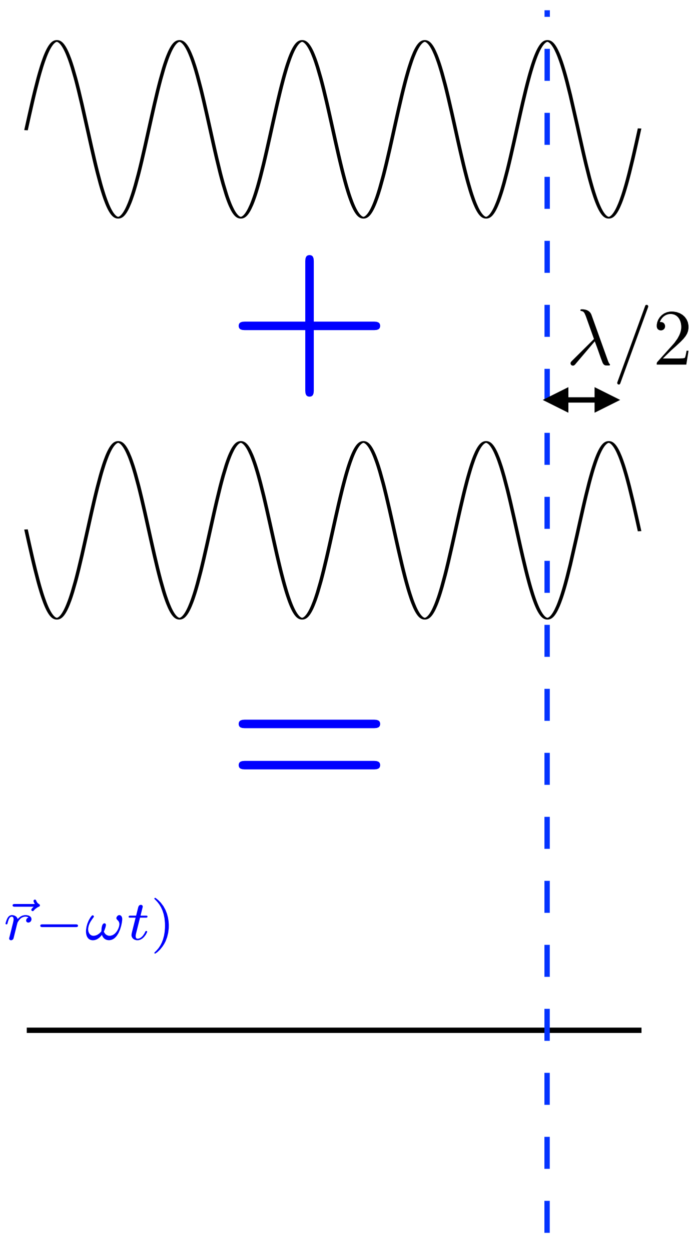
$|A|e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

+

$|B|e^{i(\vec{k}\cdot\vec{r}-\omega t+\pi)}$

=

$(|A|-|B|)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

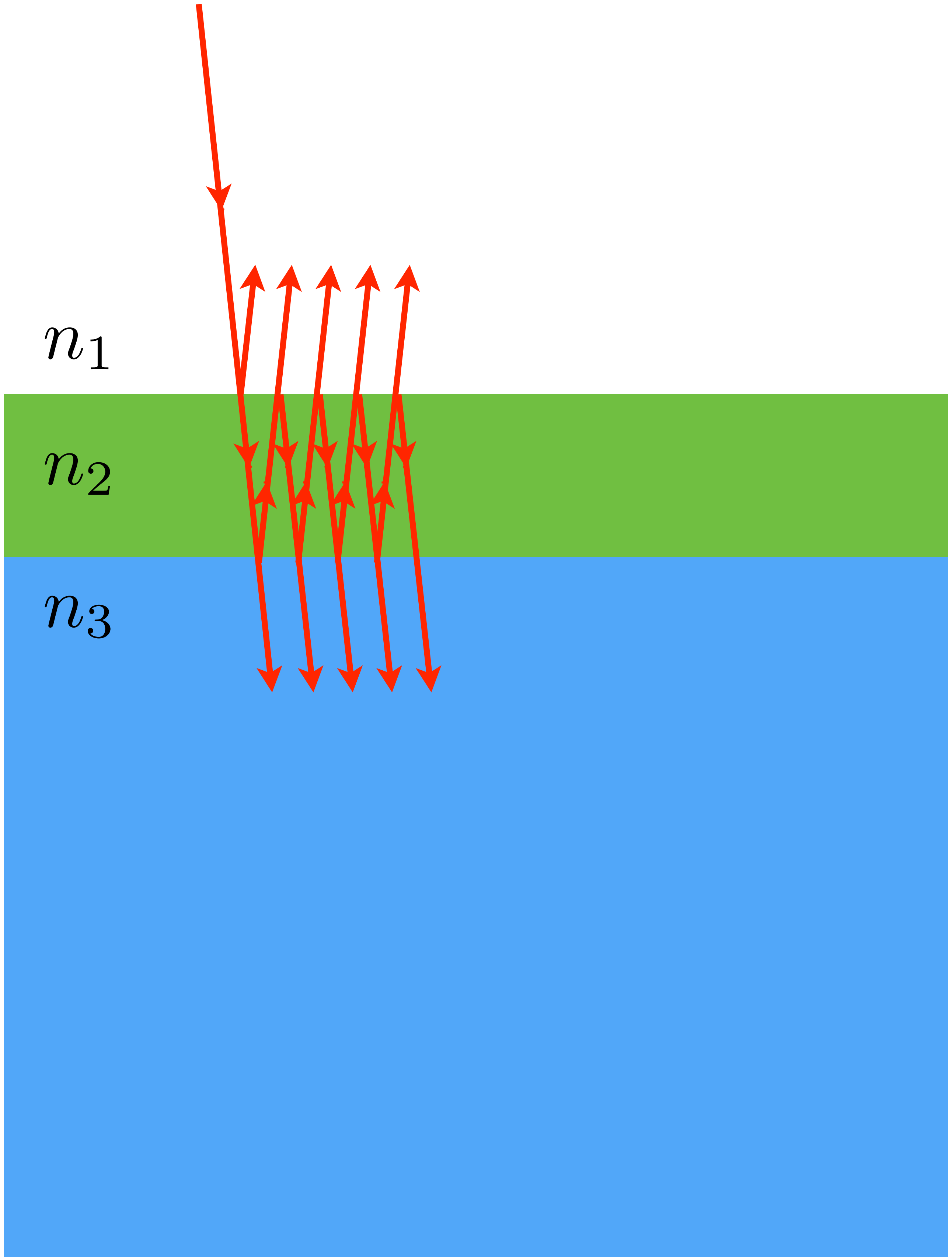


Destructive interference occurs when the two waves are out of phase: waves offset by $(m+1/2)\lambda$ **,**

$$m = 0, \pm 1, \pm 2, \ldots$$

$$e^{ik(m+1/2)\lambda} = e^{i(2\pi m+\pi)} = -1$$

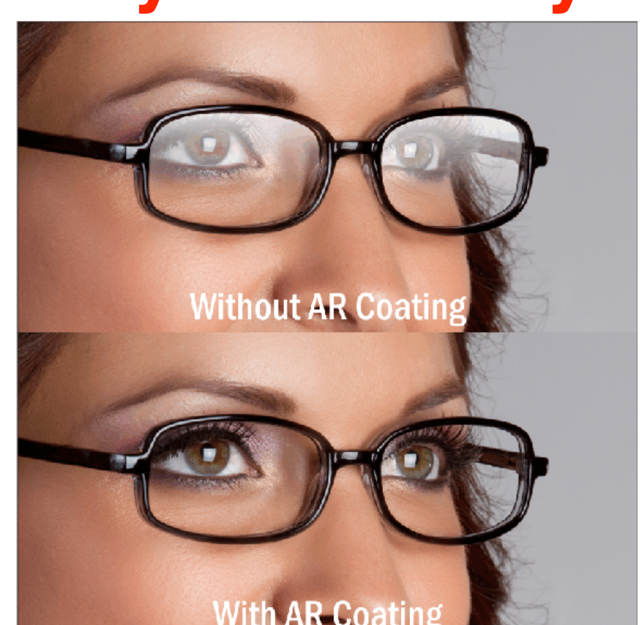
Interference on thin films



Constructive interference of reflected rays results in strongly reflected rays with very little transmission.

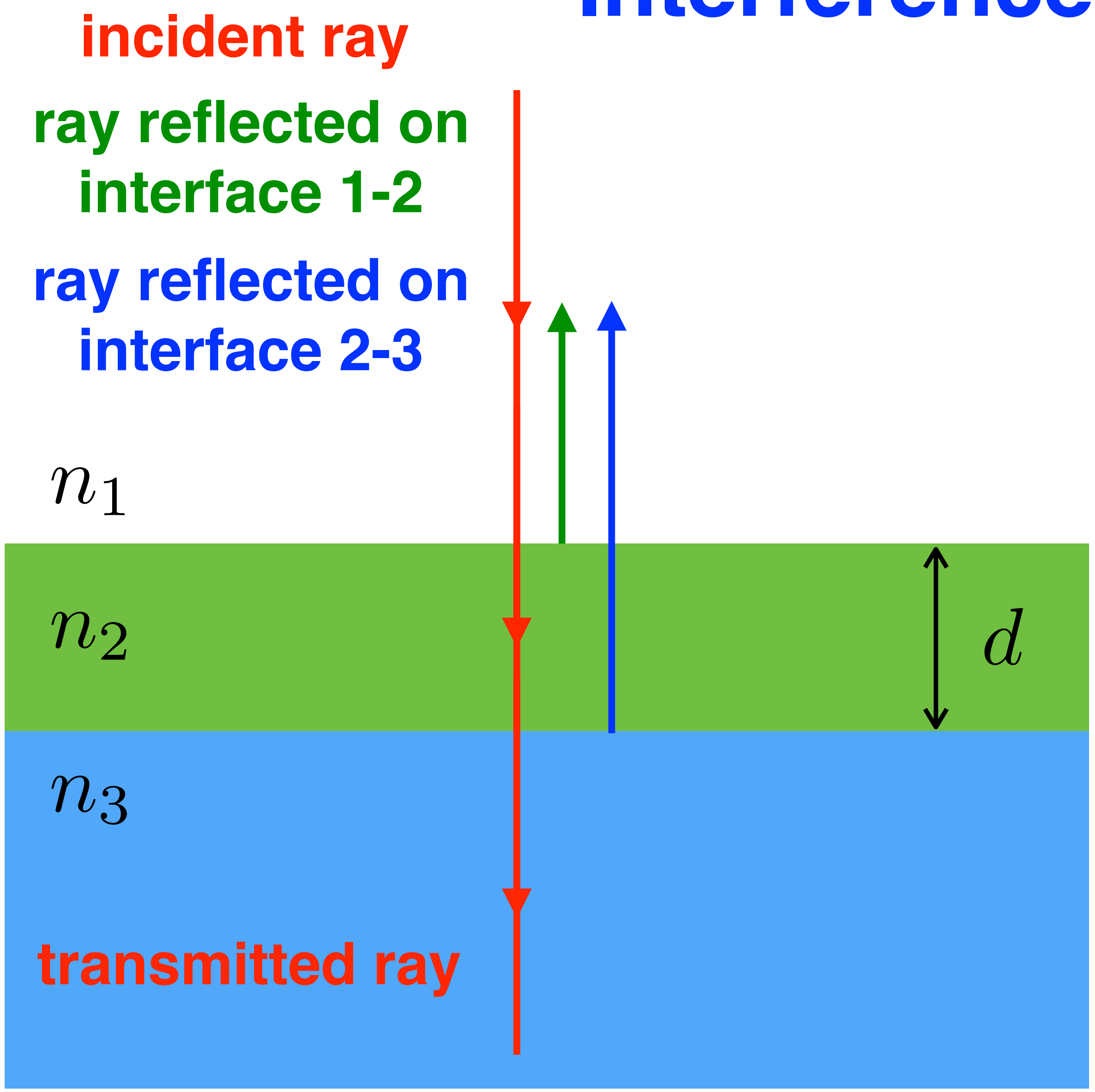
mirrors

Deconstructive interference of reflected rays results in almost perfectly transmitted rays with very little reflection.



