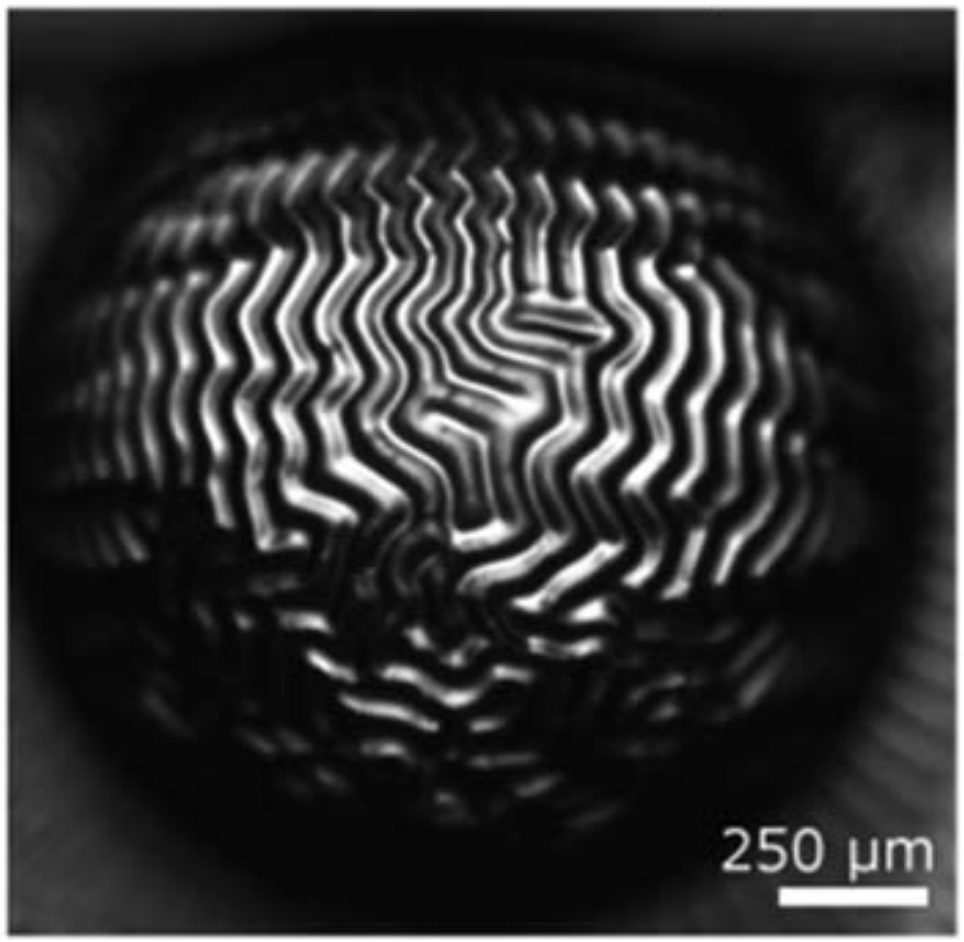
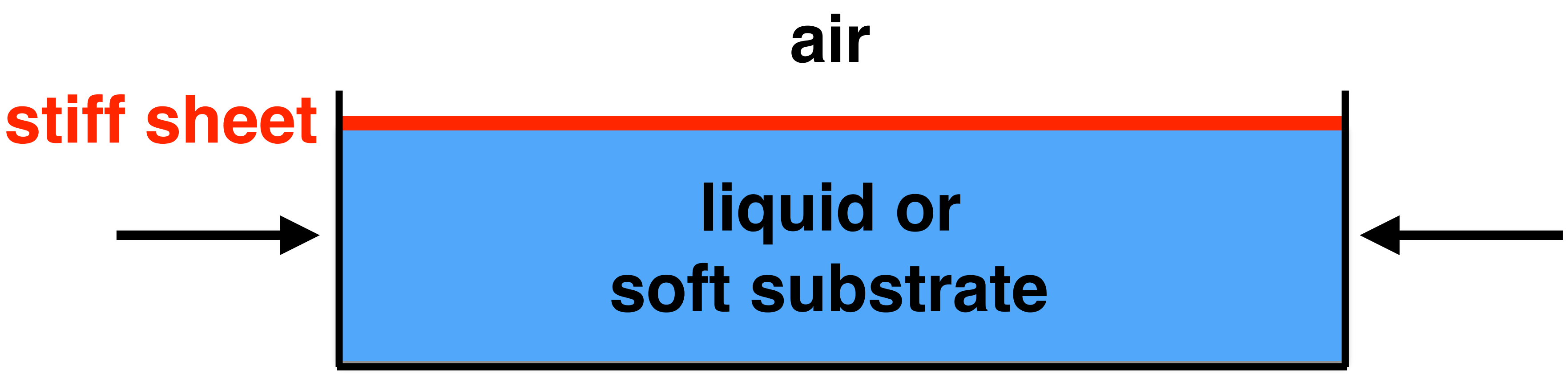
MAE 545: Lecture 5 (2/20)

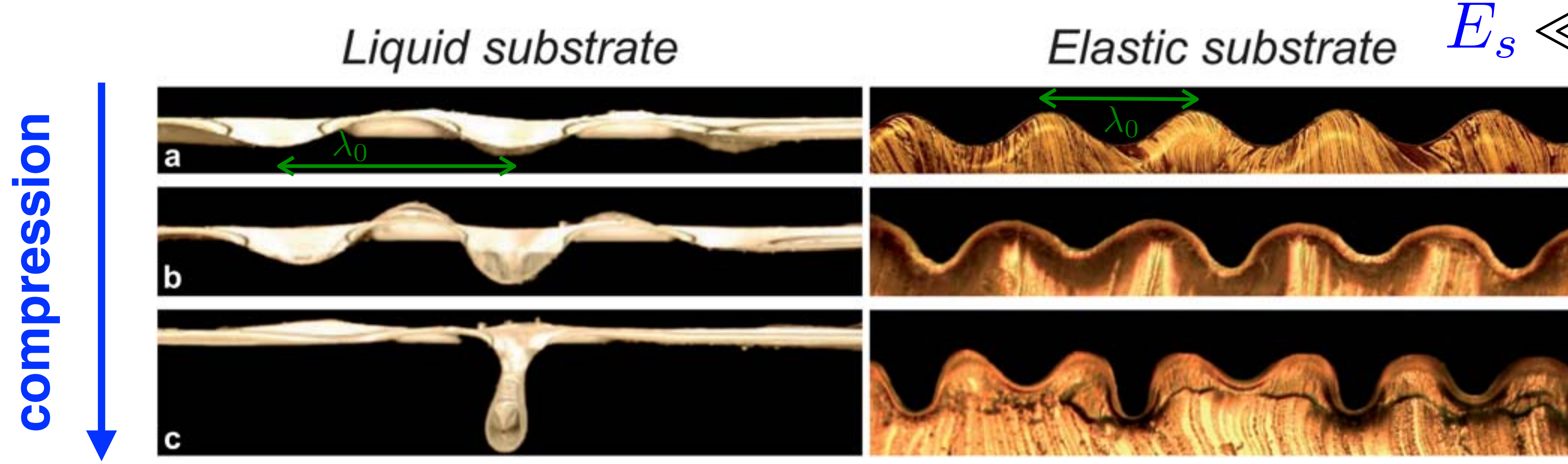
Wrinkled surfaces



Compression of stiff thin sheets on liquid and soft elastic substrates



$E_s \ll E_m$



10 μm thin sheet of polyester on water

$$\lambda_0 = 1.6\,\mathrm{cm}$$

~10 μm thin PDMS (stiffer) sheet on PDMS (softer) substrate

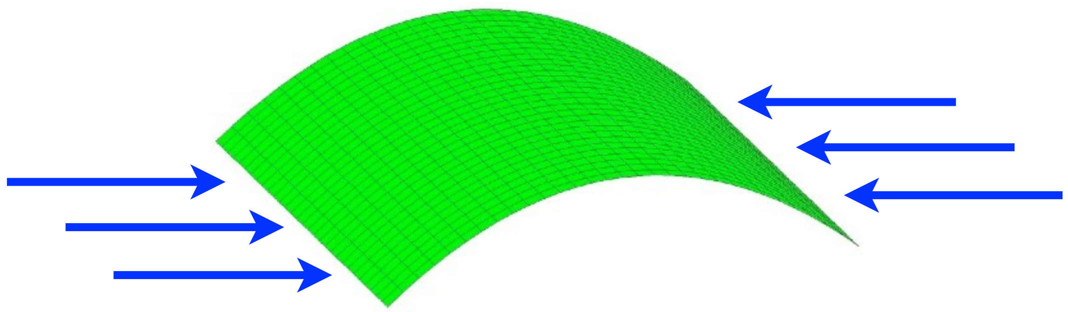
$$\lambda_0 = 70\,\mu\mathrm{m}$$

L. Pocivavsek et al., Science **320**, 912 (2008)

F. Brau et al., Soft Matter **9**, 8177 (2013)

Buckling vs wrinkling

Compressed thin sheets buckle



Compressed thin sheets on liquid and soft elastic substrates wrinkle

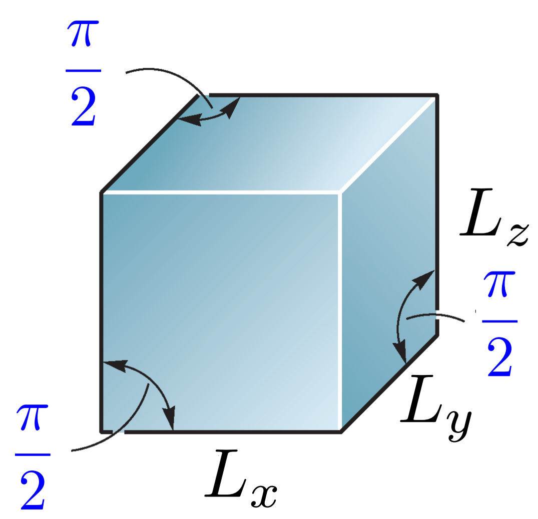
Liquid substrate

Elastic substrate

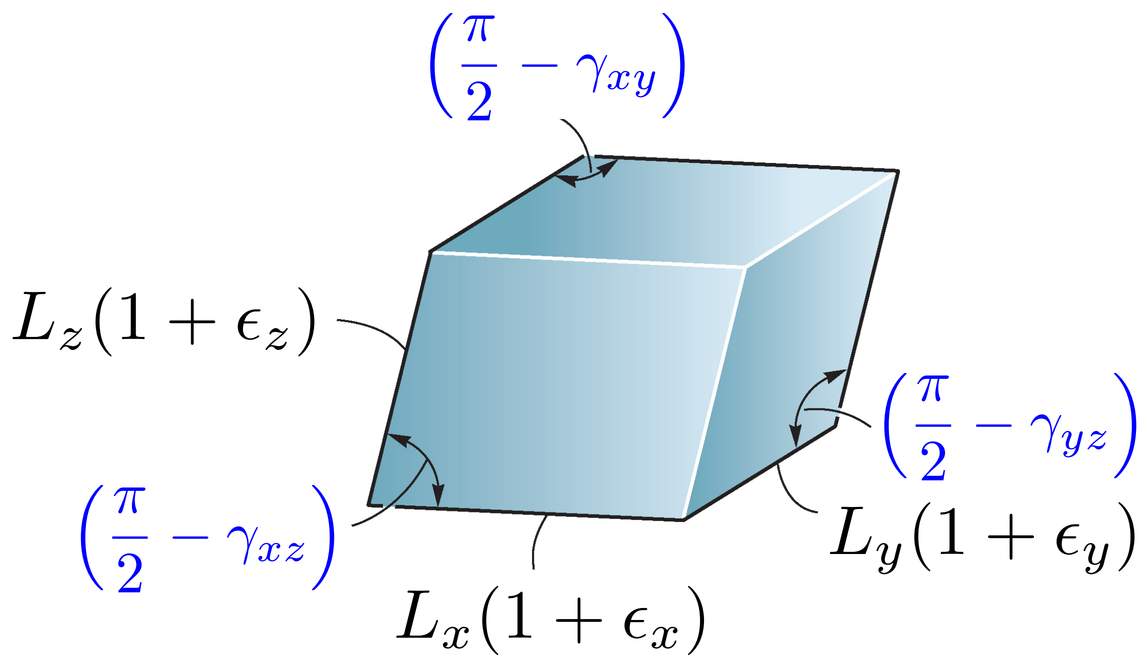
In compressed thin sheets on liquid and soft elastic substrates global buckling is suppressed, because it would result in very large energy cost associated with deformation of the liquid or soft elastic substrate!

