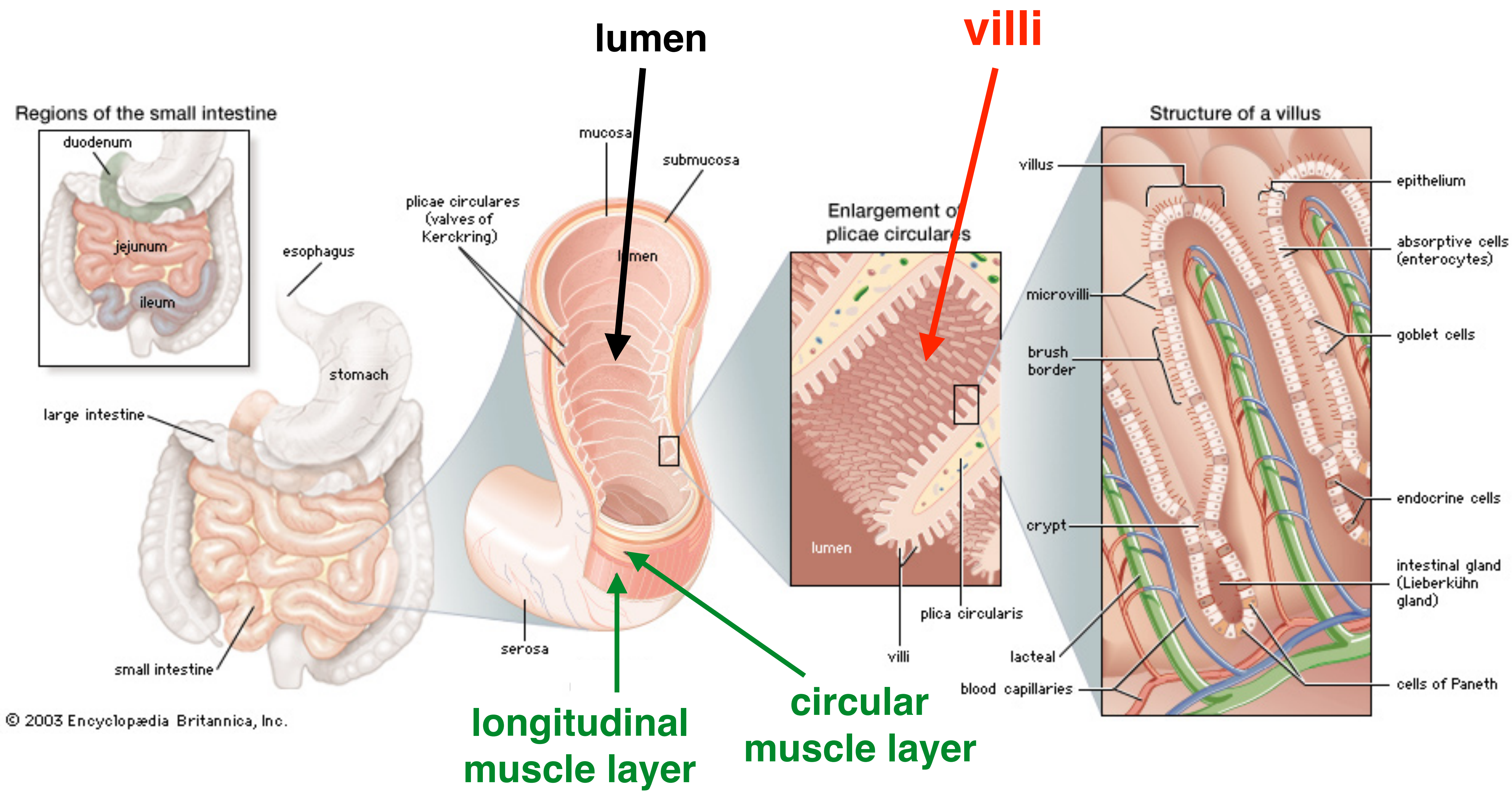
MAE 545: Lectures 7,8 (2/27)

Wrinkled surfaces

Shapes of growing sheets

How are villi formed in guts?



Villi increase internal surface area of intestine for faster absorption of digested nutrients.