antireflective coatings

Interference on thin films



difference between optical path lengths of the two reflected rays

$$OPD = 2n_2 d$$

$n_1 < n_2 < n_3$ $\quad$ $n_1 > n_2 > n_3$

no additional phase difference due to reflections

constructive interference of reflected rays $\quad OPD = m\lambda$

destructive interference of reflected rays $\quad OPD = \left(m + \frac{1}{2}\right)\lambda$

$m = 0, \pm 1, \pm 2, \ldots$

$n_1 > n_2 < n_3$ $\quad$ $n_1 < n_2 > n_3$

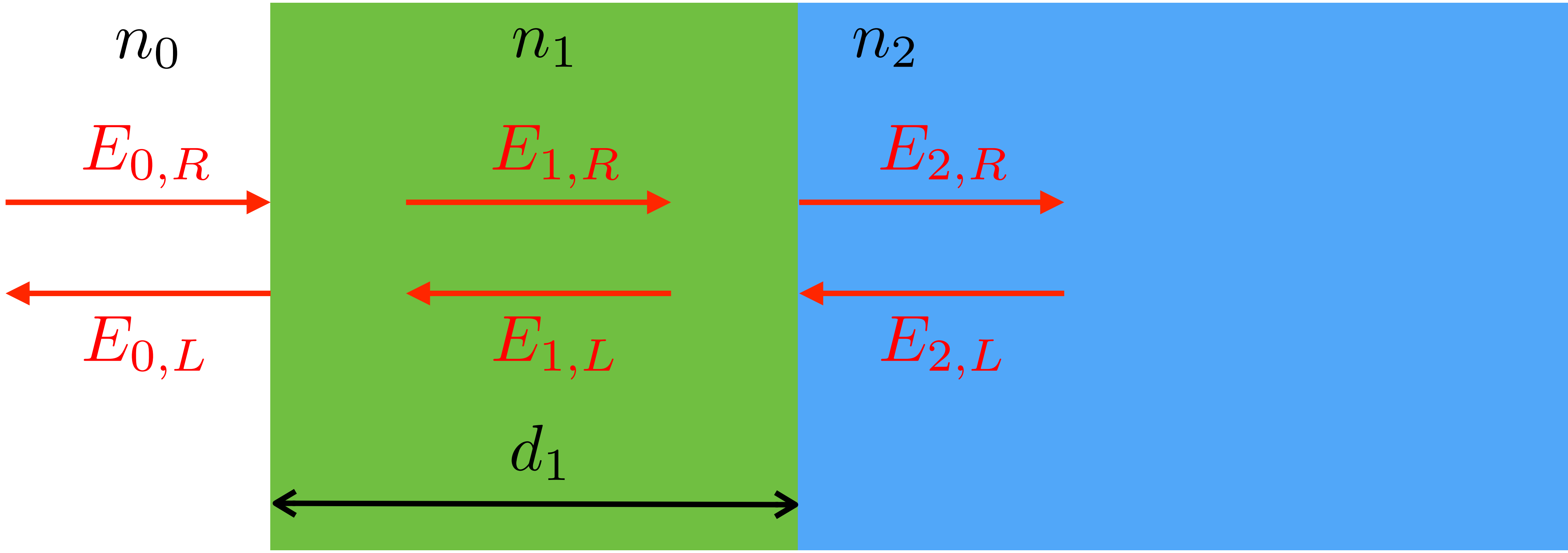
additional π phase difference due to reflections

constructive interference of reflected rays $\quad OPD = \left(m + \frac{1}{2}\right)\lambda$

destructive interference of reflected rays $\quad OPD = m\lambda$

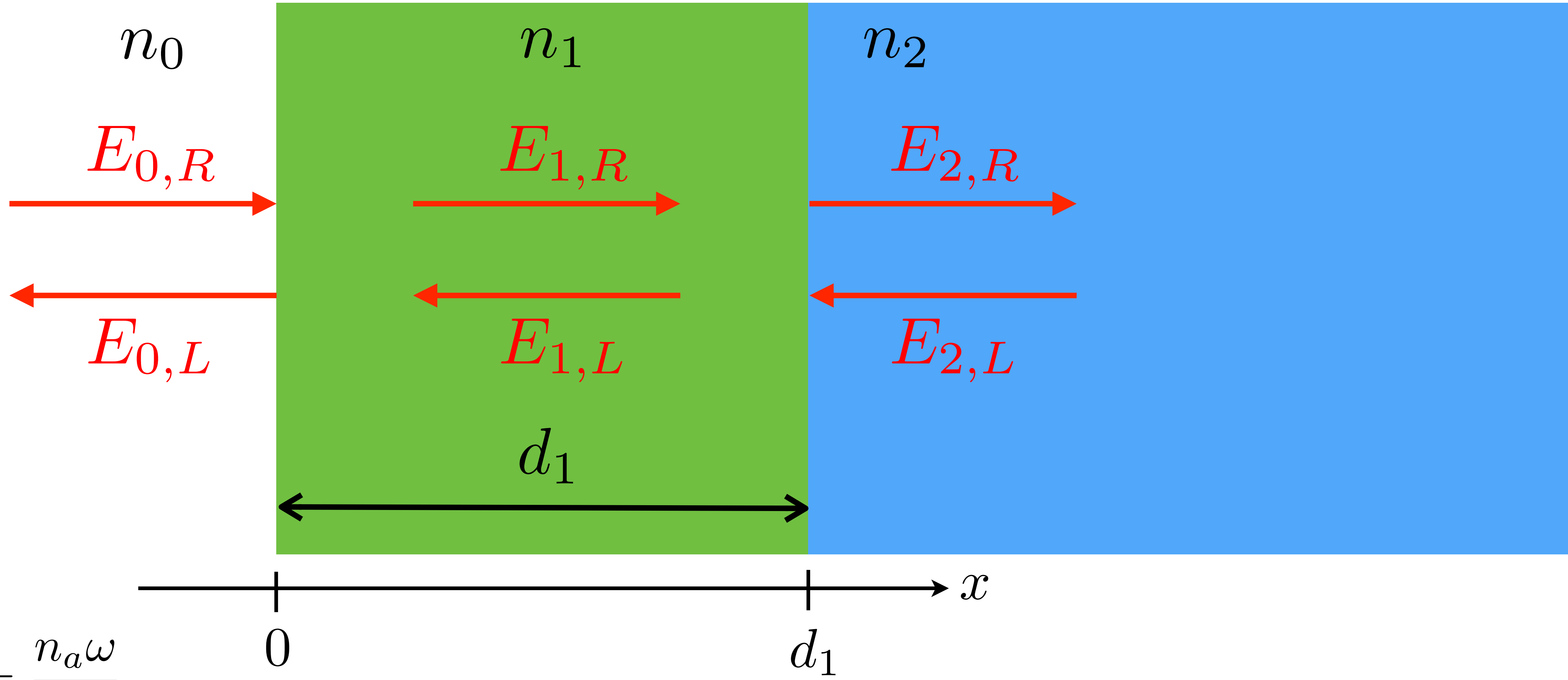
What happens for other wavelengths?

Transfer matrices



How can we relate the amplitudes of electromagnetic waves in the region 0 (white) to the amplitudes of electromagnetic waves in the region 2 (blue)?

Transfer matrices



$$k_a = \frac{2\pi n_a}{\lambda} = \frac{n_a \omega}{c_0}$$

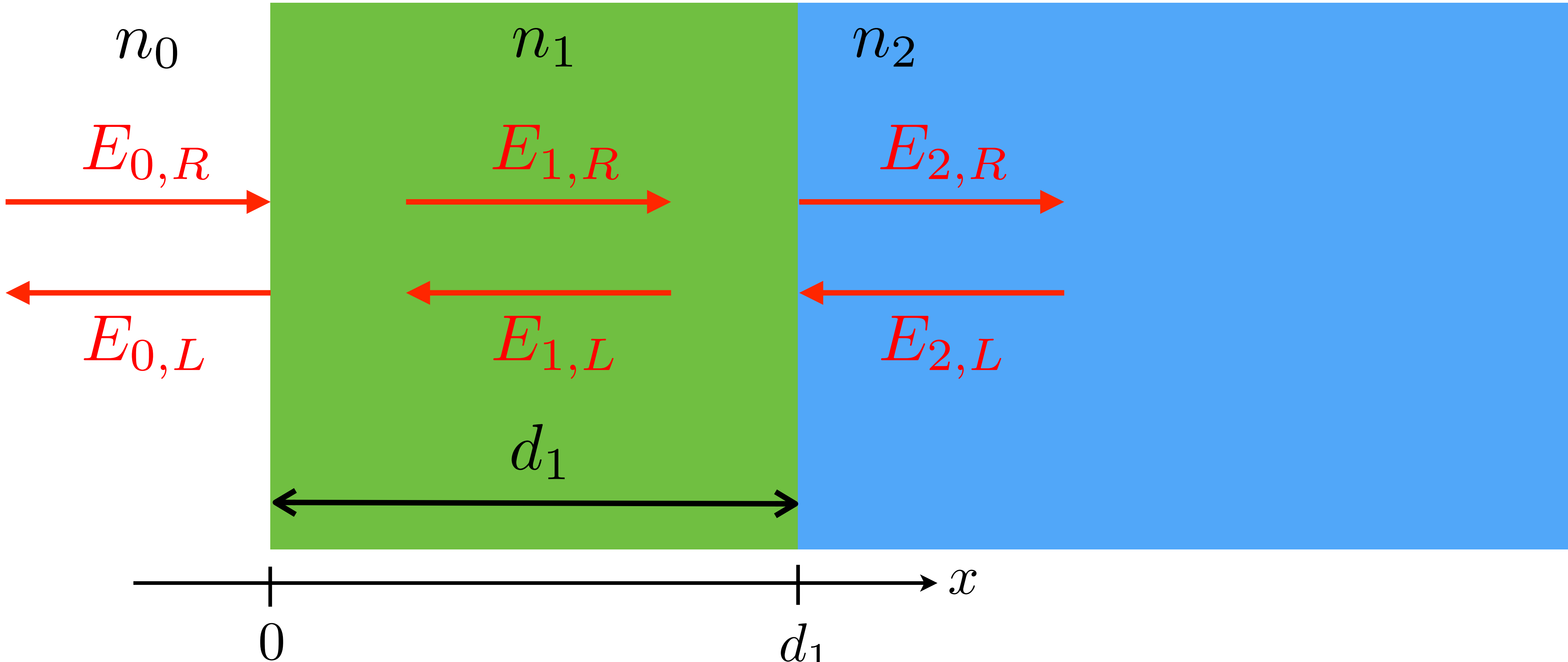
Electromagnetic waves in different regions:

$$E_0(x,t) = E_{0,R} e^{i(k_0 x - \omega t)} + E_{0,L} e^{i(-k_0 x - \omega t)}$$
$$E_1(x,t) = E_{1,R} e^{i(k_1 x - \omega t)} + E_{1,L} e^{i(-k_1 x - \omega t)}$$
$$E_2(x,t) = E_{2,R} e^{i(k_2 x - \omega t)} + E_{2,L} e^{i(-k_2 x - \omega t)}$$

Boundary conditions:

$$E_0(0,t) = E_1(0,t) \qquad E_1(d_1,t) = E_2(d_1,t)$$
$$\frac{\partial E_0}{\partial x}(0,t) = \frac{\partial E_1}{\partial x}(0,t) \qquad \frac{\partial E_1}{\partial x}(d_1,t) = \frac{\partial E_2}{\partial x}(d_1,t)$$

Transfer matrices



Boundary conditions:

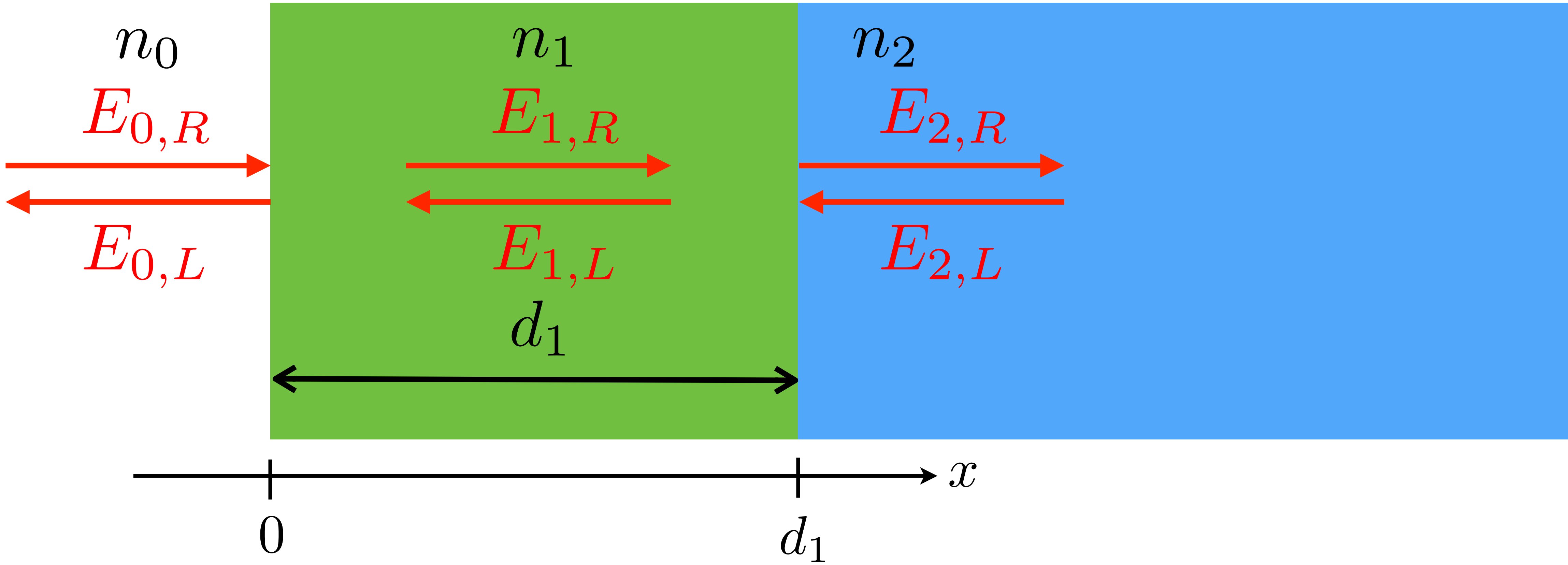
$$E_0(0,t) = E_1(0,t) \qquad E_1(d_1,t) = E_2(d_1,t)$$

$$\frac{\partial E_0}{\partial x}(0,t) = \frac{\partial E_1}{\partial x}(0,t) \qquad \frac{\partial E_1}{\partial x}(d_1,t) = \frac{\partial E_2}{\partial x}(d_1,t)$$

We would like to relate boundary conditions at two different interfaces via a transfer matrix *M_1*:

$$\begin{pmatrix} E_2(d_1,t) \\ \frac{\partial E_2}{\partial x}(d_1,t) \end{pmatrix} = M_1 \begin{pmatrix} E_0(0,t) \\ \frac{\partial E_0}{\partial x}(0,t) \end{pmatrix}$$

Transfer matrices



Electromagnetic waves in regions 1:

$$E_1(x,t) = E_{1,R}e^{i(k_1x-\omega t)} + E_{1,L}e^{i(-k_1x-\omega t)}$$

Relation between boundary conditions:

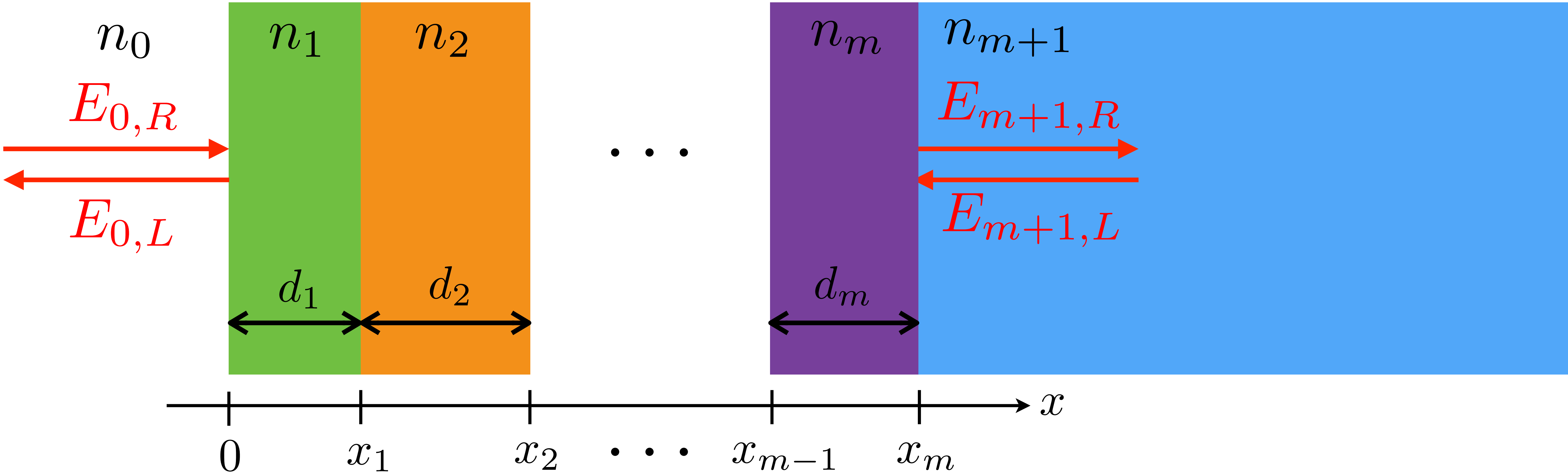
$$\begin{pmatrix} E_1(d_1,t) \\ \frac{\partial E_1}{\partial x}(d_1,t) \end{pmatrix} = \begin{pmatrix} E_2(d_1,t) \\ \frac{\partial E_2}{\partial x}(d_1,t) \end{pmatrix} = M_1 \begin{pmatrix} E_0(0,t) \\ \frac{\partial E_0}{\partial x}(0,t) \end{pmatrix} = M_1 \begin{pmatrix} E_1(0,t) \\ \frac{\partial E_1}{\partial x}(0,t) \end{pmatrix}$$

$$\begin{pmatrix} E_{1,R}e^{ik_1d_1} + E_{1,L}e^{-ik_1d_1} \\ ik_1E_{1,R}e^{ik_1d_1} - ik_1E_{1,L}e^{-ik_1d_1} \end{pmatrix} = M_1 \begin{pmatrix} E_{1,R} + E_{1,L} \\ ik_1E_{1,R} - ik_1E_{1,L} \end{pmatrix}$$

Transfer matrix M_1 can be obtained by solving equations above:

$$M_1 = \begin{pmatrix} \cos(k_1d_1), & \frac{\sin(k_1d_1)}{k_1} \\ -k_1\sin(k_1d_1), & \cos(k_1d_1) \end{pmatrix}$$