Energy costs associated with deformation of 3D solid element

undeformed element



deformed element



$$\boxed{\text{energy} \sim \text{volume} \times \text{modulus} \times \text{strain}^2}$$

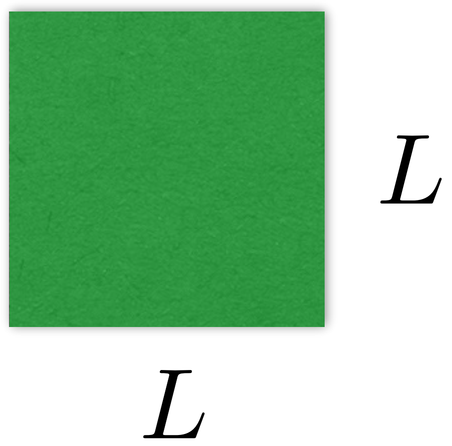
Young's modulus E

normal strain ϵ

shear strain γ

In plane deformations of thin sheets

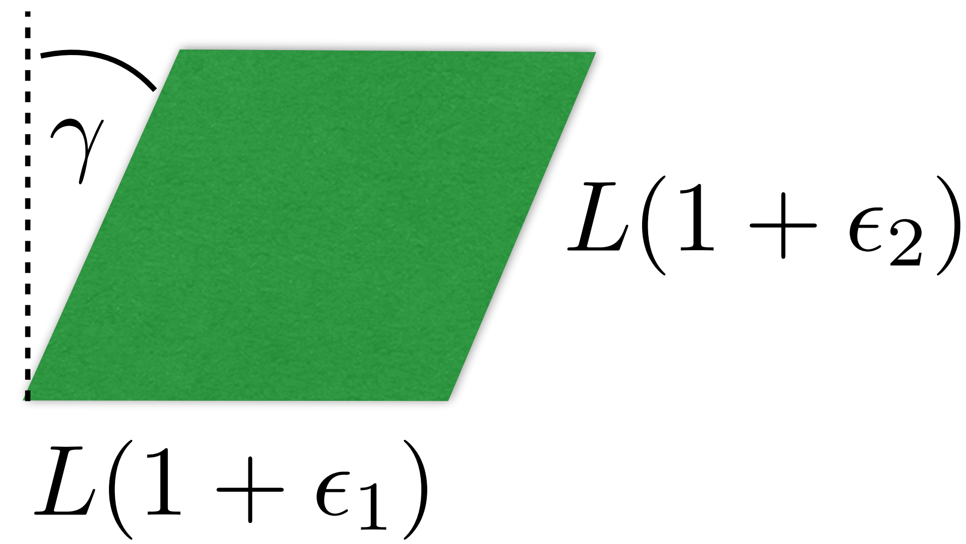
undeformed patch of thin sheet



sheet thickness t

Young's modulus E

deformed patch of thin sheet



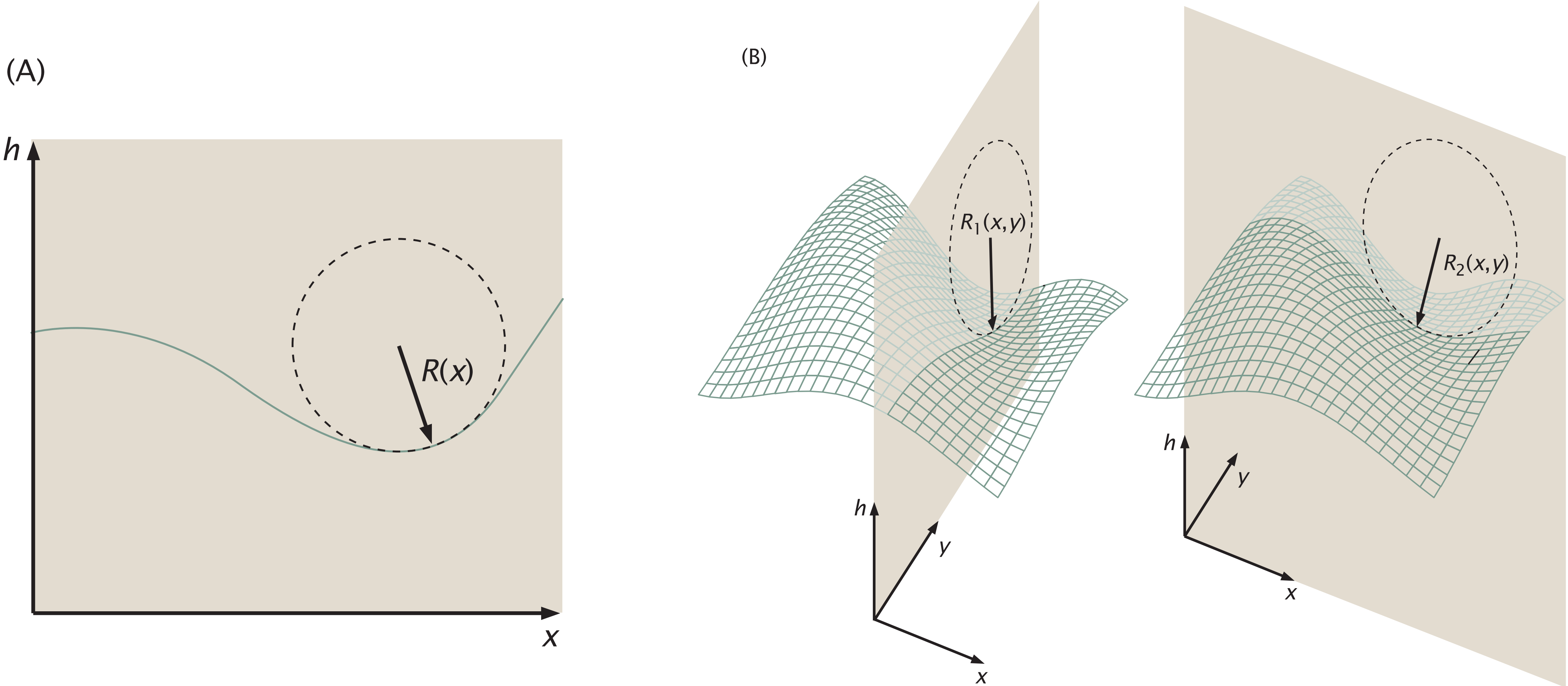
$$\boxed{\text{energy} \sim \text{area} \times \text{modulus} \times \text{strain}^2}$$

2D modulus Et

normal strain ϵ

shear strain γ

Curvature of surfaces



curvature of space curves

$$\frac{1}{R} = \frac{h''}{\left(1+h'^2\right)^{3/2}} \approx h''$$

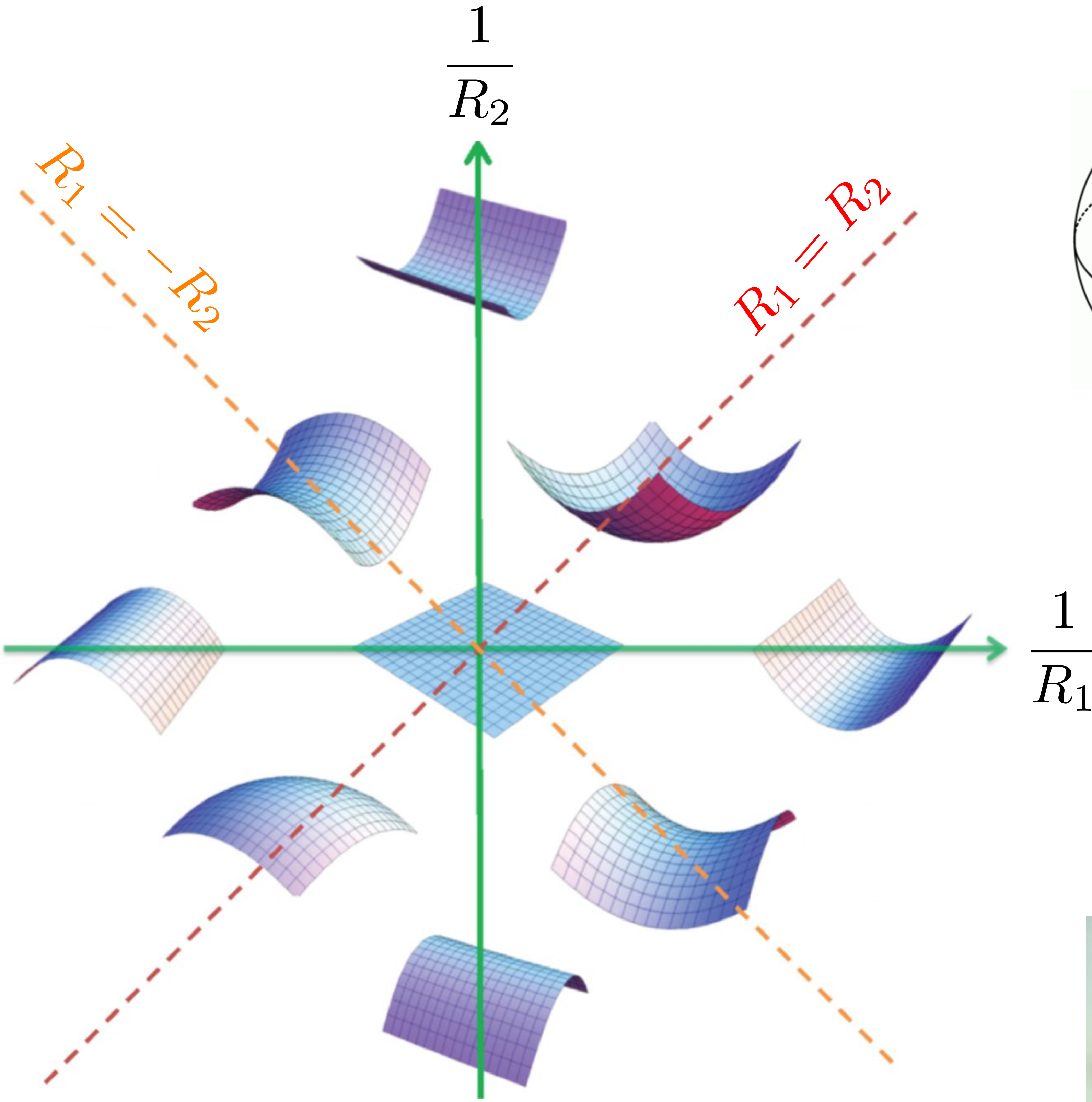
R. Phillips et al., Physical Biology of the Cell

curvature tensor for surfaces

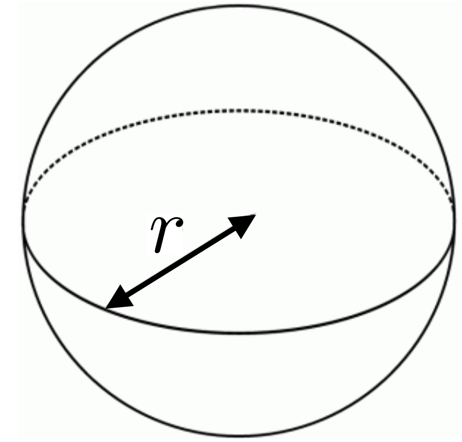
$$K_{ij} \approx \begin{pmatrix} \dfrac{\partial^2 h}{\partial x^2}, & \dfrac{\partial^2 h}{\partial x \partial y} \\ \dfrac{\partial^2 h}{\partial x \partial y}, & \dfrac{\partial^2 h}{\partial y^2} \end{pmatrix}$$

maximal and minimal curvatures (principal curvatures) correspond to the eigenvalues of curvature tensor

Surfaces of various principal curvatures

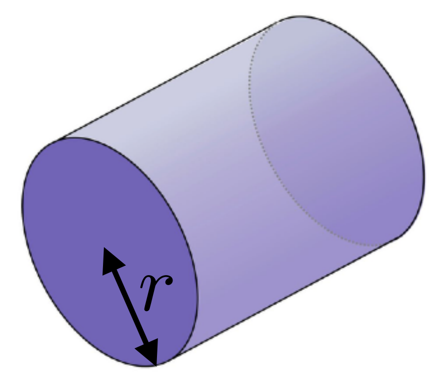


sphere



$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{r}$$

cylinder



$$\frac{1}{R_1} = \frac{1}{r}$$

$$\frac{1}{R_2} = 0$$

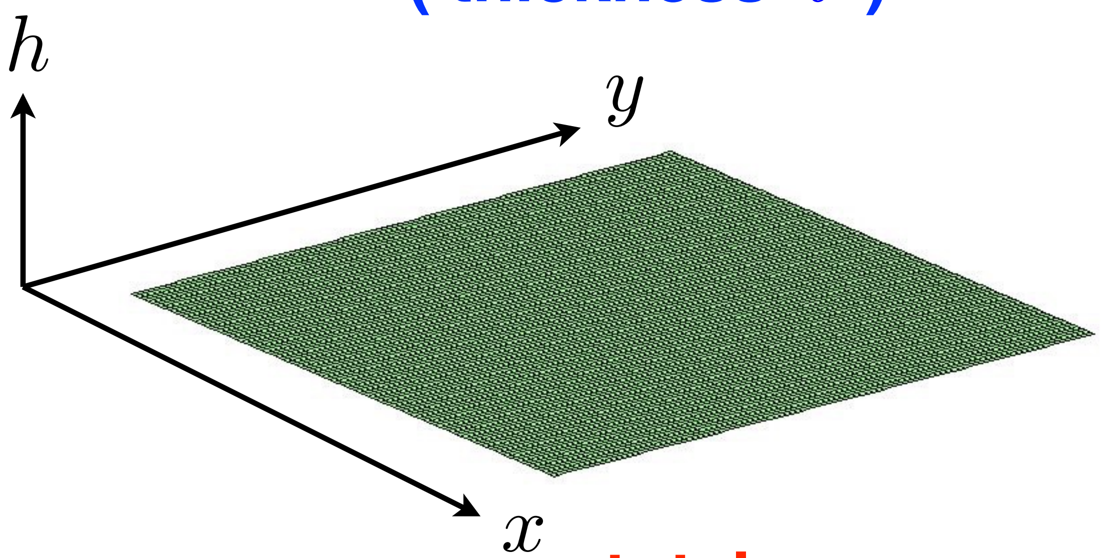
potato chips = "saddle"