Transfer matrices



Transfer matrix for *m* layers:

$$\begin{pmatrix} E_{m+1}(x_m,t) \\ \frac{\partial E_{m+1}}{\partial x}(x_m,t) \end{pmatrix} = M_m \begin{pmatrix} E_m(x_{m-1},t) \\ \frac{\partial E_m}{\partial x}(x_{m-1},t) \end{pmatrix} = M_m M_{m-1} \begin{pmatrix} E_{m-1}(x_{m-2},t) \\ \frac{\partial E_{m-1}}{\partial x}(x_{m-2},t) \end{pmatrix} = \dots$$

$$\begin{pmatrix} E_{m+1}(x_m,t) \\ \frac{\partial E_{m+1}}{\partial x}(x_m,t) \end{pmatrix} = M \begin{pmatrix} E_0(0,t) \\ \frac{\partial E_0}{\partial x}(0,t) \end{pmatrix}$$

$$M = M_m \cdot \ldots \cdot M_2 \cdot M_1$$

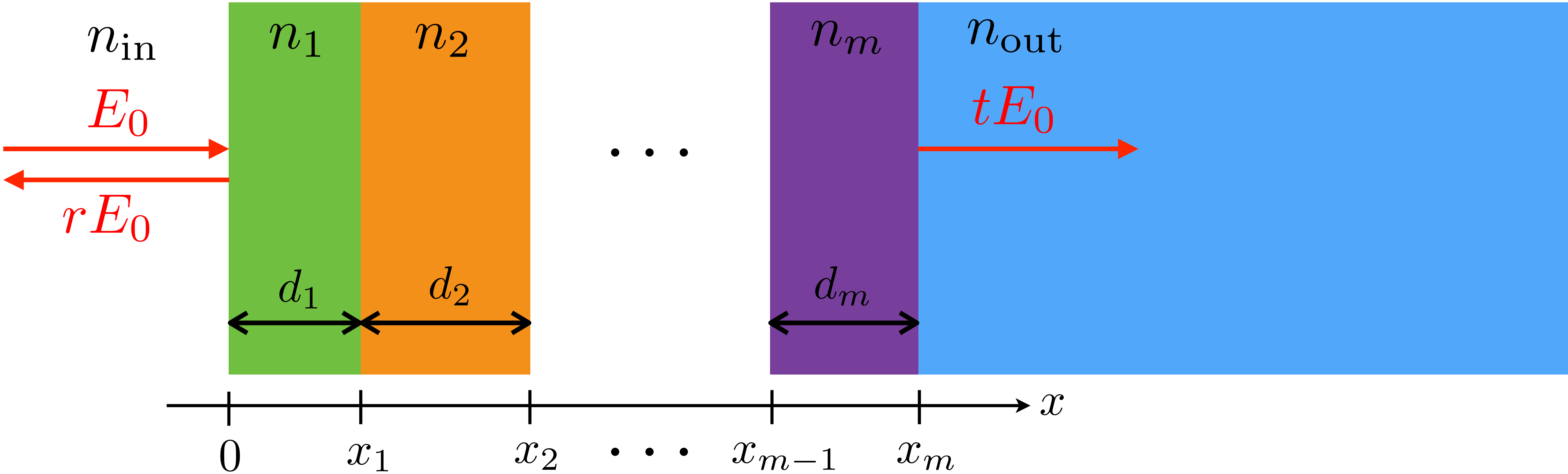
Note:

$$M_a = \begin{pmatrix} \cos(k_a d_a), & \frac{\sin(k_a d_a)}{k_a} \\ -k_a \sin(k_a d_a), & \cos(k_a d_a) \end{pmatrix}$$

$$\det(M) = \det(M_a) = 1$$

$$k_a = \frac{2\pi n_a}{\lambda} = \frac{n_a \omega}{c_0}$$

Transfer matrices



Incoming and outgoing electromagnetic waves:

$$E_{\rm in}(x,t) = E_0 e^{i(k_{\rm in}x-\omega t)} + rE_0 e^{i(-k_{\rm in}x-\omega t)}$$

$$E_{\rm out}(x,t) = tE_0 e^{i(k_{\rm out}x-\omega t)}$$

$$\begin{pmatrix} E_{\rm out}(x_m,t) \\ \frac{\partial E_{\rm out}}{\partial x}(x_m,t) \end{pmatrix} = \begin{pmatrix} M_{11}, & M_{12} \\ M_{21}, & M_{22} \end{pmatrix} \begin{pmatrix} E_{\rm in}(0,t) \\ \frac{\partial E_{\rm in}}{\partial x}(0,t) \end{pmatrix}$$

Amplitudes of reflected and transmitted waves:

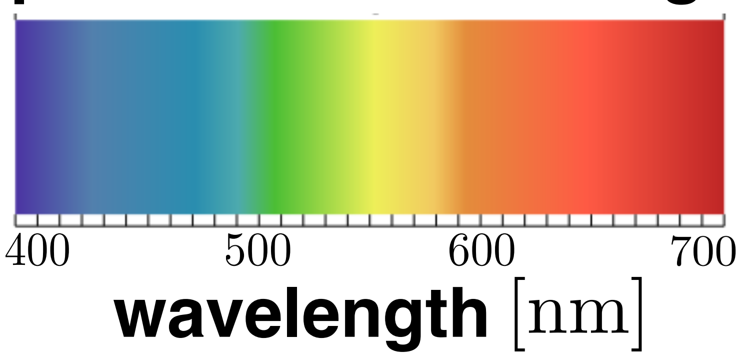
$$r = \frac{(M_{21} + k_{\rm in}k_{\rm out}M_{12}) + i(k_{\rm in}M_{22} - k_{\rm out}M_{11})}{(-M_{21} + k_{\rm in}k_{\rm out}M_{12}) + i(k_{\rm out}M_{11} + k_{\rm in}M_{22})}$$

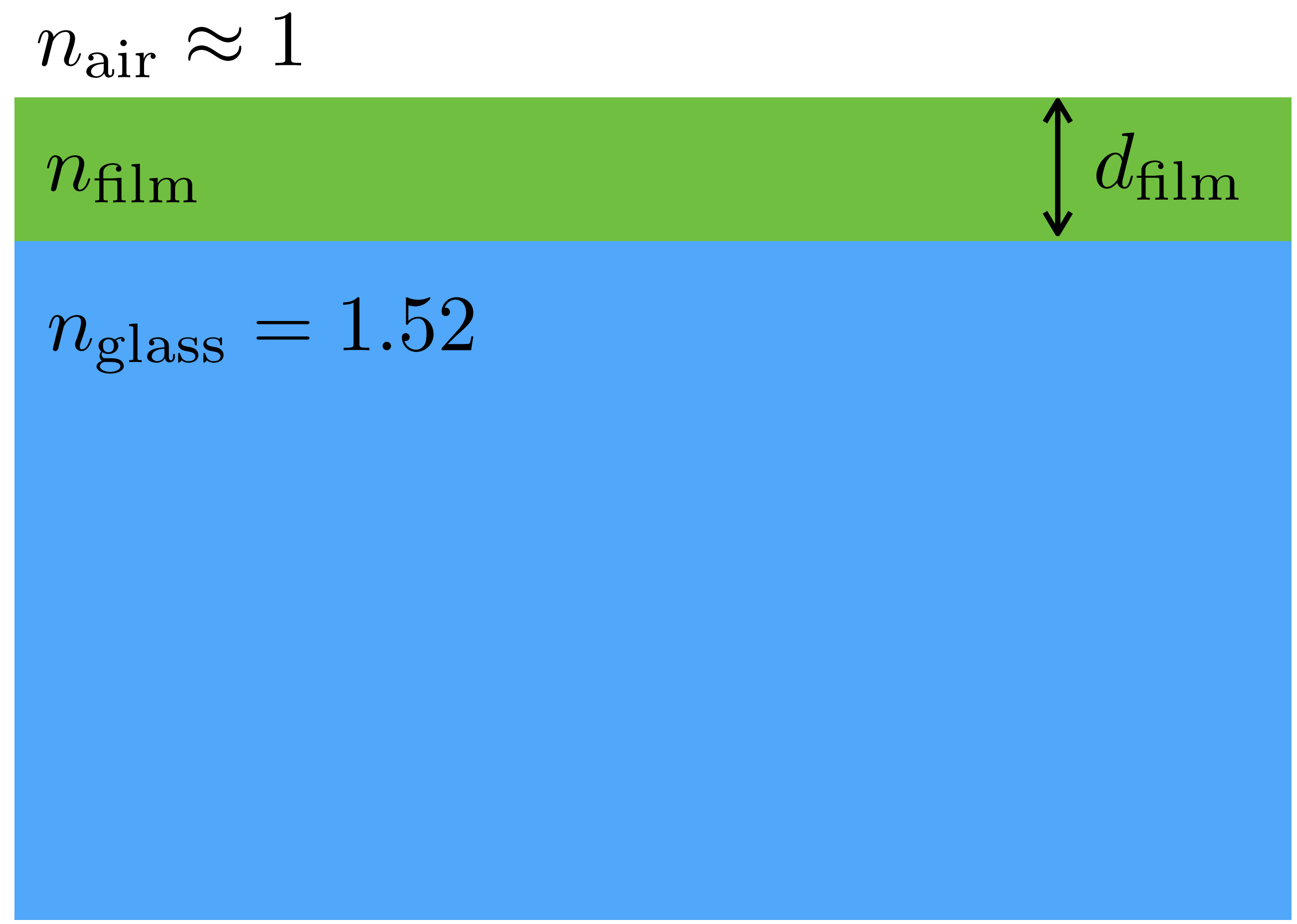
$$t = \frac{2ik_{\rm in}e^{-ix_m k_{\rm out}}}{(-M_{21} + k_{\rm in}k_{\rm out}M_{12}) + i(k_{\rm out}M_{11} + k_{\rm in}M_{22})}$$

Example: antireflective coating

We would like to design a thin film coating for glasses that minimizes reflection of visible light.

spectrum of visible light





Assume that thin film is made of MgF_2 that can be easily applied with physical vapor deposition:

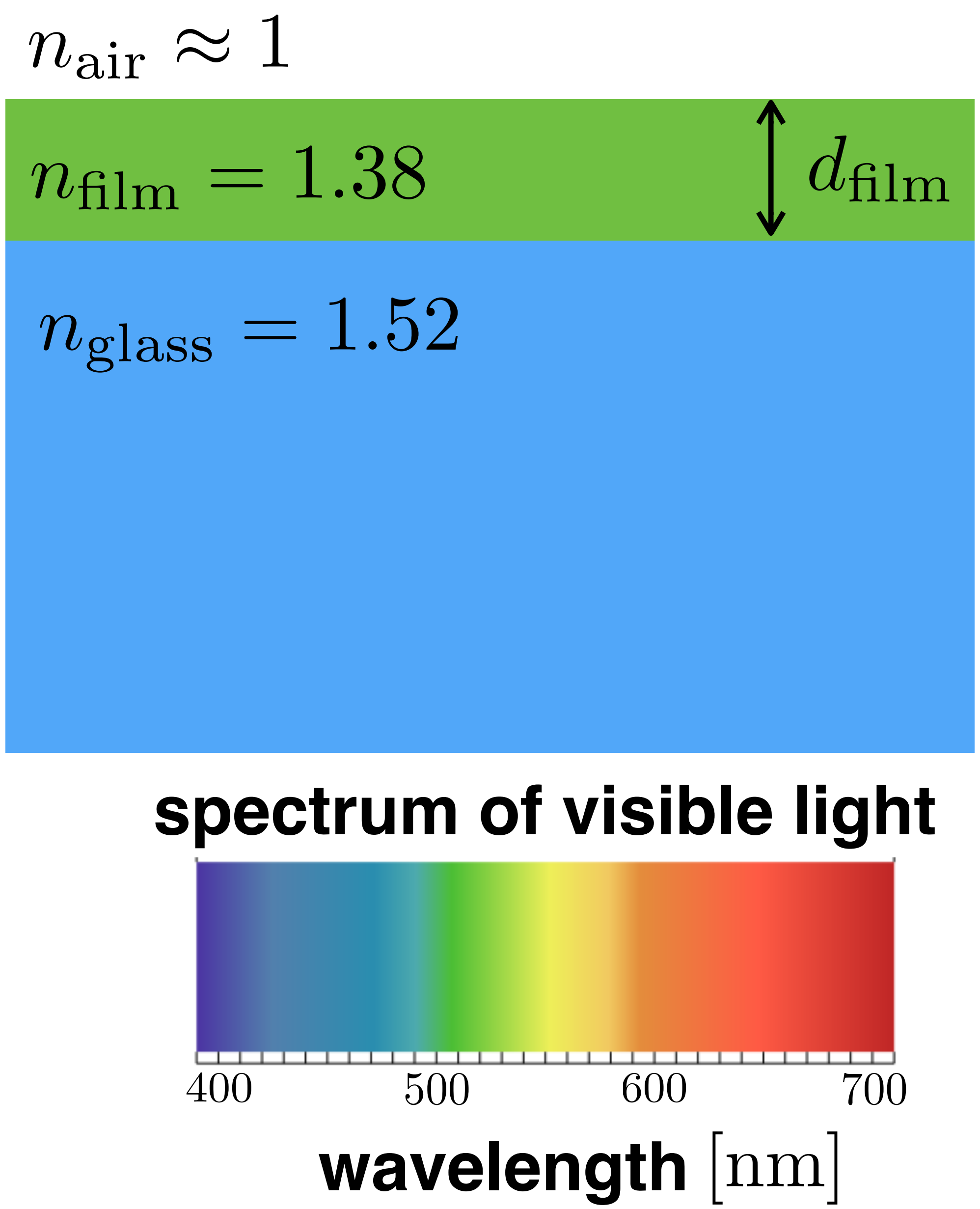
$$n_{\mathrm{film}} = 1.38$$

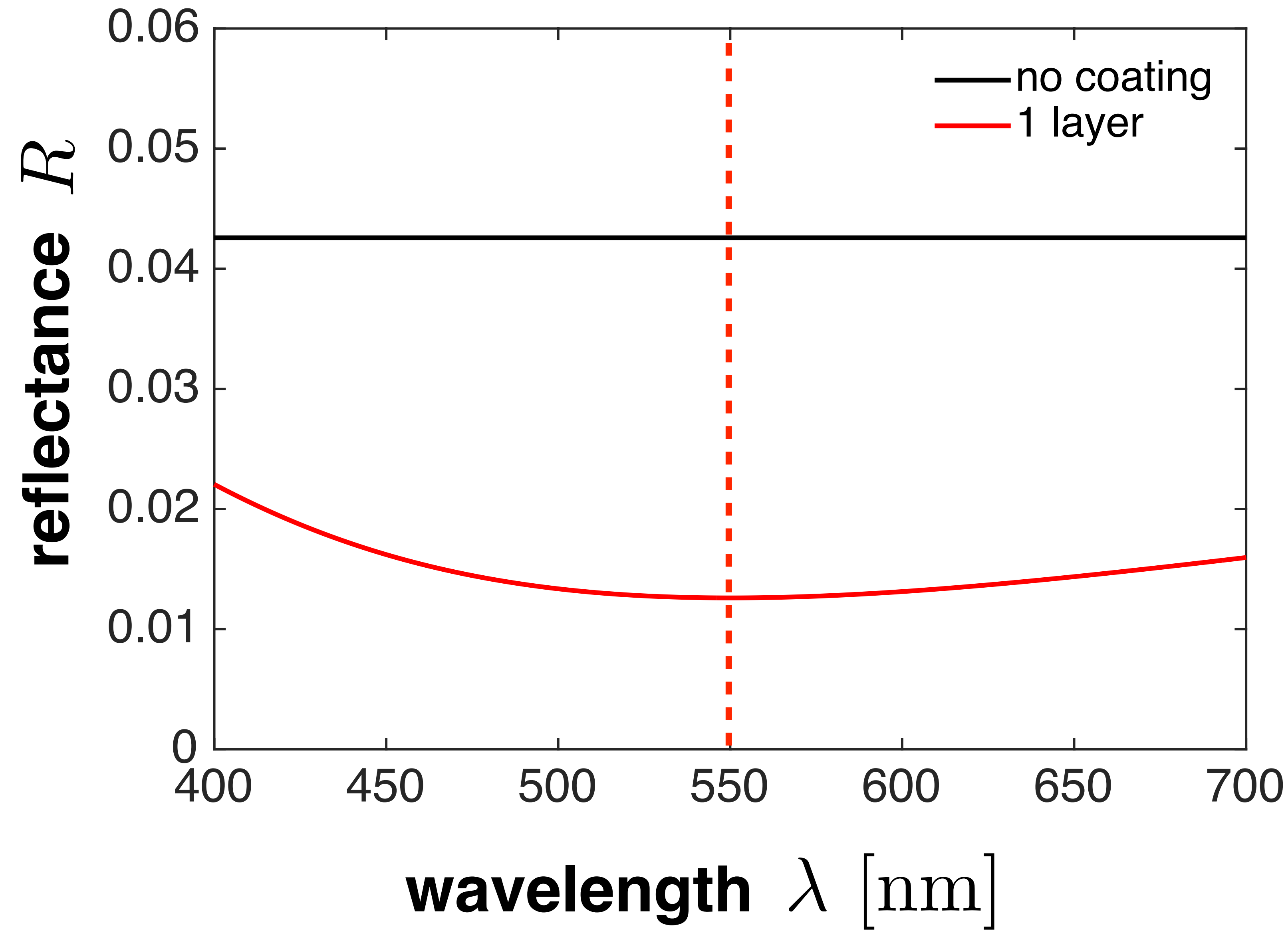
Note: the condition for deconstructive interference of reflected rays can be satisfied only for discrete set of wavelengths λ_0 :

$$2 d_{\mathrm{film}} n_{\mathrm{film}} = \left(m + \frac{1}{2} \right) \lambda_0$$

$$m = 0, 1, 2, \ldots$$

Example: antireflective coating

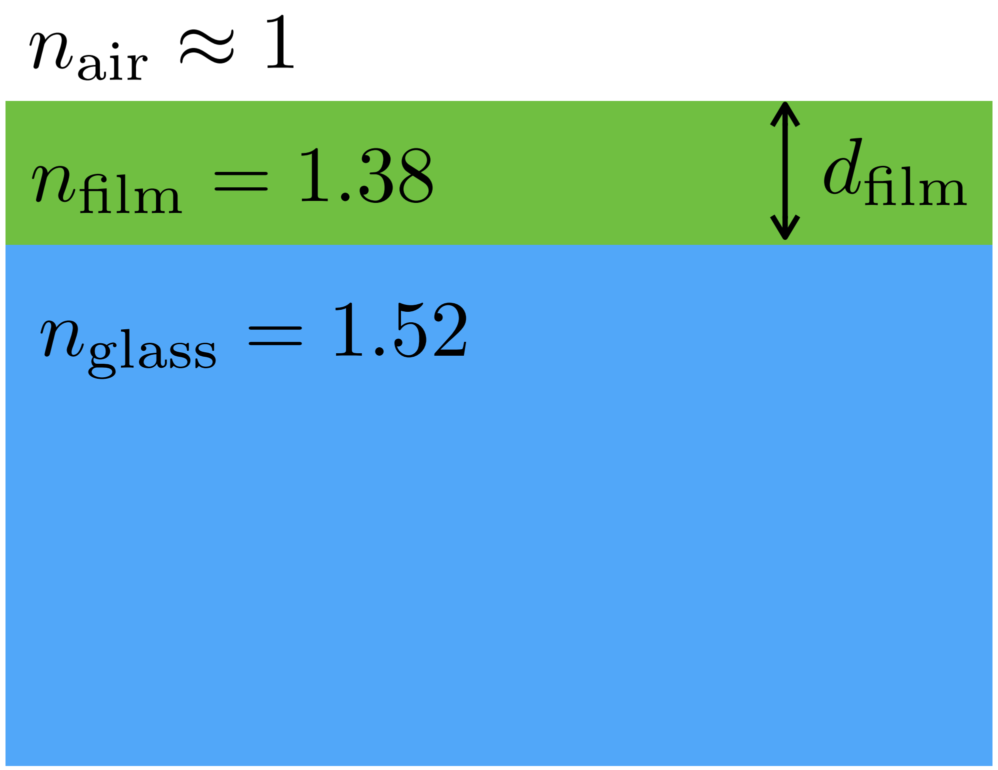


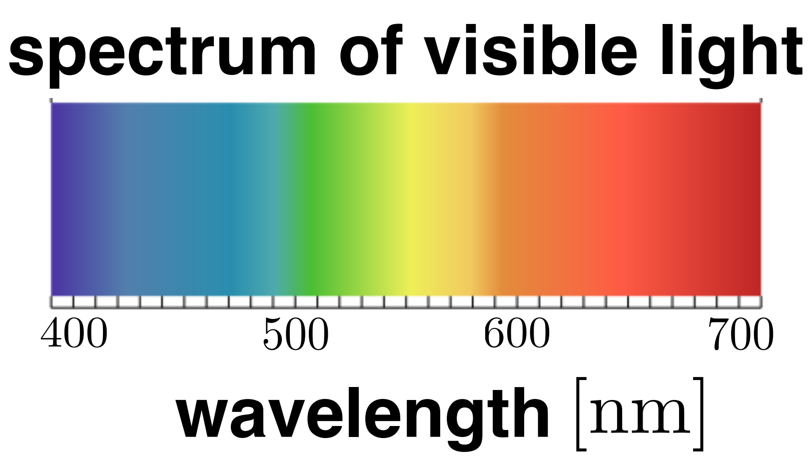


Use film thickness that corresponds to the destructive interference for the wavelength in the middle of the visible spectrum $\lambda_{\text{target}} = 550\,\text{nm}$:

$$d_{\text{film}} = \frac{\lambda_{\text{target}}}{4 n_{\text{film}}} = 100\,\text{nm}$$

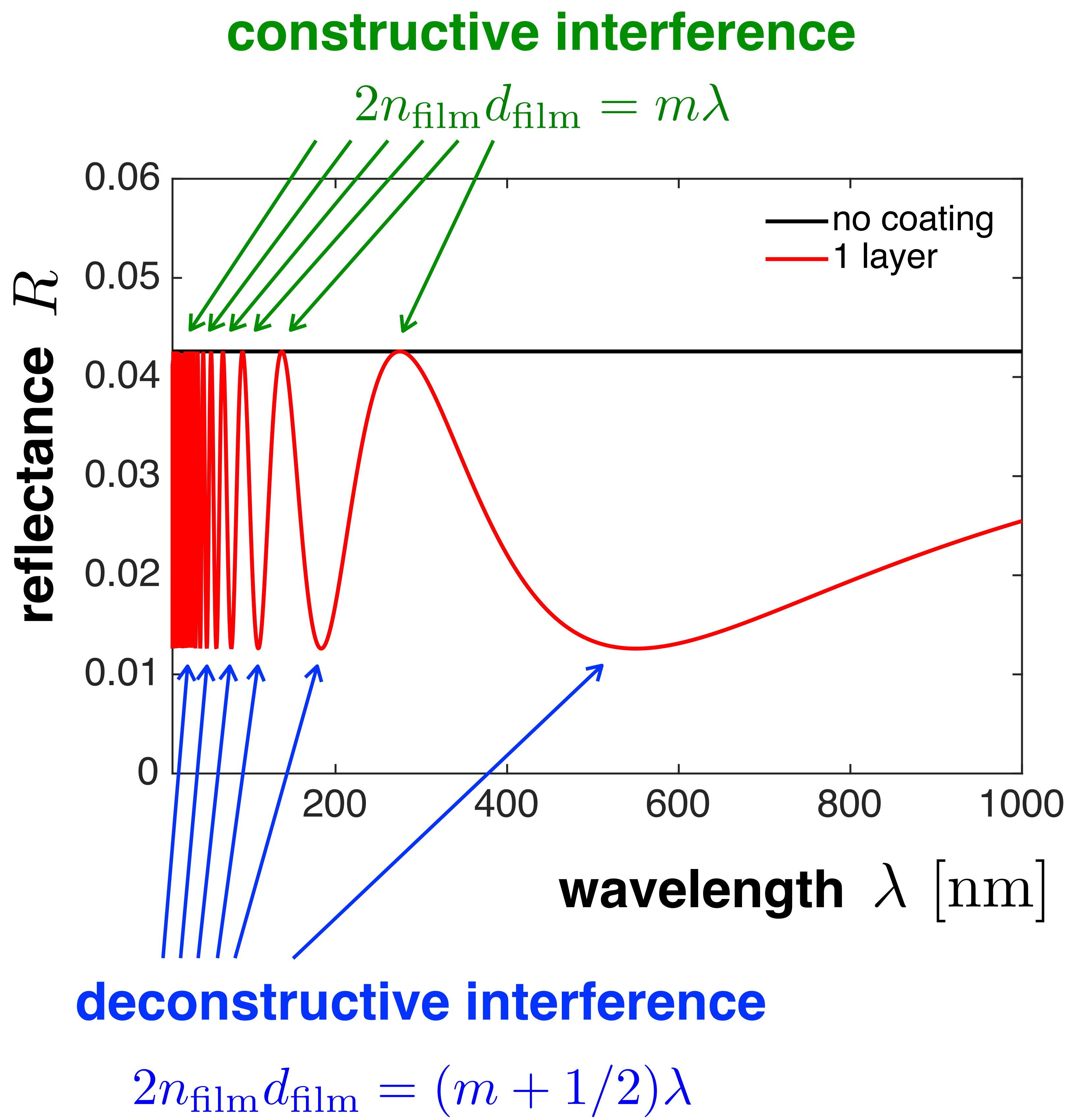
Example: antireflective coating



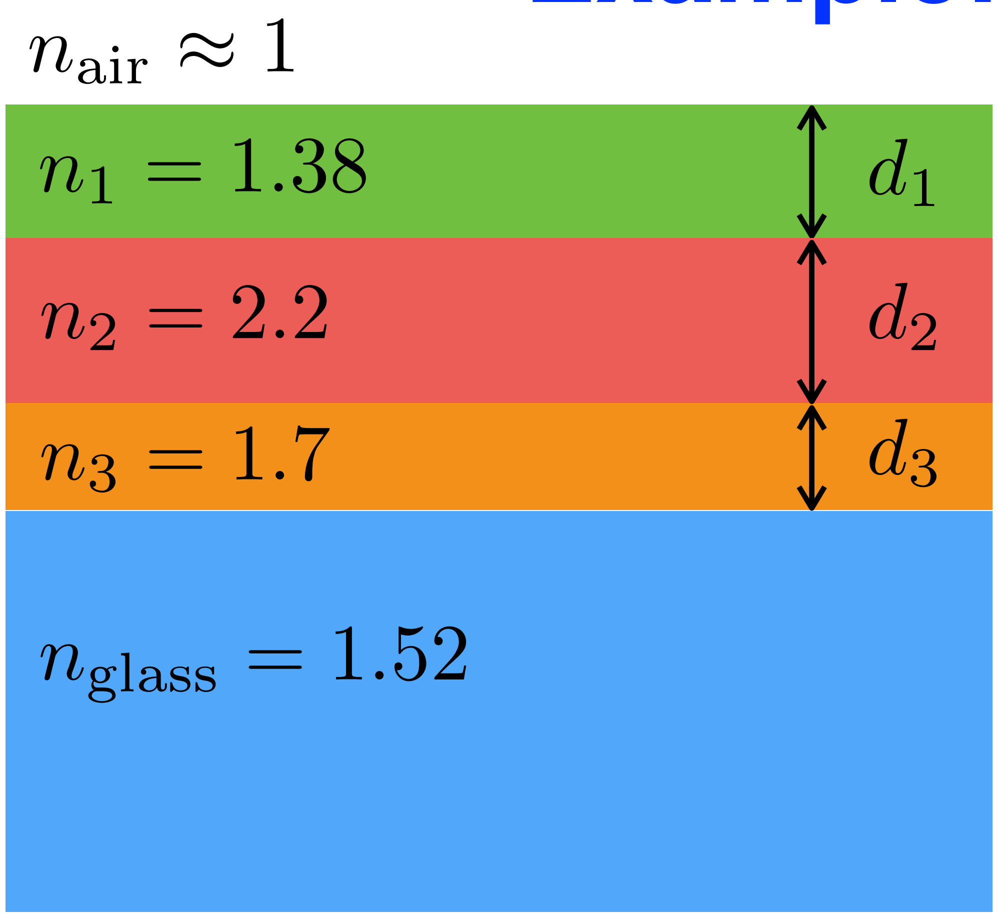


$$\lambda_{\text{target}} = 550\,\text{nm}$$

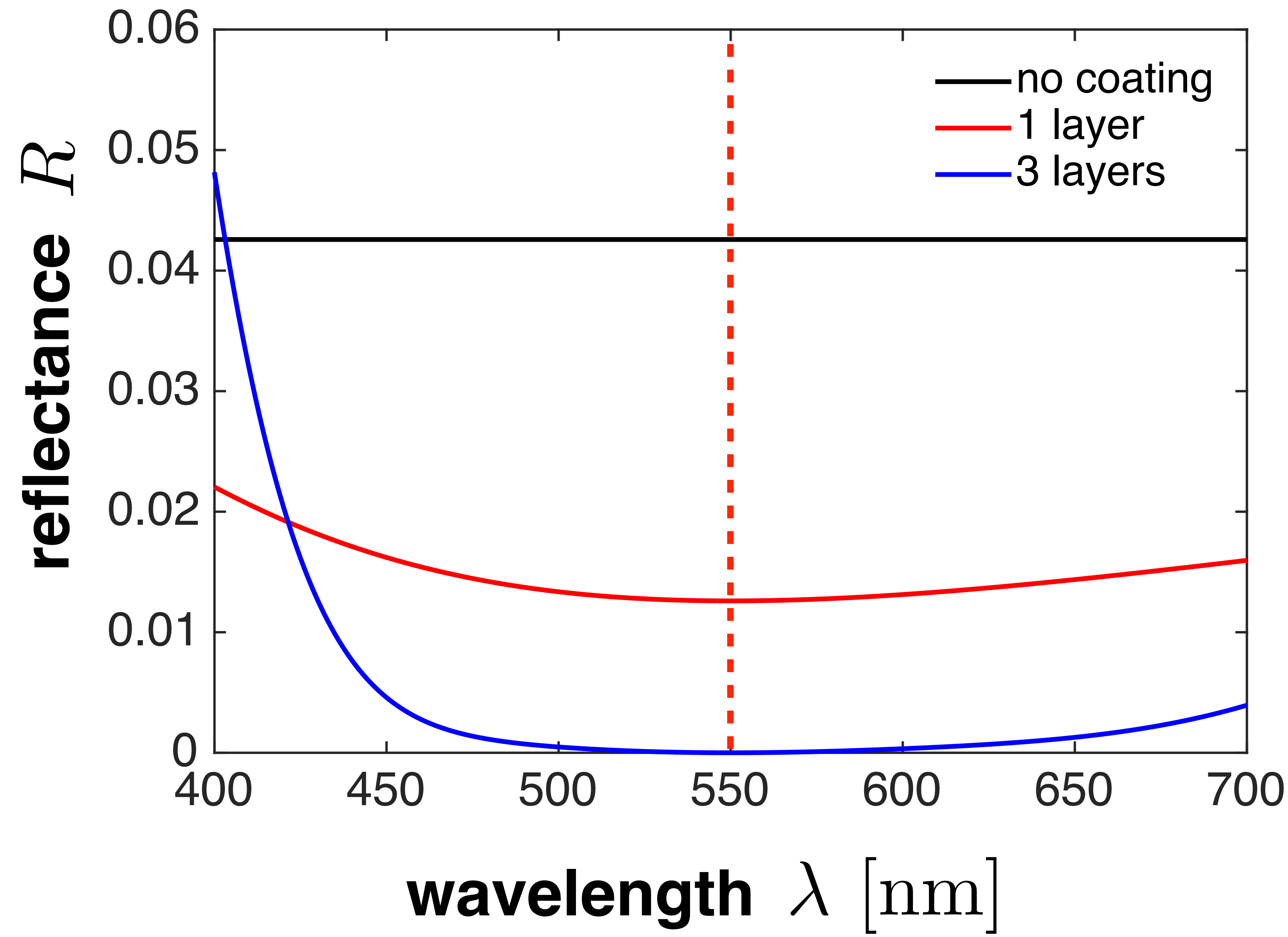
$$d_{\text{film}} = \frac{\lambda_{\text{target}}}{4 n_{\text{film}}} = 100\,\text{nm}$$