$$\frac{1}{R_1} > 0$$

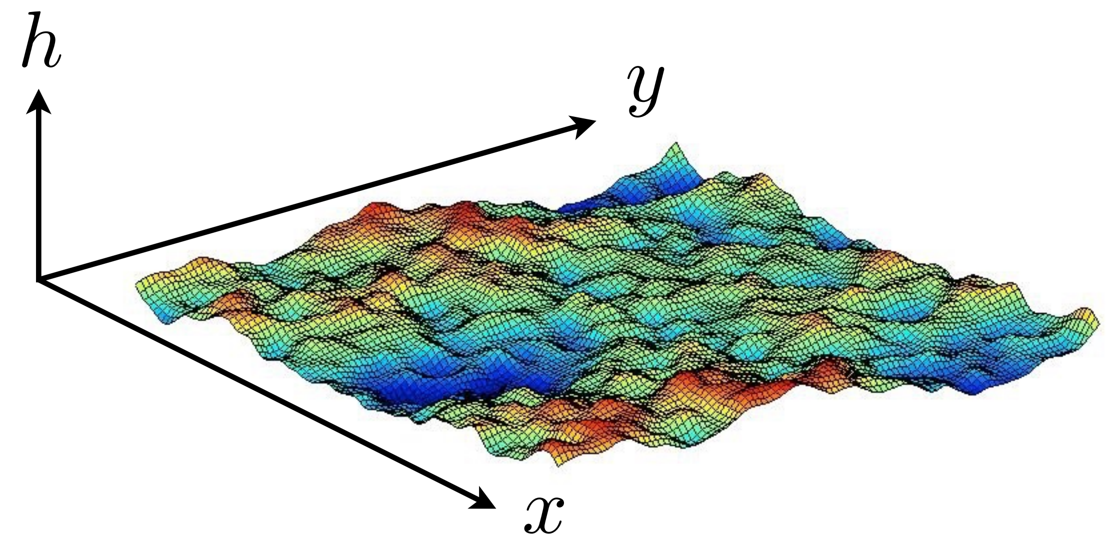
$$\frac{1}{R_2} < 0$$

Bending energy cost for thin sheets

undeformed thin sheet
(thickness t)



deformed thin sheet



total mean curvature **Gaussian curvature**

$$U = \int dA \left[\frac{\kappa}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^2 + \kappa_G \frac{1}{R_1 R_2} \right]$$

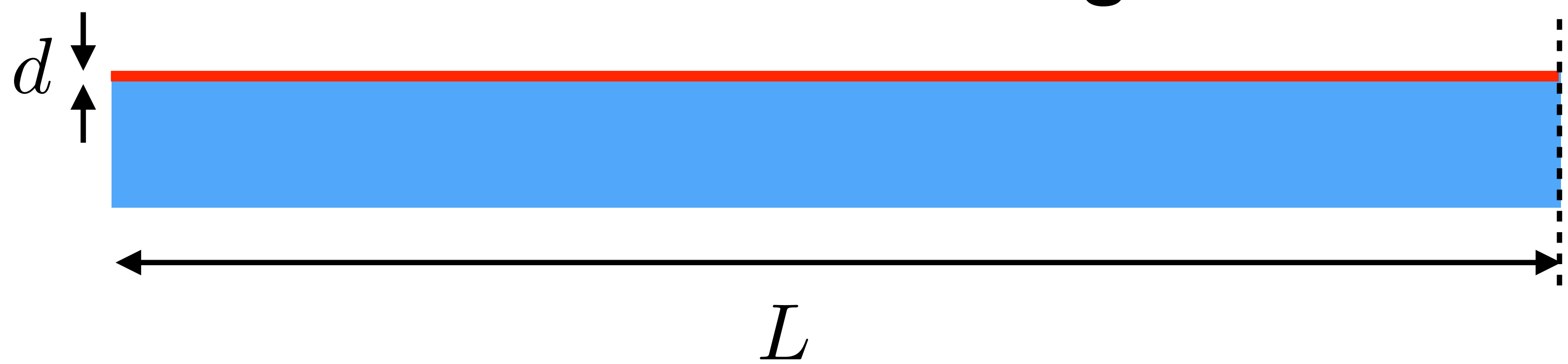
$$U \approx \int dx dy \left[\frac{\kappa}{2} \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right)^2 + \kappa_G \det \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right) \right] \quad x_i, x_j \in \{x, y\}$$

bending rigidity (flexural rigidity) $\kappa = \frac{Et^3}{12(1-\nu^2)}$

Gauss bending rigidity $\kappa_G = -\frac{Et^3}{12(1+\nu)}$

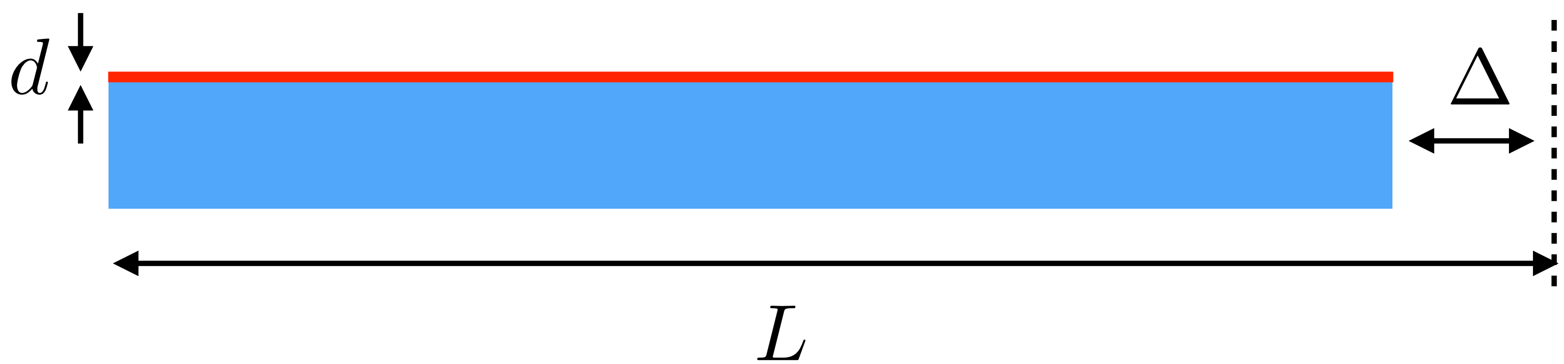
Compression of stiff thin membranes on liquid substrates

initial undeformed configuration



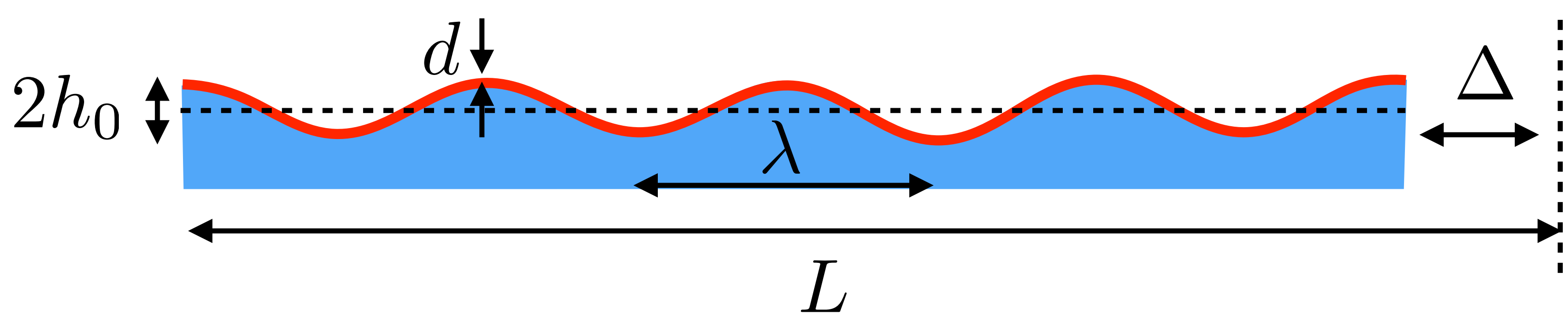
Consider the energy cost for two different scenarios:

1.) thin membrane is compressed (no bending)

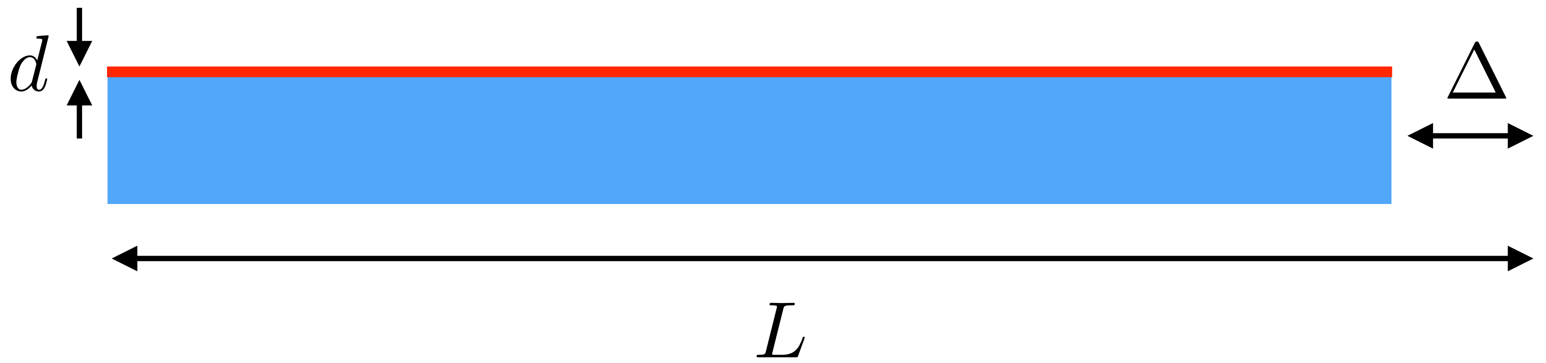


2.) thin membrane is wrinkled (no compression)

+ additional potential energy of liquid



Compression of stiff thin membranes on liquid substrates



compression energy of thin membrane

$$\boxed{U_c \sim A \times E_m d \times \epsilon^2}$$

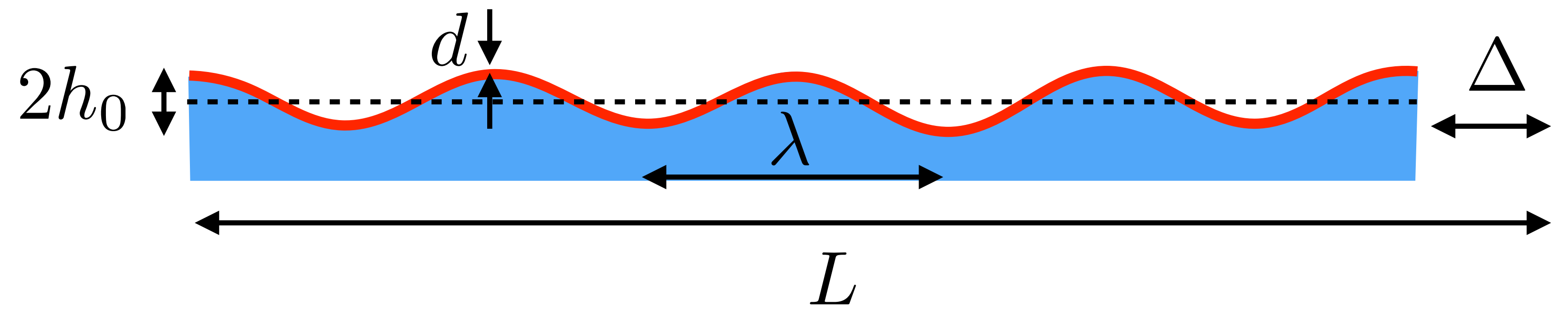
membrane area	membrane 3D Young's modulus	strain	liquid density
$A = WL$	E_m	$\epsilon = \dfrac{\Delta}{L}$	ρ

Note: upon compression the liquid surface also raises, but we will measure the potential energy relative to this new height!