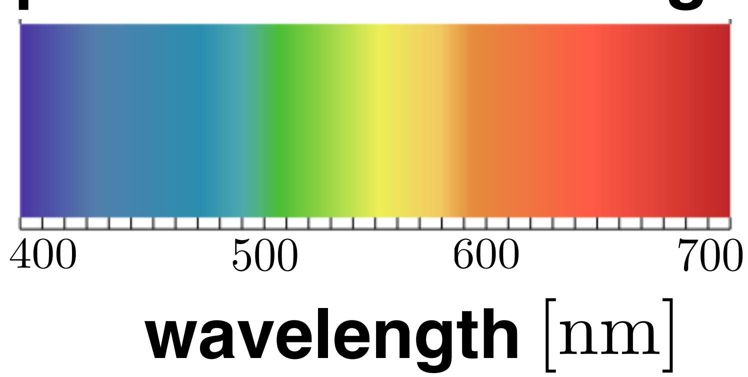
Example: antireflective coating



Multiple layers of coating significantly reduce the reflectance of visible spectrum!



spectrum of visible light

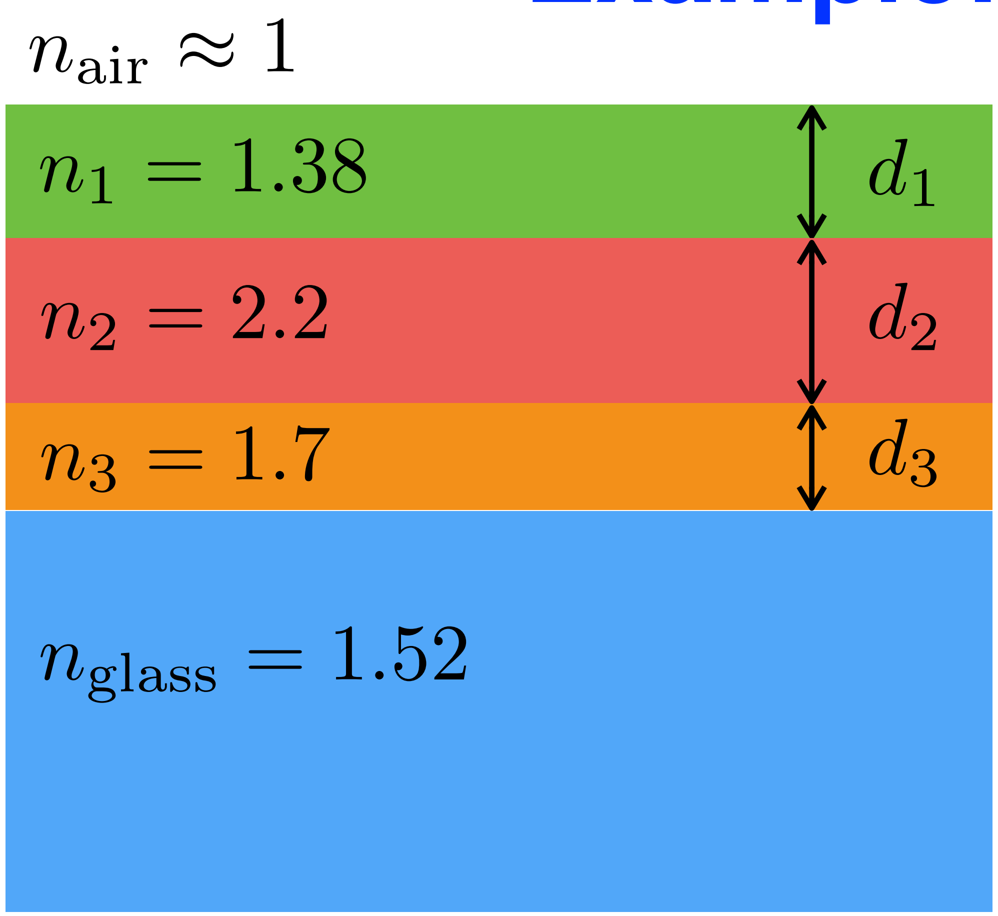


Use film thicknesses that correspond to the destructive interference for the wavelength in the middle of the visible spectrum $\lambda_{\mathrm{target}} = 550\,\mathrm{nm}$:

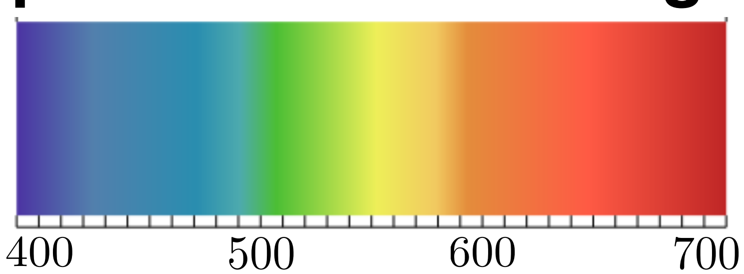
$$d_1 = \lambda_{\mathrm{target}}/(4n_1)$$
$$d_2 = \lambda_{\mathrm{target}}/(2n_2)$$
$$d_3 = \lambda_{\mathrm{target}}/(4n_3)$$

note the additional phase difference!

Example: antireflective coating



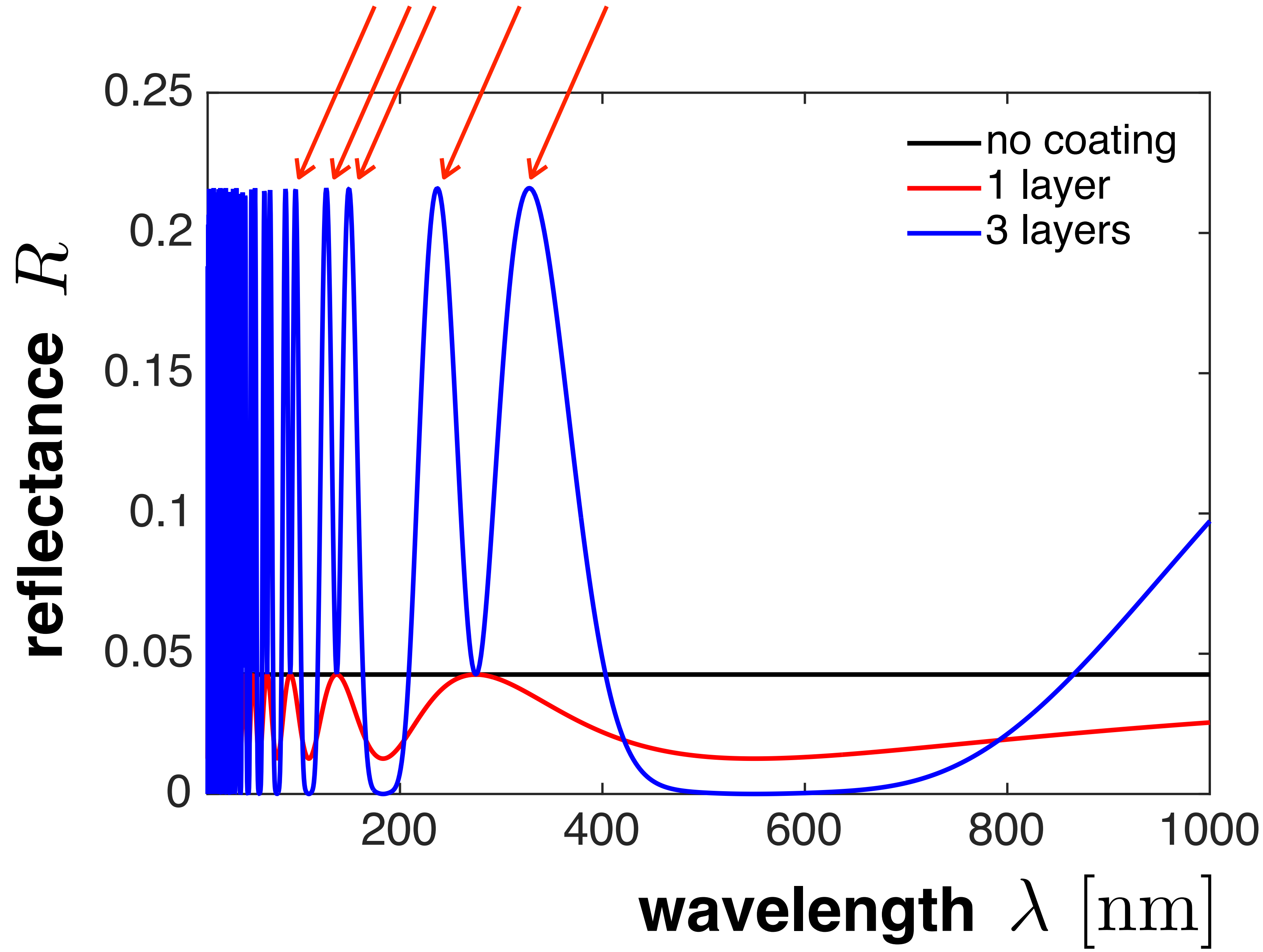
spectrum of visible light



wavelength [nm]

$$\lambda_{\mathrm{target}} = 550\,\mathrm{nm}$$
$$d_1 = \lambda_{\mathrm{target}}/(4n_1)$$
$$d_2 = \lambda_{\mathrm{target}}/(2n_2)$$
$$d_3 = \lambda_{\mathrm{target}}/(4n_3)$$

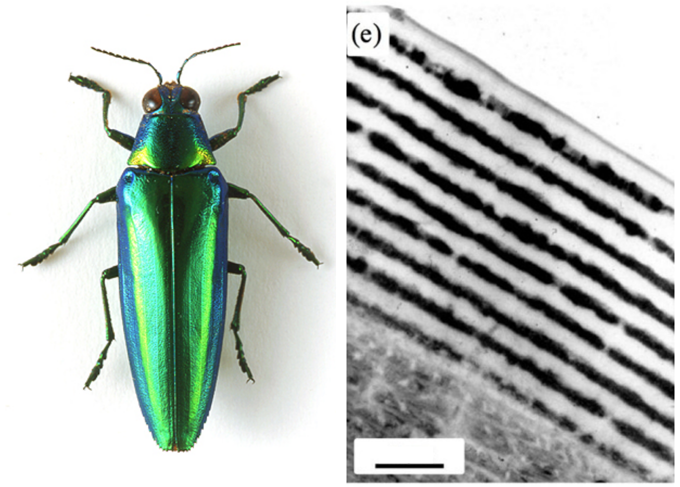
Multiple layers of coating significantly enhance reflectance of certain wavelengths outside the visible spectrum!