Compression of stiff thin membranes on liquid substrates

assumed profile

$$h(s) = h_0 \cos(2\pi s/\lambda)$$



projected length assuming that membrane doesn't stretch

$$L - \Delta = \int_0^L ds \sqrt{1 - h'(s)^2} \approx \int_0^L ds\, (1 - h'(s)^2/2) \approx L\left(1 - \frac{\pi^2 h_0^2}{\lambda^2}\right)$$

amplitude of wrinkles

$$\boxed{h_0 = \frac{\lambda}{\pi}\sqrt{\frac{\Delta}{L}} = \frac{\lambda\sqrt{\epsilon}}{\pi}}$$

bending energy of stiff membrane

$$U_b \sim A \times \kappa \times \frac{1}{R^2} \sim A \times E_m d^3 \times \frac{h_0^2}{\lambda^4} \sim \frac{A E_m d^3 \epsilon}{\lambda^2}$$

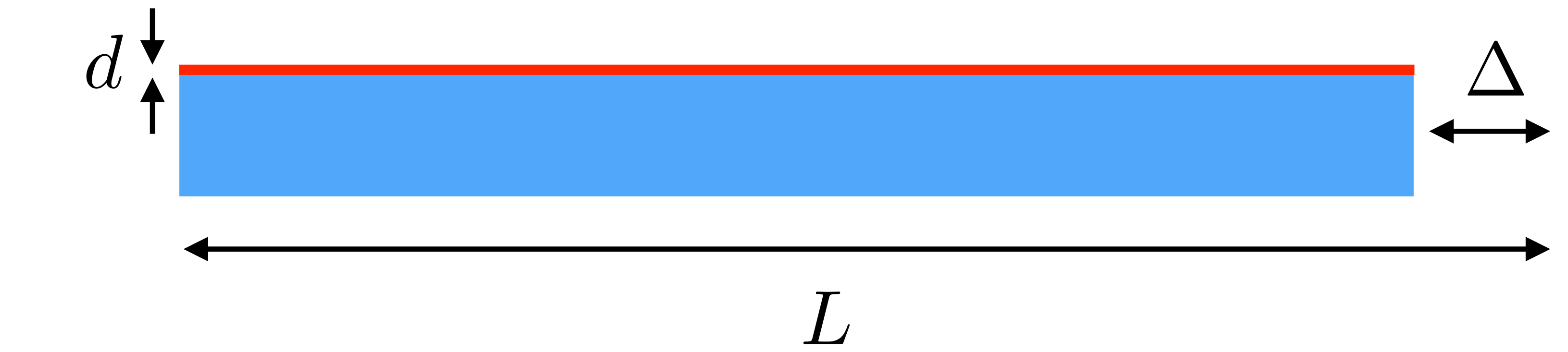
potential energy of liquid

$$U_p \sim m \times g \times \Delta h \sim \rho \times A h_0 \times g \times h_0 \sim A \rho g \lambda^2 \epsilon$$

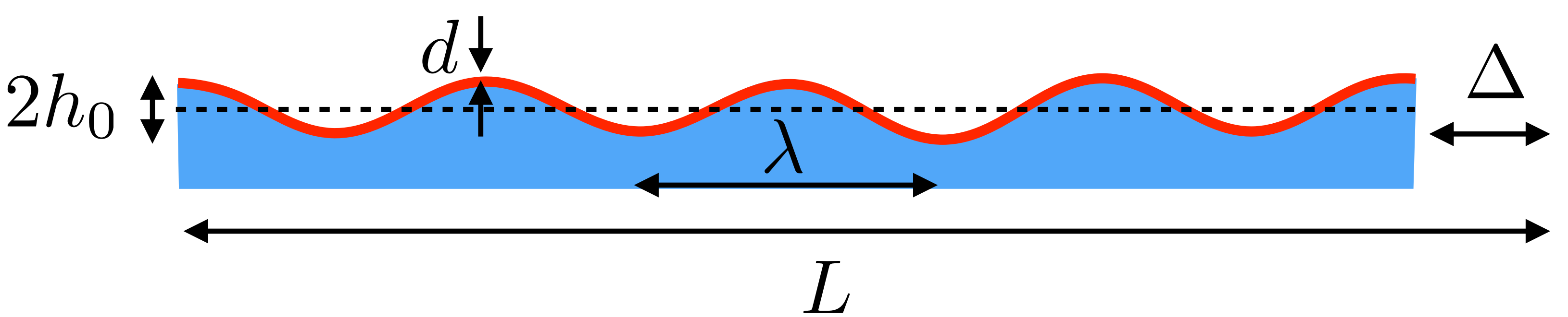
minimize total energy (U_b+U_p) with respect to λ

$$\boxed{\lambda \sim \left(\frac{E_m d^3}{\rho g}\right)^{1/4}} \quad \longrightarrow \quad \boxed{U_b, U_p \sim A\epsilon\sqrt{E_m d^3 \rho g}}$$

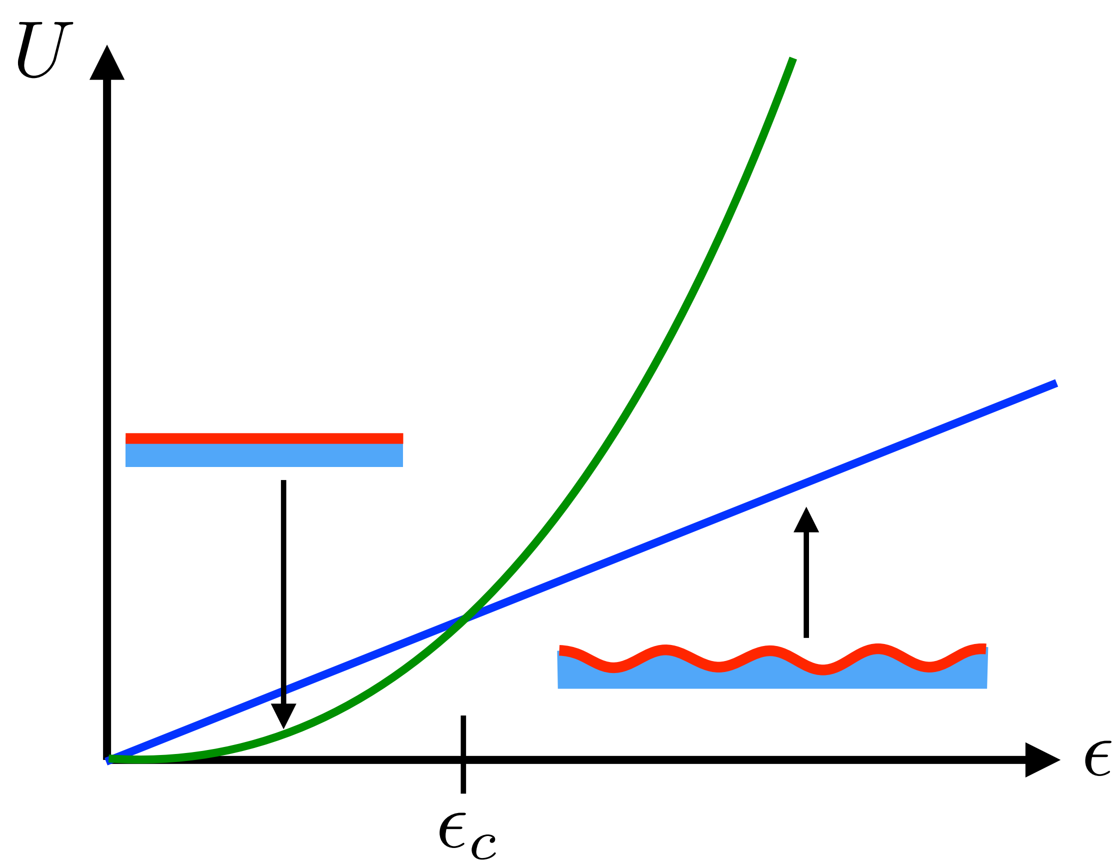
Compression of stiff thin membranes on liquid substrates



$$U_c \sim A \times E_m d \times \epsilon^2$$



$$U_b, U_p \sim A\epsilon\sqrt{E_m d^3 \rho g}$$



wrinkles are stable above the critical strain

$$\boxed{\epsilon > \epsilon_c \sim \sqrt{\frac{\rho g d}{E_m}}}$$

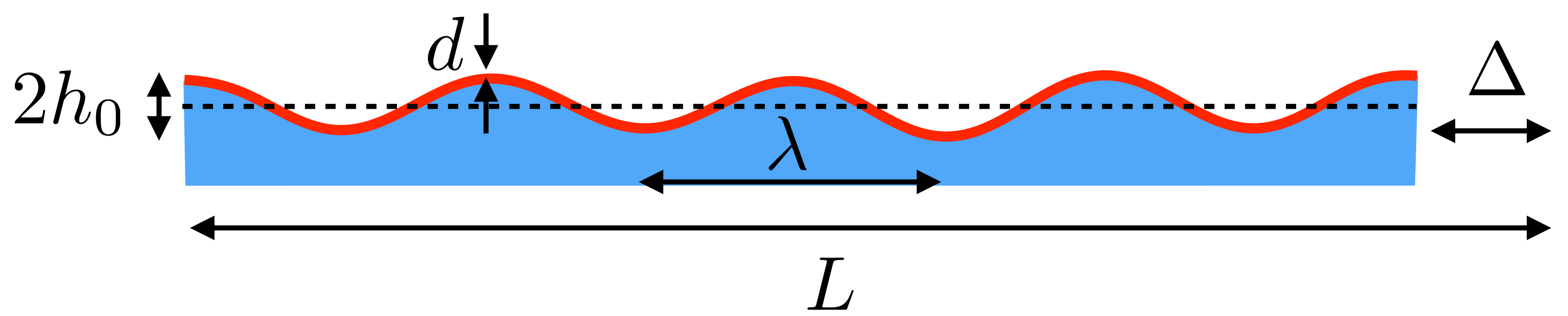
wavelength of wrinkles

$$\boxed{\lambda \sim \left(\frac{E_m d^3}{\rho g}\right)^{1/4}}$$

amplitude of wrinkles at the critical strain

$$\boxed{h_0^* \sim \lambda\sqrt{\epsilon_c} \sim d}$$

Compression of stiff thin membranes on liquid substrates



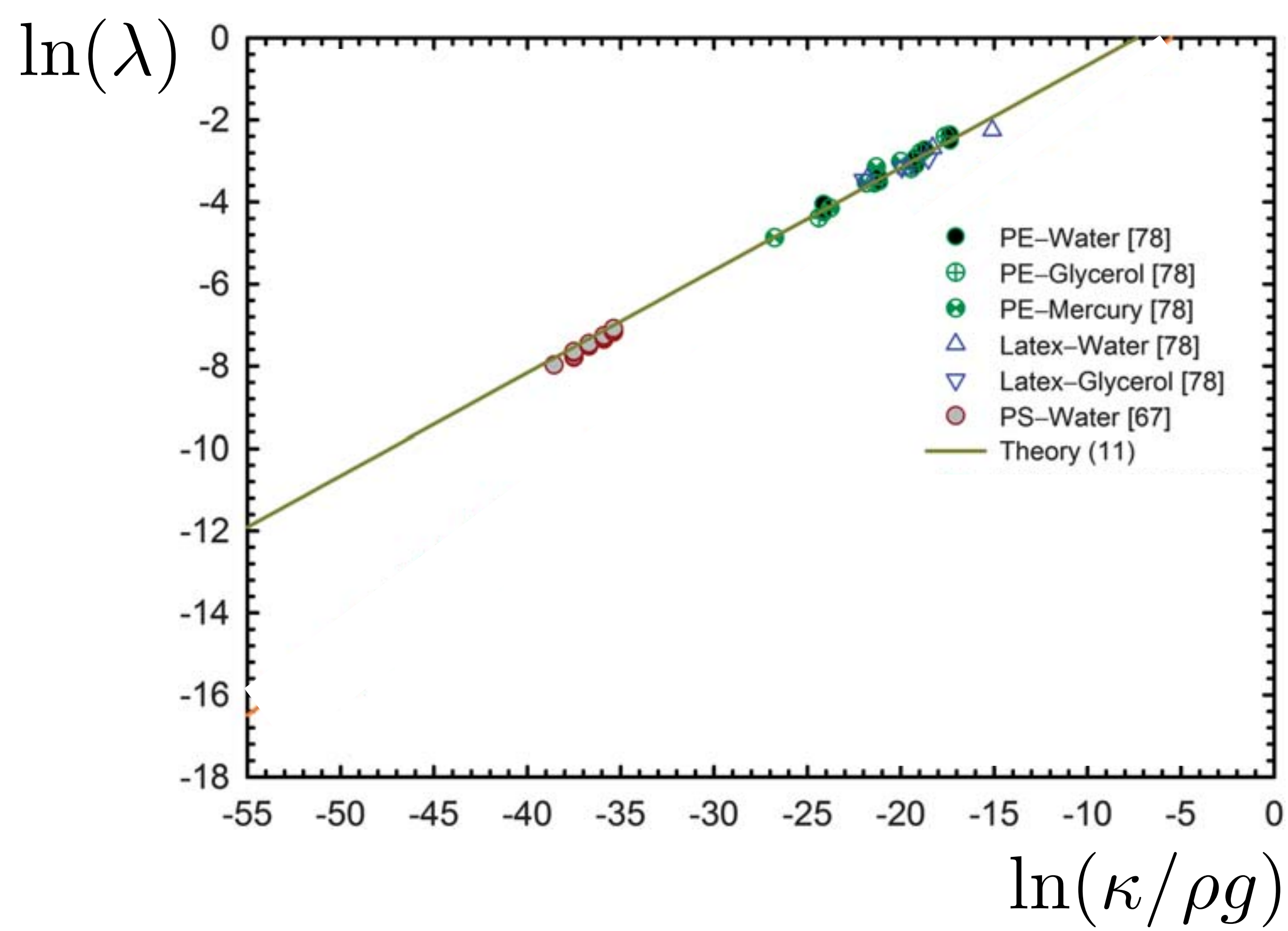
scaling analysis

$$\boxed{\lambda \sim \left(\frac{E_m d^3}{\rho g}\right)^{1/4}}$$

exact result

$$\boxed{\lambda = 2\pi \left(\frac{\kappa}{\rho g}\right)^{1/4}}$$

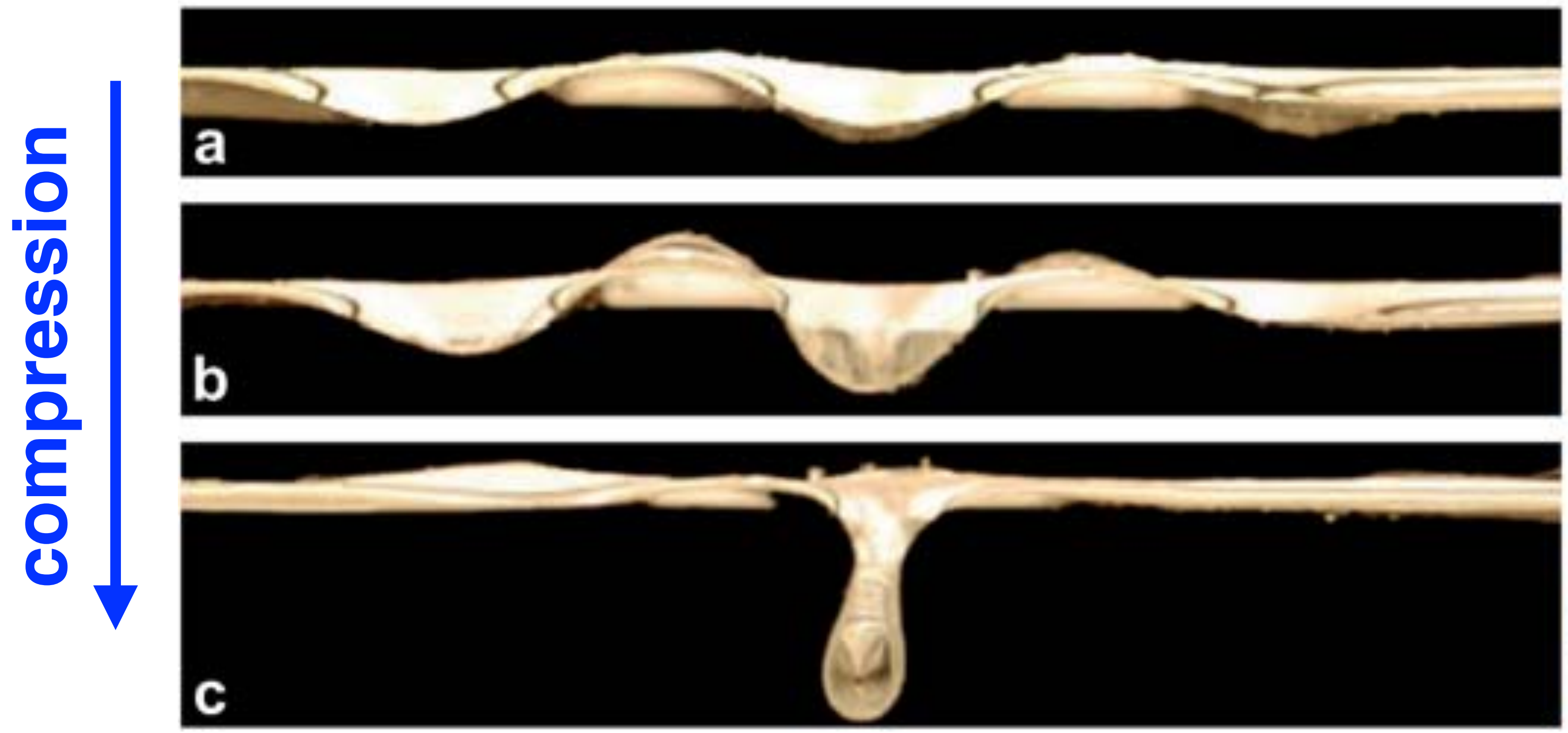
$$\kappa = \frac{E_m d^3}{12(1-\nu_m^2)}$$



F. Brau et al., Soft Matter **9**, 8177 (2013)

Compression of stiff thin membranes on liquid substrates

How to go beyond the simple scaling analysis to determine the nonlinear post-buckling behavior?



Find shape profile h(s) that minimizes total energy

$$U_b + U_p = W \int_0^L \frac{ds}{2} \left[\frac{\kappa h''^2}{(1+h'^2)^3} + \rho g h^2 \sqrt{1-h'^2} \right]$$

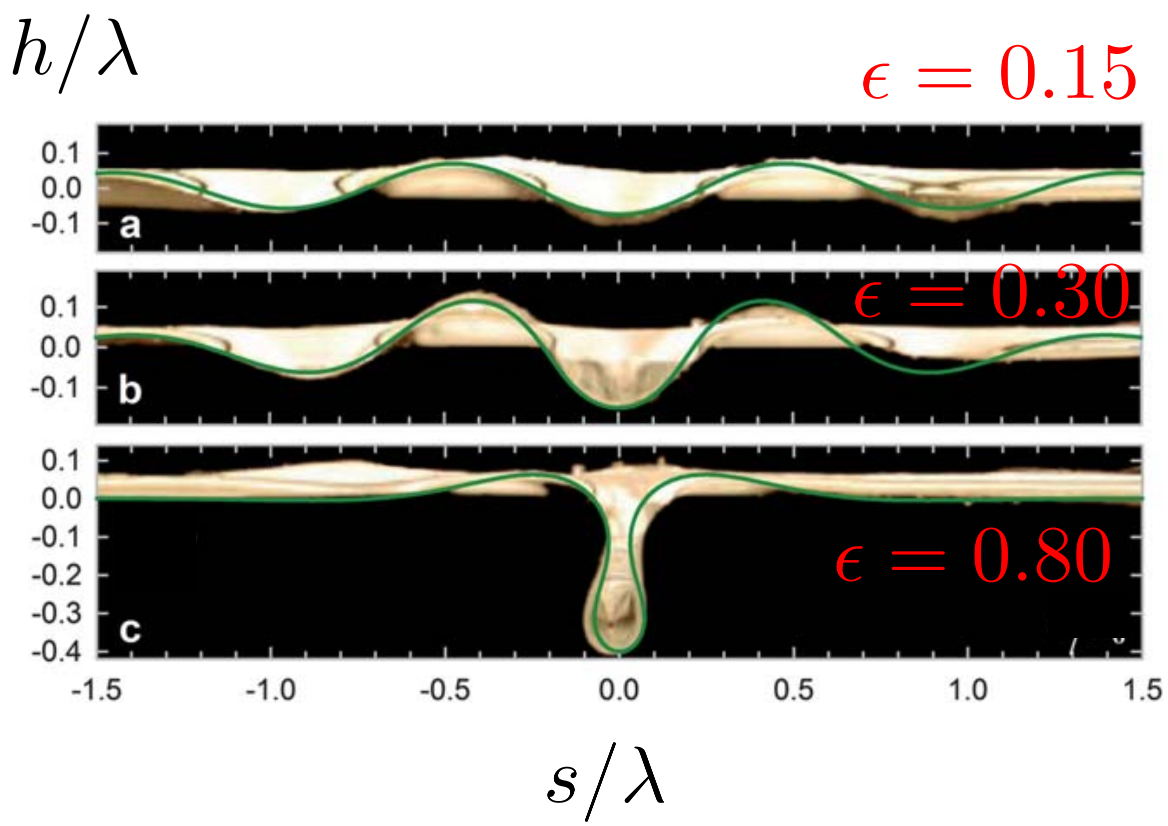
subject to constraint

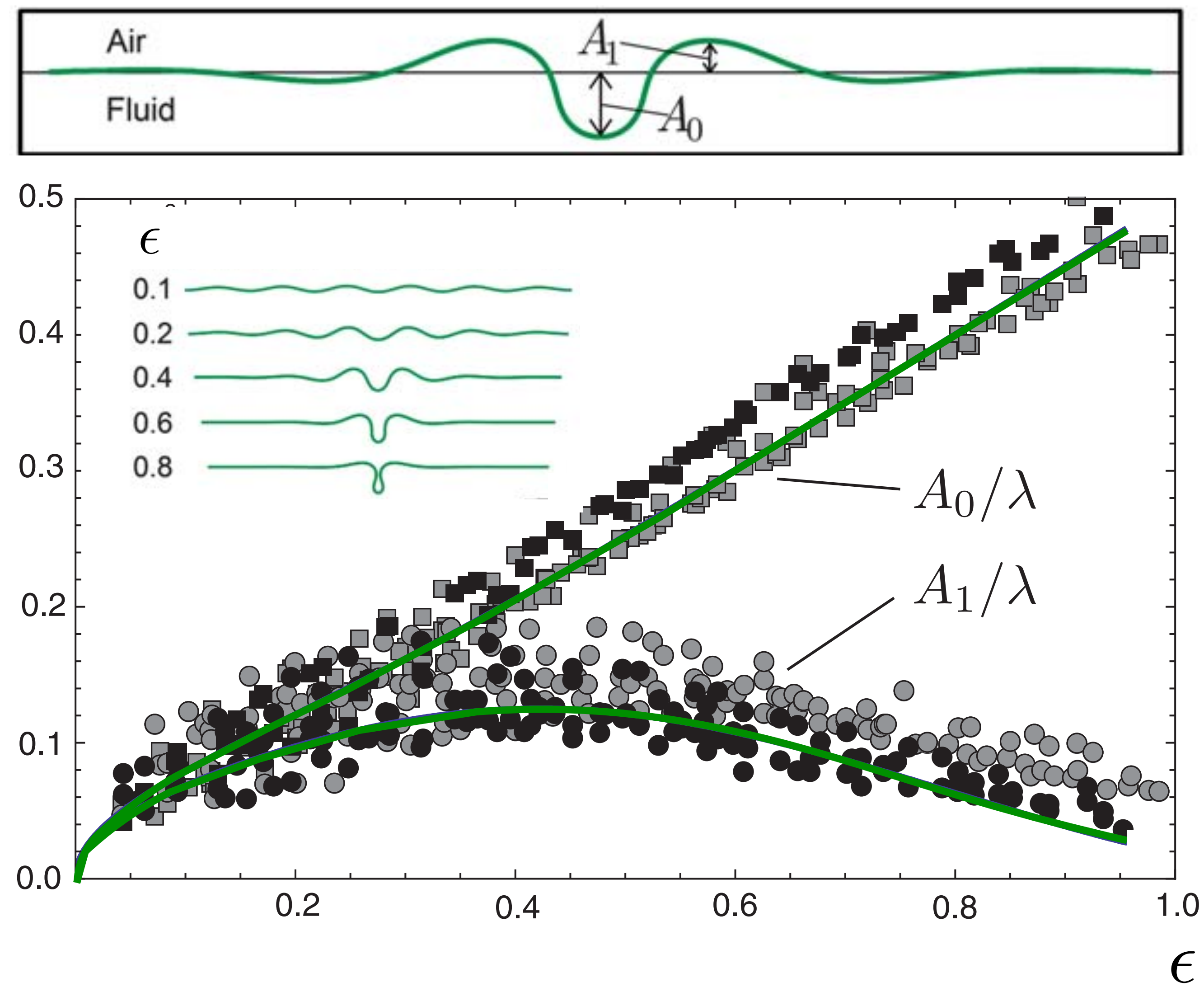
$$L - \Delta = \int_0^L ds \sqrt{1-h'^2}$$

F. Brau et al., Soft Matter **9**, 8177 (2013)

Compression of stiff thin membranes on liquid substrates

Comparison between theory (infinite membrane) and experiment



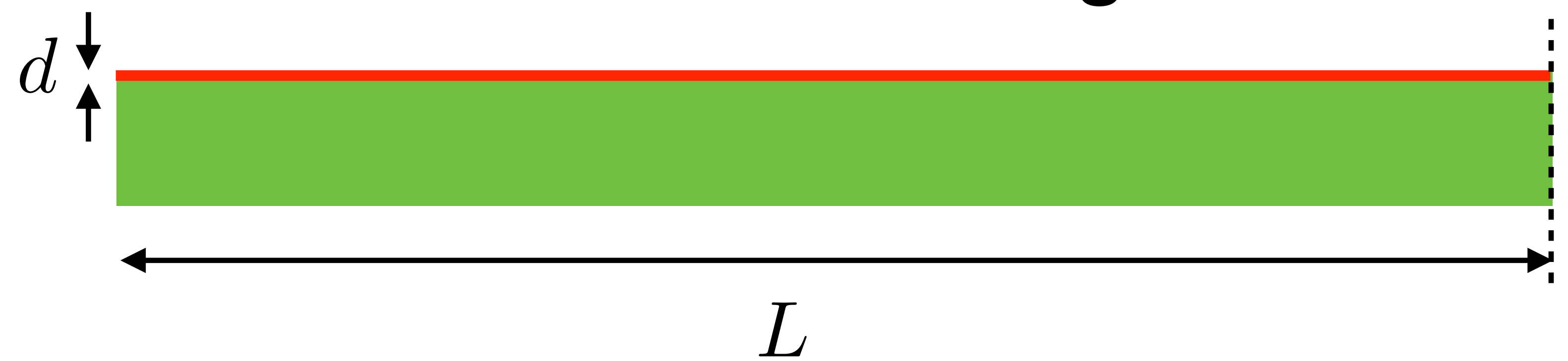


L. Pocivavsek et al., Science **320**, 912 (2008)

F. Brau et al., Soft Matter **9**, 8177 (2013)

Compression of stiff thin membranes on soft elastic substrates

initial undeformed configuration



Consider the energy cost for two different scenarios:

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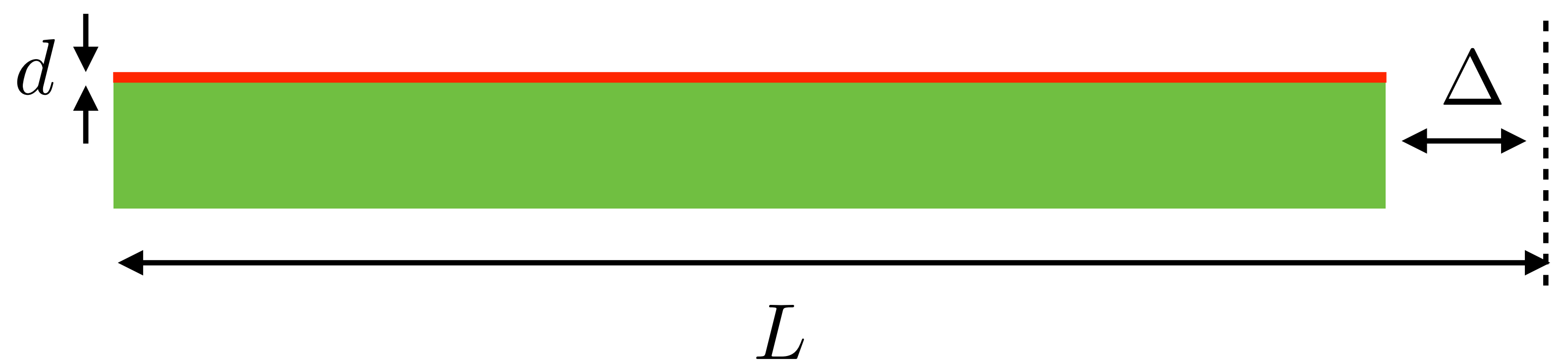
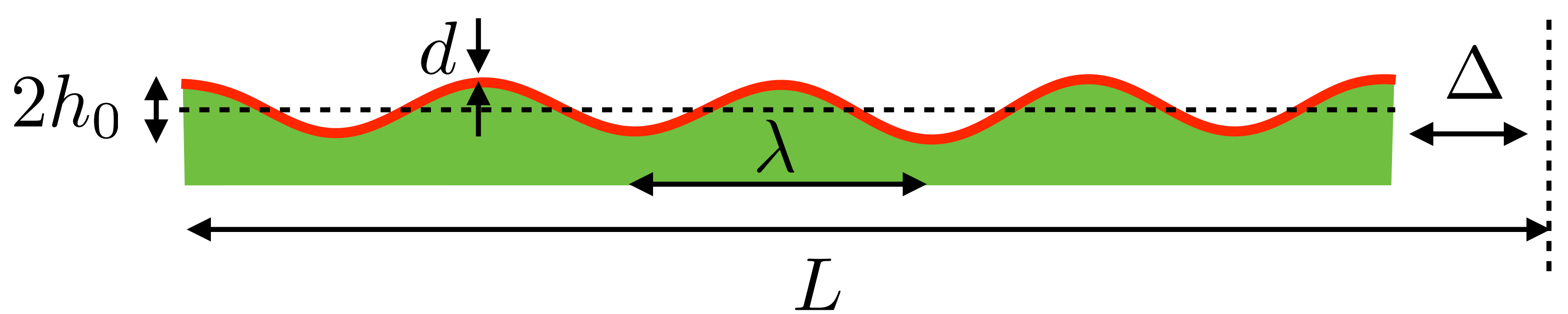
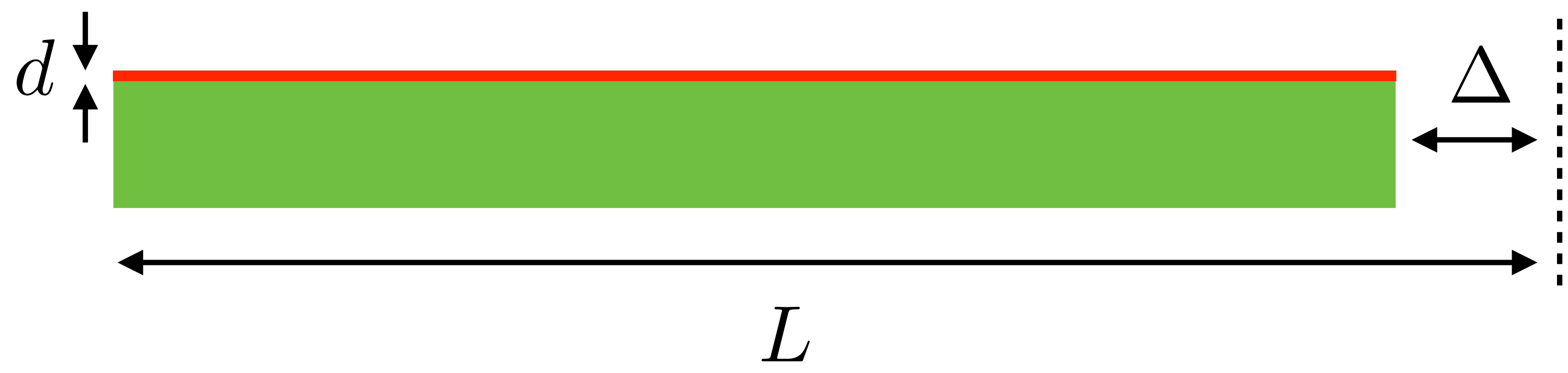


2.) thin membrane is wrinkled (no compression)

additional elastic energy for deformed substrate



Compression of stiff thin membranes on soft elastic substrates



compression energy of thin membrane

$$\boxed{U_c \sim A \times E_m d \times \epsilon^2}$$

membrane area	membrane 3D Young's modulus	strain	substrate 3D Young's modulus
$A = WL$	E_m	$\epsilon = \dfrac{\Delta}{L}$	E_s

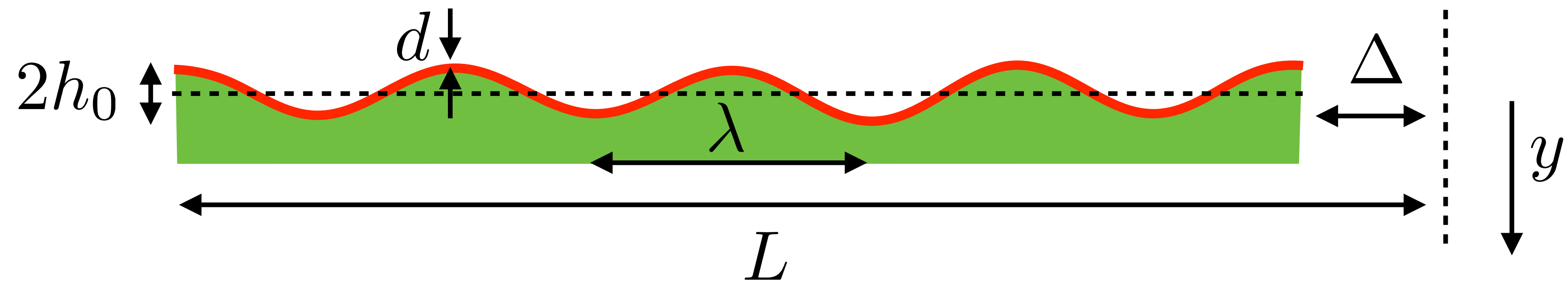
$$E_s \ll E_m$$

Note: soft elastic substrate is also compressed, but we will measure the substrate elastic energy relative to this base value!

Compression of stiff thin membranes on soft elastic substrates

assumed profile

$$h(s) = h_0 \cos(2\pi s/\lambda)$$

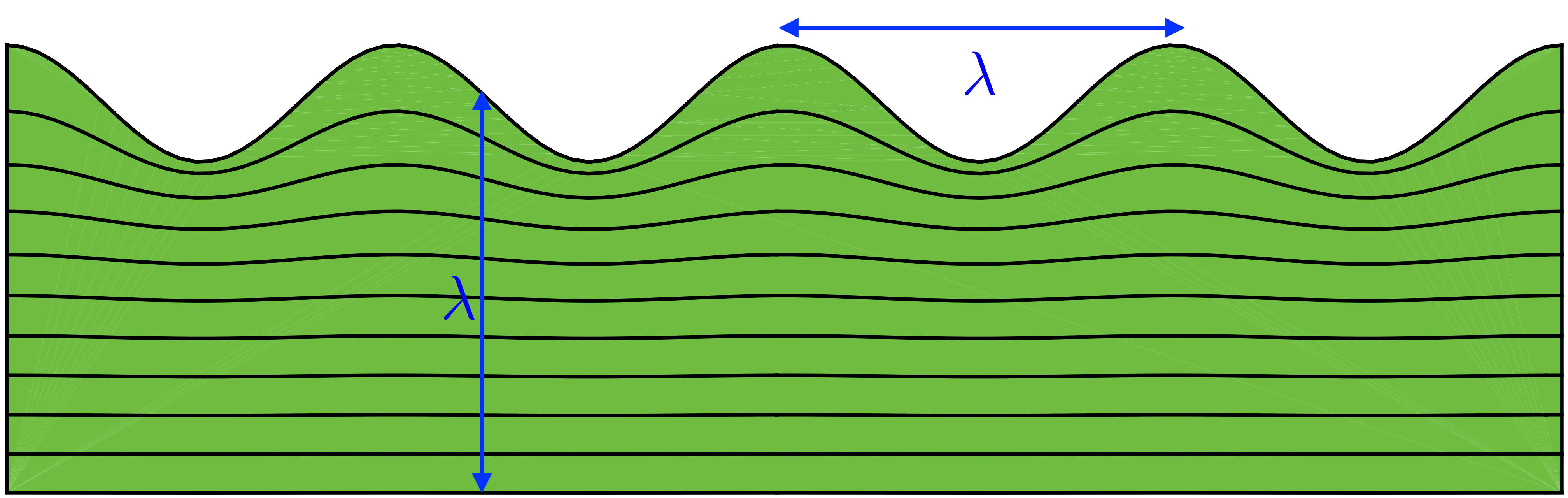


amplitude of wrinkles

$$\boxed{h_0 = \frac{\lambda}{\pi}\sqrt{\frac{\Delta}{L}} = \frac{\lambda\sqrt{\epsilon}}{\pi}}$$

deformation of the soft substrate decays exponentially away from the surface

$$h(s,y) \approx h_0 \cos(2\pi s/\lambda) e^{-2\pi y/\lambda}$$

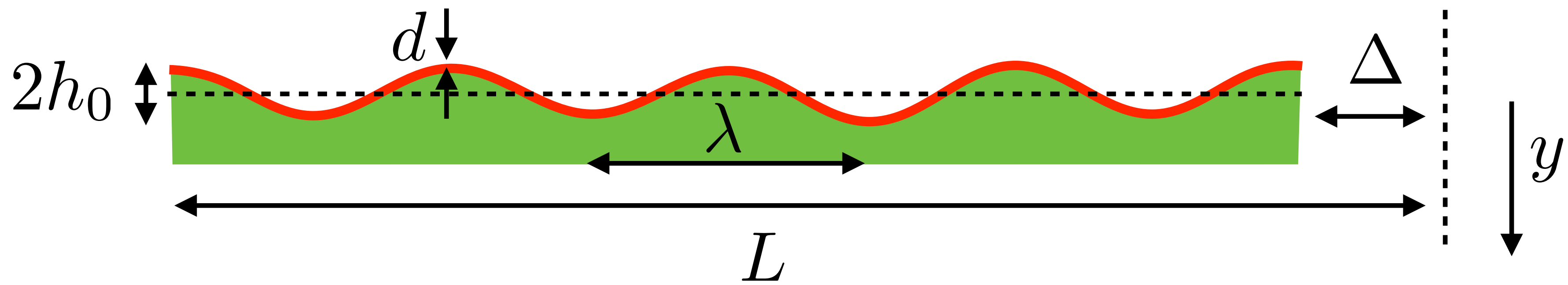


F. Brau et al., Nat. Phys. **7**, 56 (2010)

Compression of stiff thin membranes on soft elastic substrates

assumed profile

$$h(s) = h_0 \cos(2\pi s/\lambda)$$



amplitude of wrinkles

$$\boxed{h_0 = \frac{\lambda}{\pi}\sqrt{\frac{\Delta}{L}} = \frac{\lambda\sqrt{\epsilon}}{\pi}}$$

deformation of the soft substrate decays exponentially away from the surface

$$h(s,y) \approx h_0 \cos(2\pi s/\lambda) e^{-2\pi y/\lambda}$$

bending energy of stiff membrane

$$U_b \sim A \times \kappa \times \frac{1}{R^2} \sim A \times E_m d^3 \times \frac{h_0^2}{\lambda^4} \sim \frac{A E_m d^3 \epsilon}{\lambda^2}$$