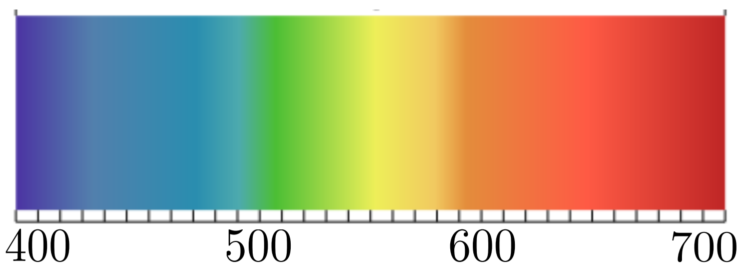


Additional peaks (minima) correspond to the constructive (deconstructive) interference for rays scattered on different combination of interfaces.

Example: structural color

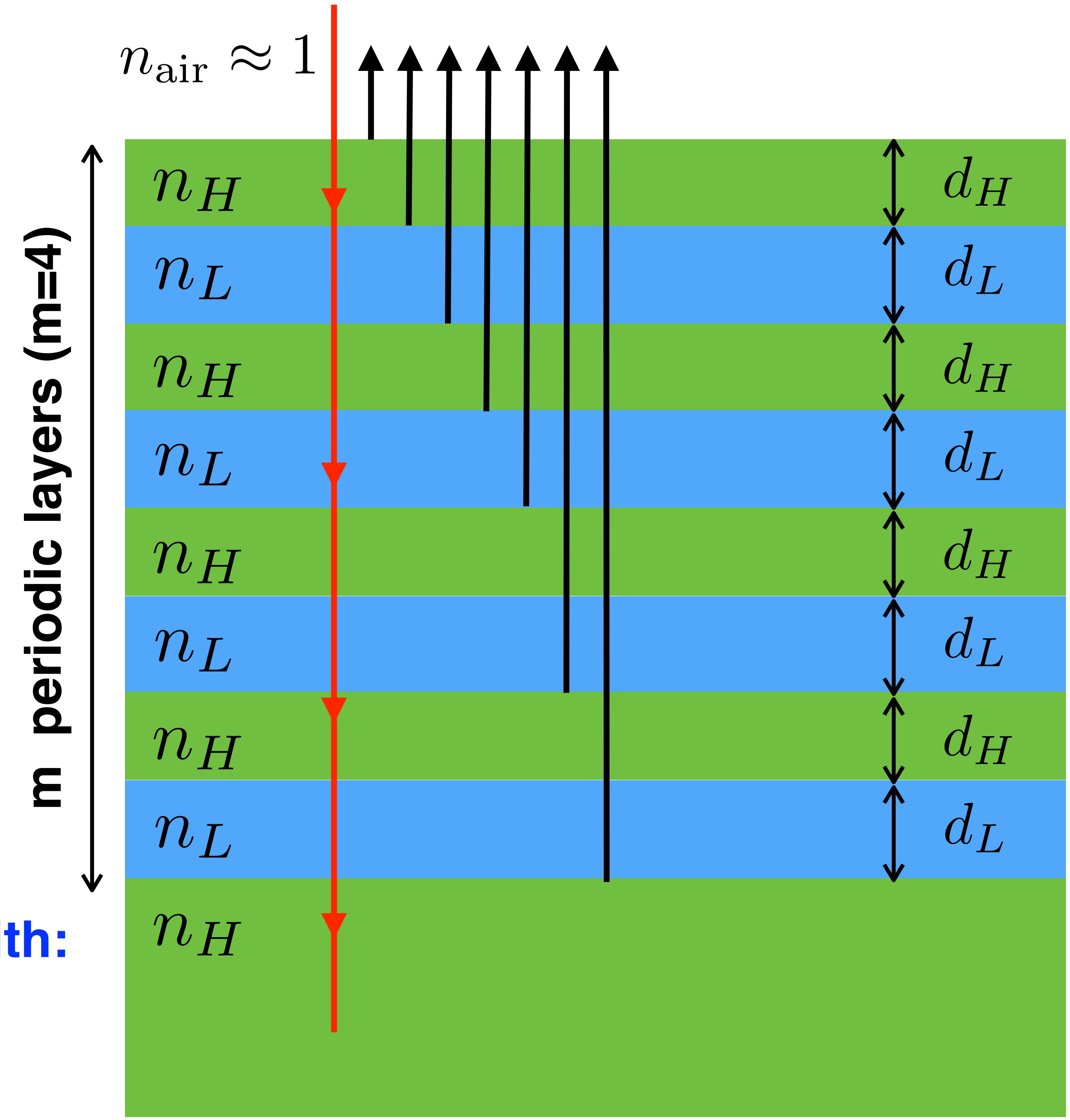
Chrysochroa raja bettle





wavelength [nm]

We would like to design periodic structure, which preferentially reflects green color with $\lambda_0 = 500\,\mathrm{nm}$.



Typical refraction indices:

$$n_H = 1.69 \qquad n_L = 1.56$$

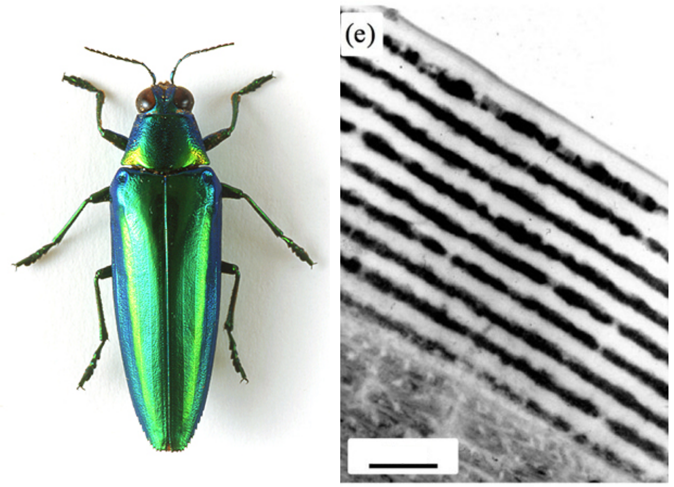
Constructive interference of reflected rays can be achieved with:

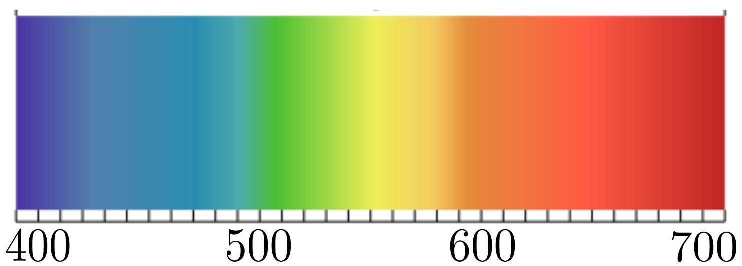
$$d_H = \frac{\lambda_0}{4n_H} = 74\,\mathrm{nm}$$

$$d_L = \frac{\lambda_0}{4n_L} = 80\,\mathrm{nm}$$

Example: structural color

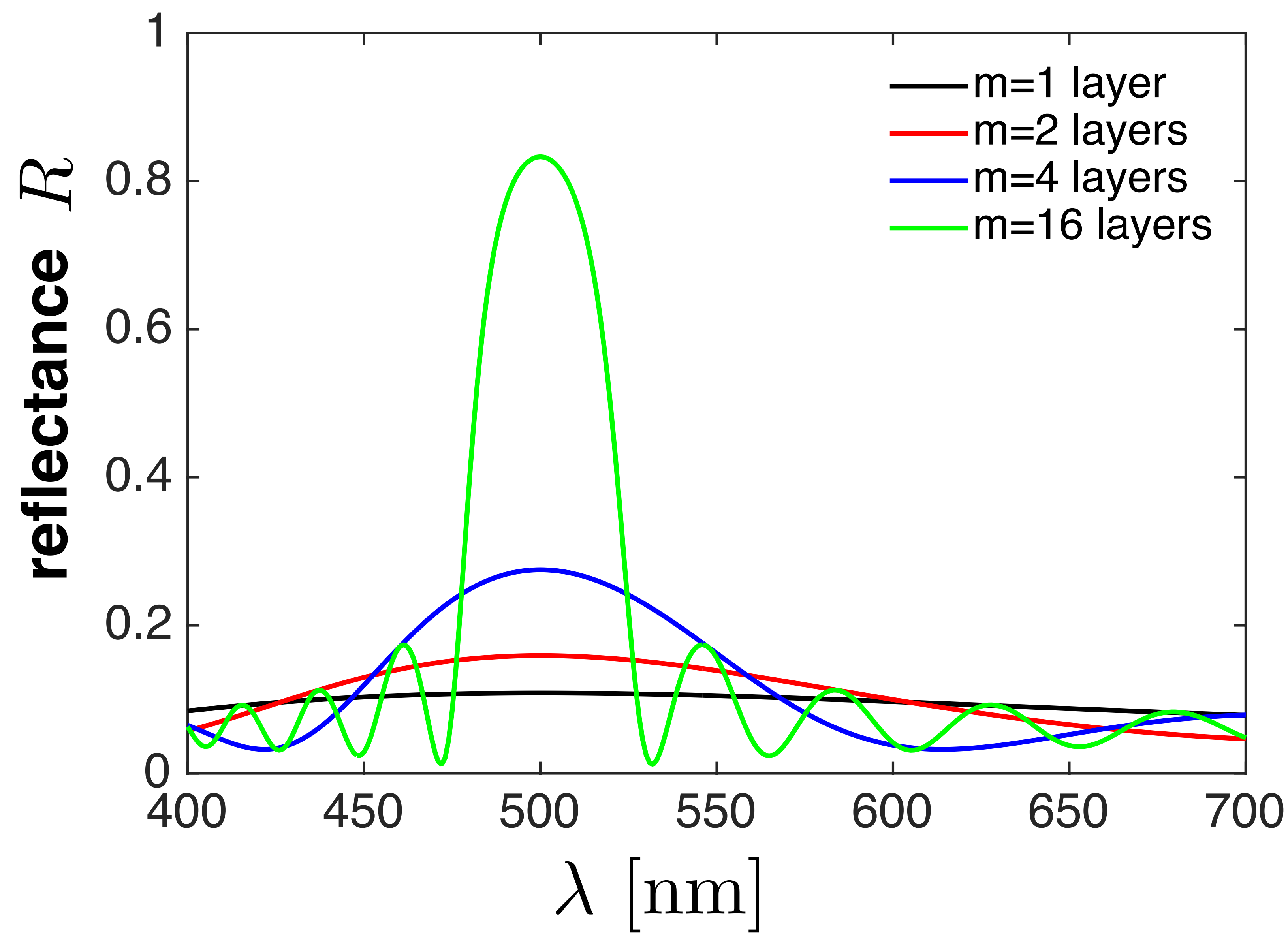
Chrysochroa raja bettle





wavelength $[\mathrm{nm}]$

Multiple periodic layers increase the reflectance of target wavelength $\lambda_0 = 500\,\mathrm{nm}$**!**



In periodic structures high reflectance is achieved for a range of wavelengths around the target wavelength. This range is called band gap.