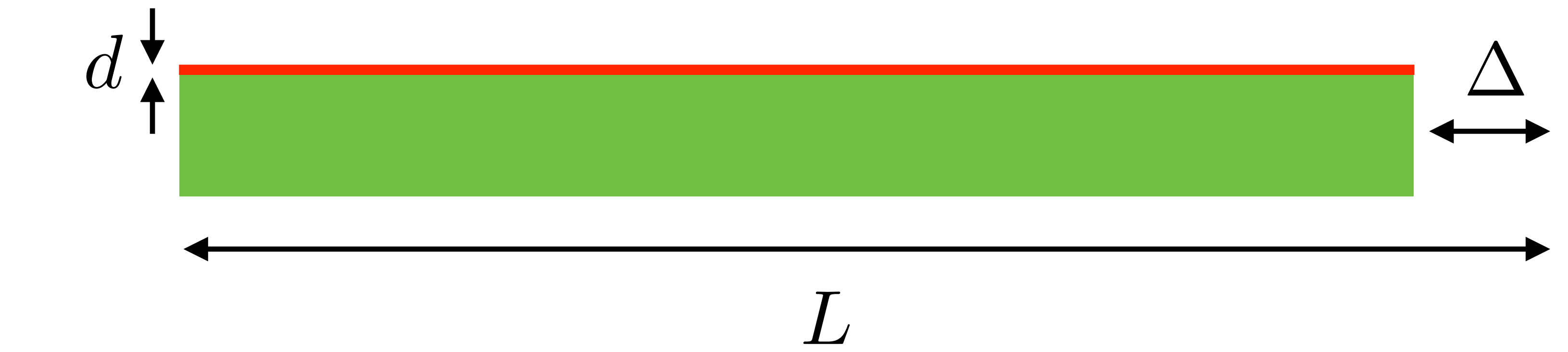
deformation energy of soft substrate

$$U_s \sim V \times E_s \times \epsilon_s^2 \sim A\lambda \times E_s \times \frac{h_0^2}{\lambda^2} \sim A E_s \lambda \epsilon$$

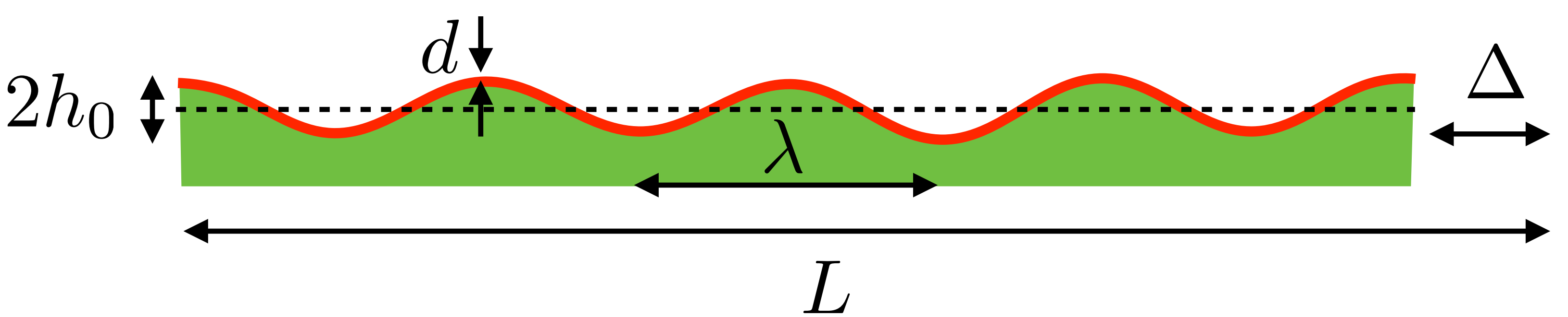
minimize total energy (U_b+U_s) with respect to λ

$$\rightarrow \boxed{\lambda \sim d\left(\frac{E_m}{E_s}\right)^{1/3}} \rightarrow \boxed{U_b, U_s \sim A d\epsilon \left(E_s^2 E_m\right)^{1/3}}$$

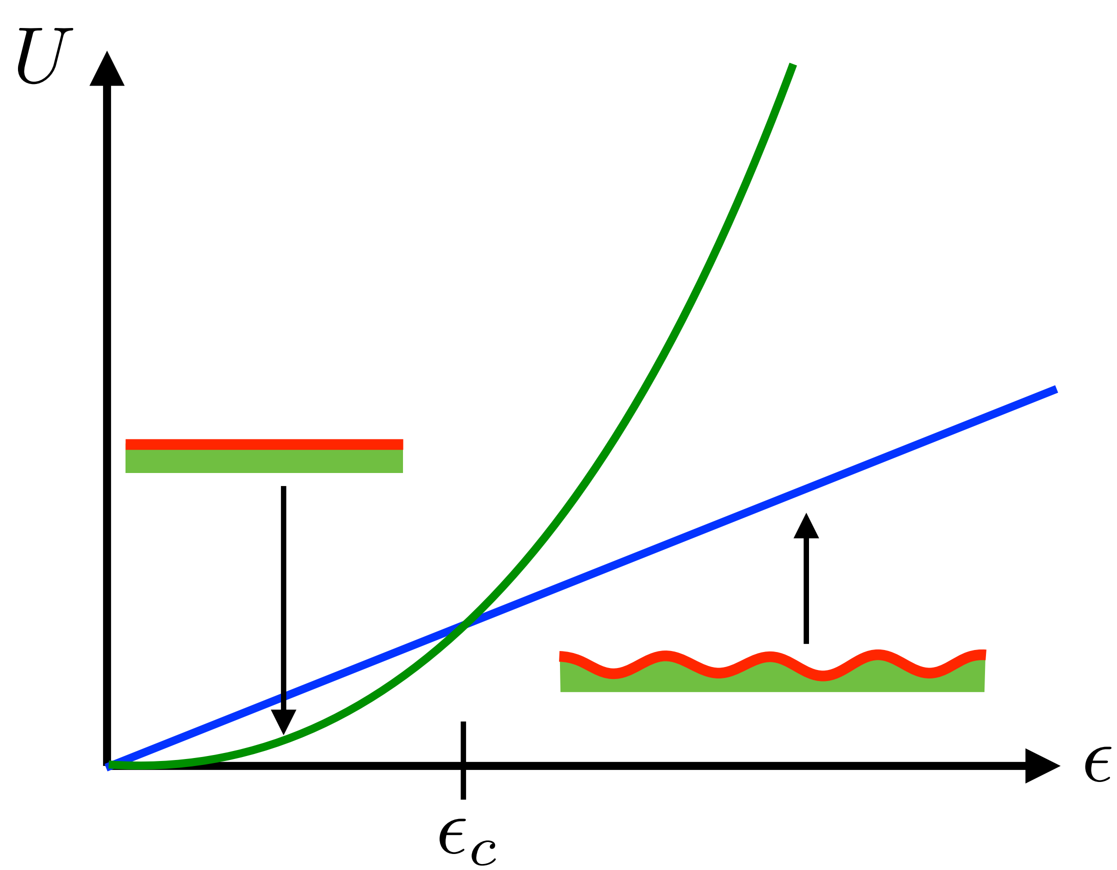
Compression of stiff thin membranes on soft elastic substrates



$$U_c \sim A \times E_m d \times \epsilon^2$$



$$U_b, U_s \sim A d\epsilon \left(E_s^2 E_m\right)^{1/3}$$



wrinkles are stable for large strains

$$\boxed{\epsilon > \epsilon_c \sim \left(\frac{E_s}{E_m}\right)^{2/3}}$$

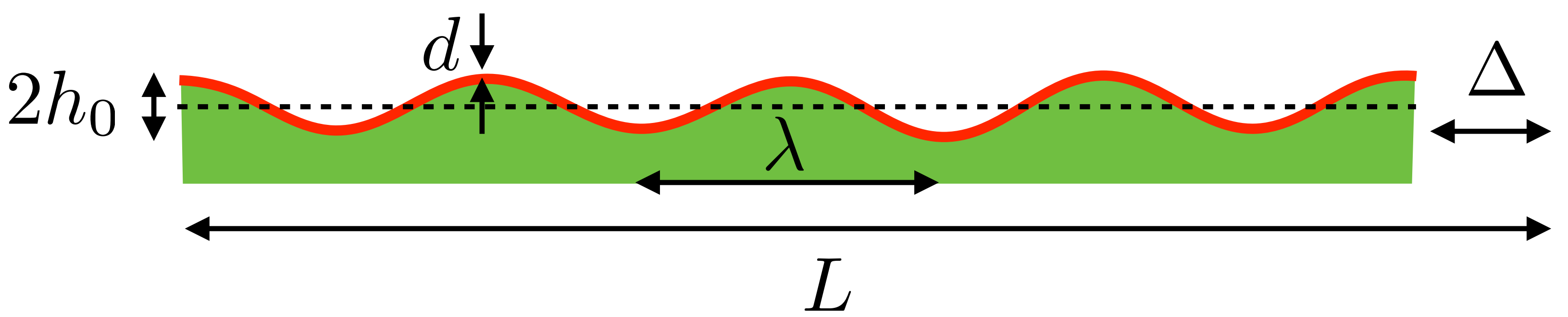
wavelength of wrinkles

$$\boxed{\lambda \sim d \left(\frac{E_m}{E_s}\right)^{1/3}}$$

amplitude of wrinkles at the critical strain

$$\boxed{h_0^* \sim \lambda \sqrt{\epsilon_c} \sim d}$$

Compression of stiff thin membranes on liquid and soft elastic substrates

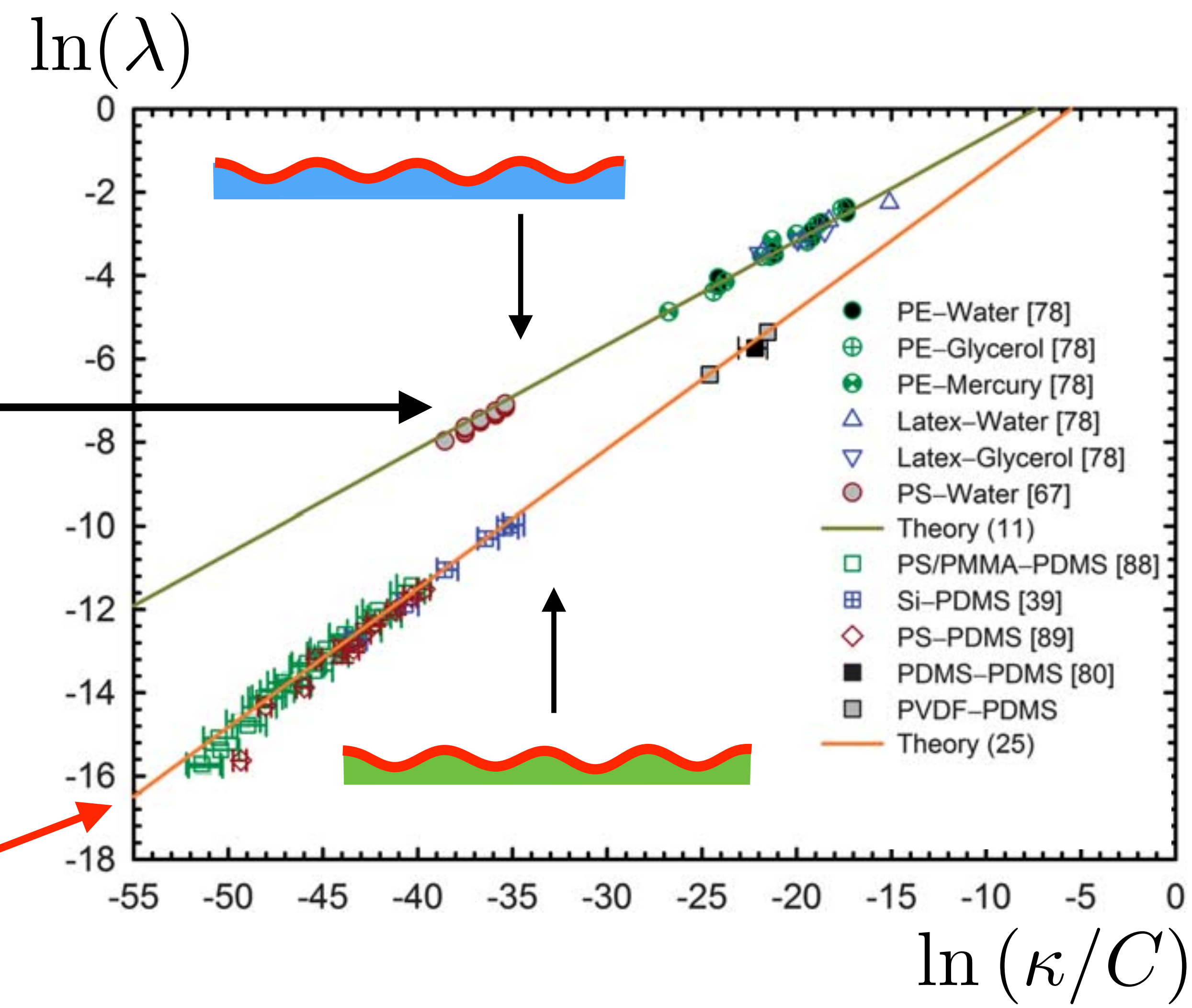


wavelength of wrinkles on liquid substrates

$$\lambda = 2\pi \left(\frac{\kappa}{\rho g} \right)^{1/4}$$

wavelength of wrinkles on soft elastic substrates

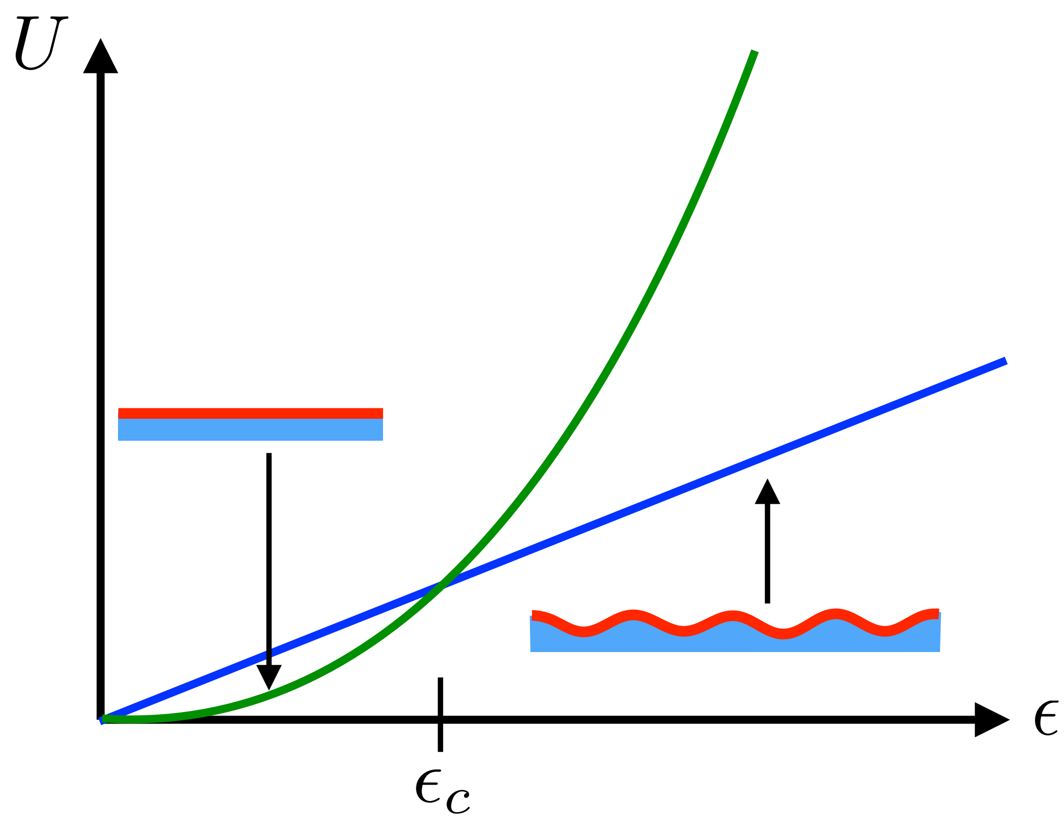
$$\lambda = 2\pi \left(\frac{3\kappa}{E_s} \right)^{1/3}$$



F. Brau et al., Soft Matter **9**, 8177 (2013)

Compression of stiff thin sheets on liquid and soft elastic substrates

liquid substrate



wrinkles are stable above the critical strain

$$\epsilon > \epsilon_c \sim \sqrt{\frac{\rho g d}{E_m}}$$

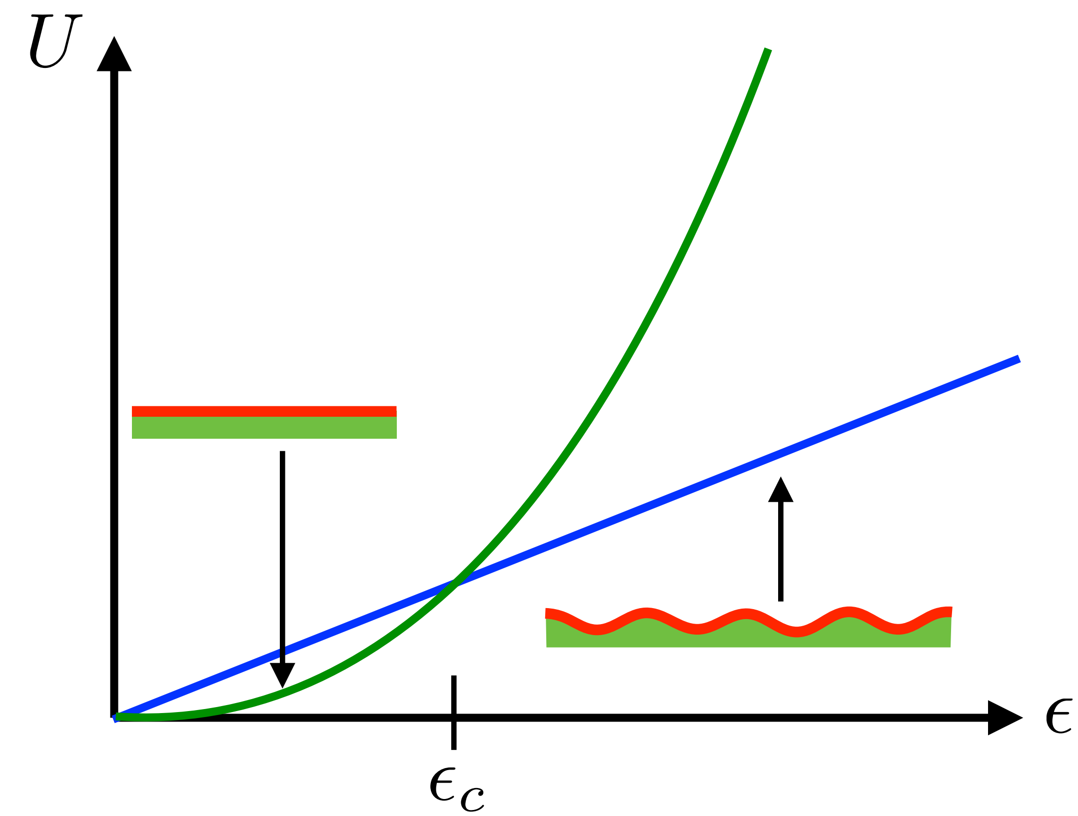
wavelength of wrinkles

$$\lambda \sim \left(\frac{E_m d^3}{\rho g}\right)^{1/4}$$

amplitude of wrinkles

$$h_0 \sim \lambda\sqrt{\epsilon}$$

soft elastic substrate $E_s \ll E_m$



wrinkles are stable for large strains

$$\epsilon > \epsilon_c \sim \left(\frac{E_s}{E_m}\right)^{2/3}$$

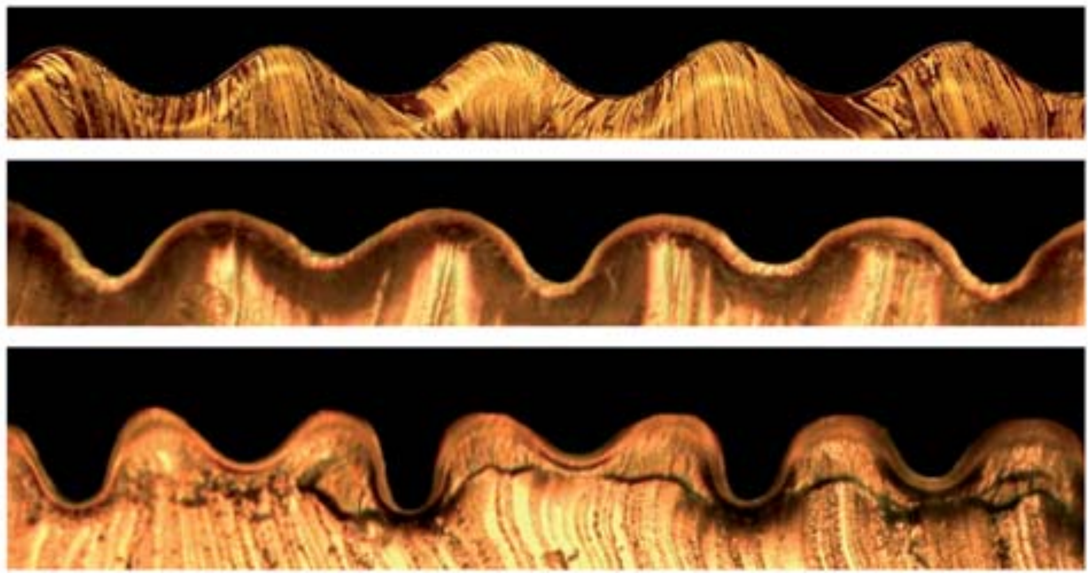
wavelength of wrinkles

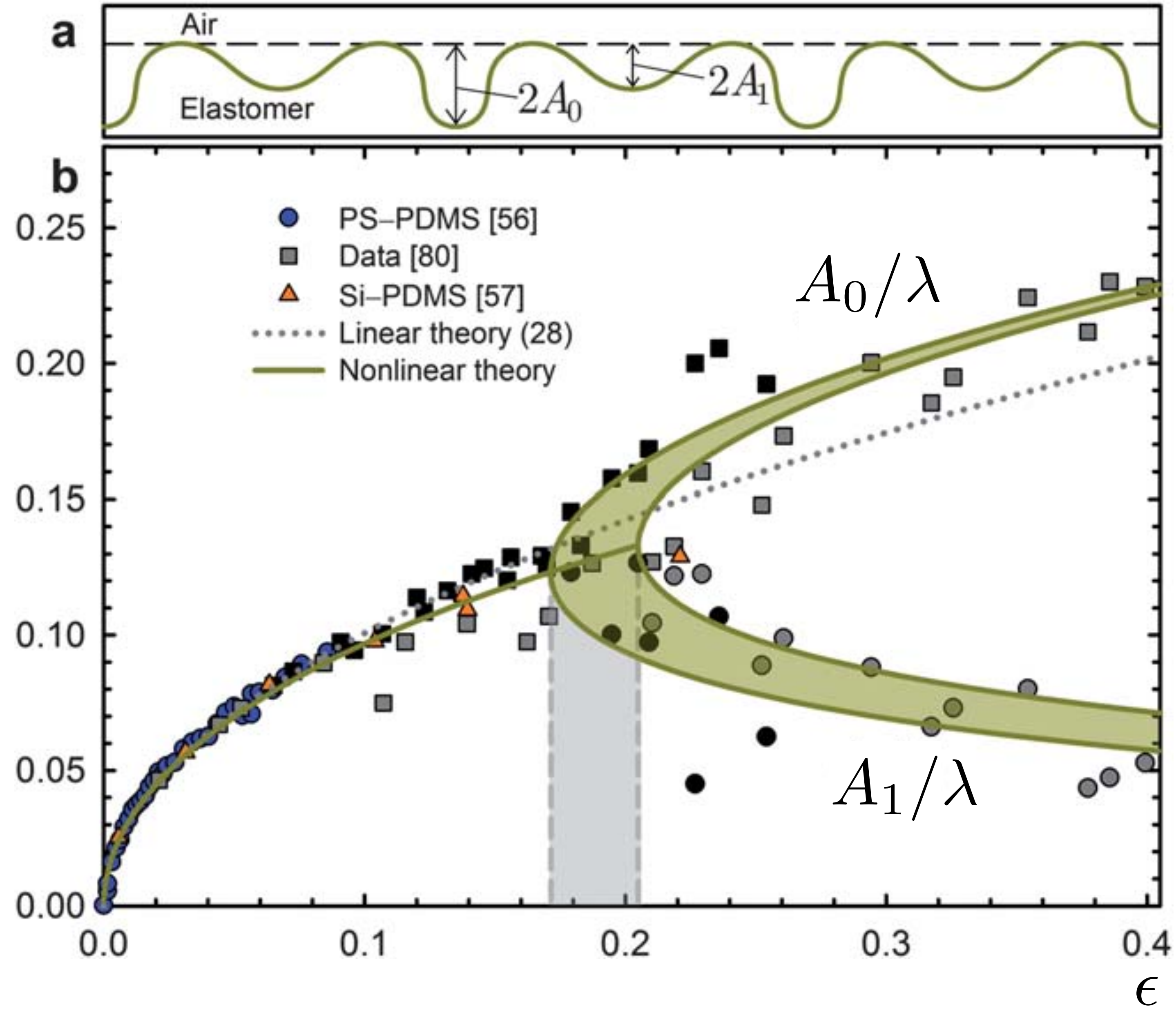
$$\lambda \sim d\left(\frac{E_m}{E_s}\right)^{1/3}$$

amplitude of wrinkles

$$h_0 \sim \lambda\sqrt{\epsilon}$$

Compression of stiff thin membranes on soft elastic substrates





In order to explain period doubling (quadrupling, ...) one has to take into account the full nonlinear deformation of the soft substrate!

F. Brau et al., Soft Matter **9**, 8177 (2013)