Lumen patterns in chick embryo

DAPI marks cell nuclei

aSMA marks smooth muscle actin

E…: age of chick embryo in days

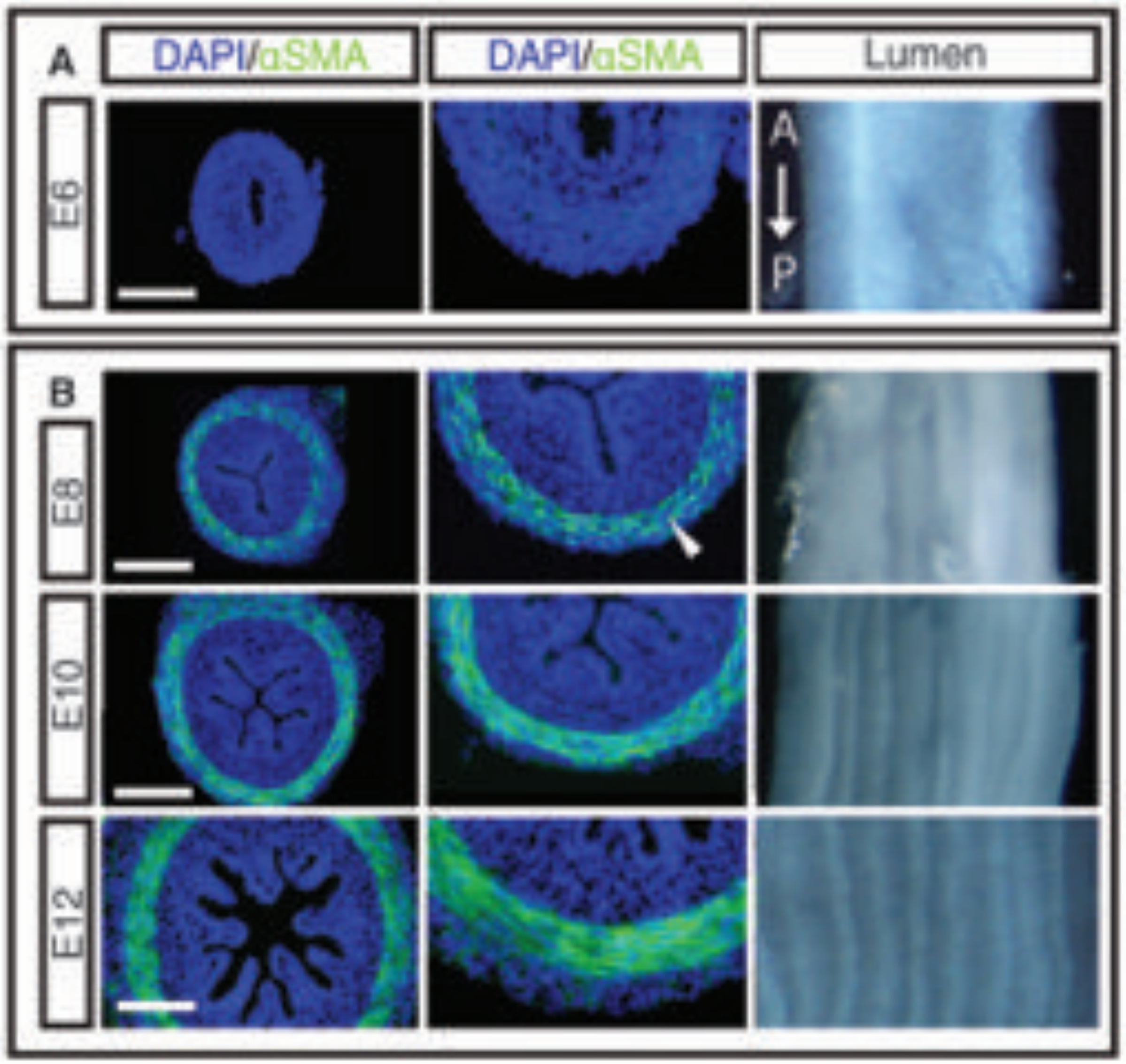
Stiff muscles grow slower than softer mesenchyme and endoderm layers

$E6$

$E8$

$E10$

$E12$

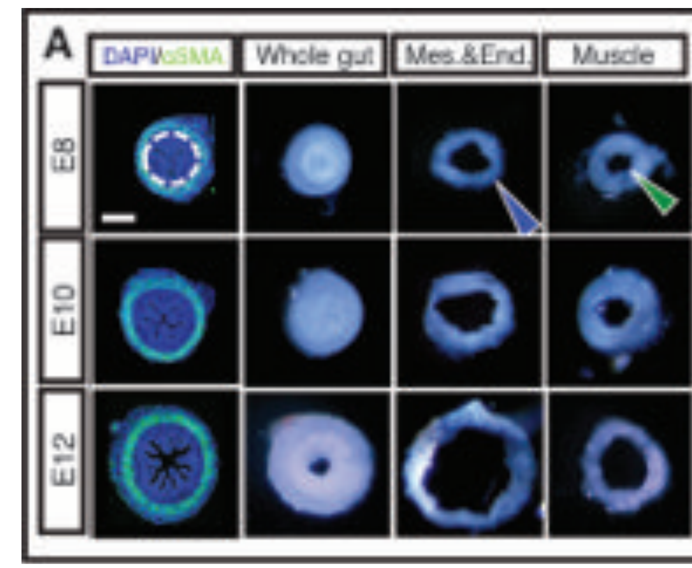


$100\mu m$

$E8$

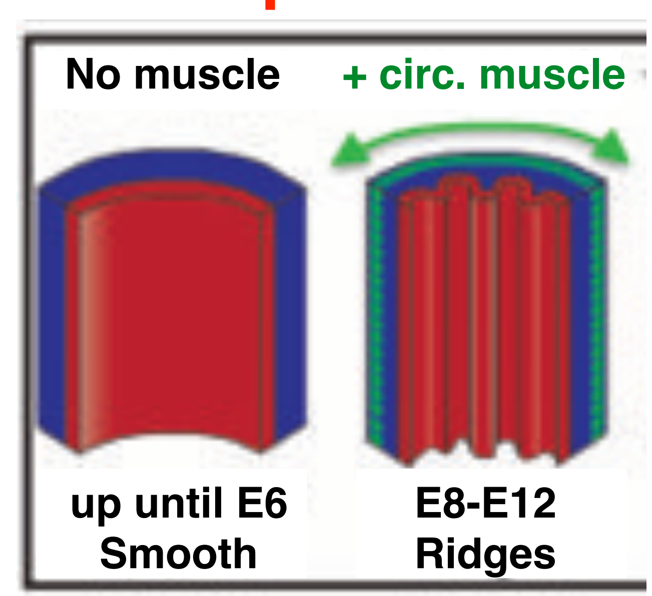
$E10$

$E12$



radial compression due to differential growth produces striped wrinkles

endoderm
mesenchyme
muscle



A. Shyer et al., Science **342**, 212 (2013)

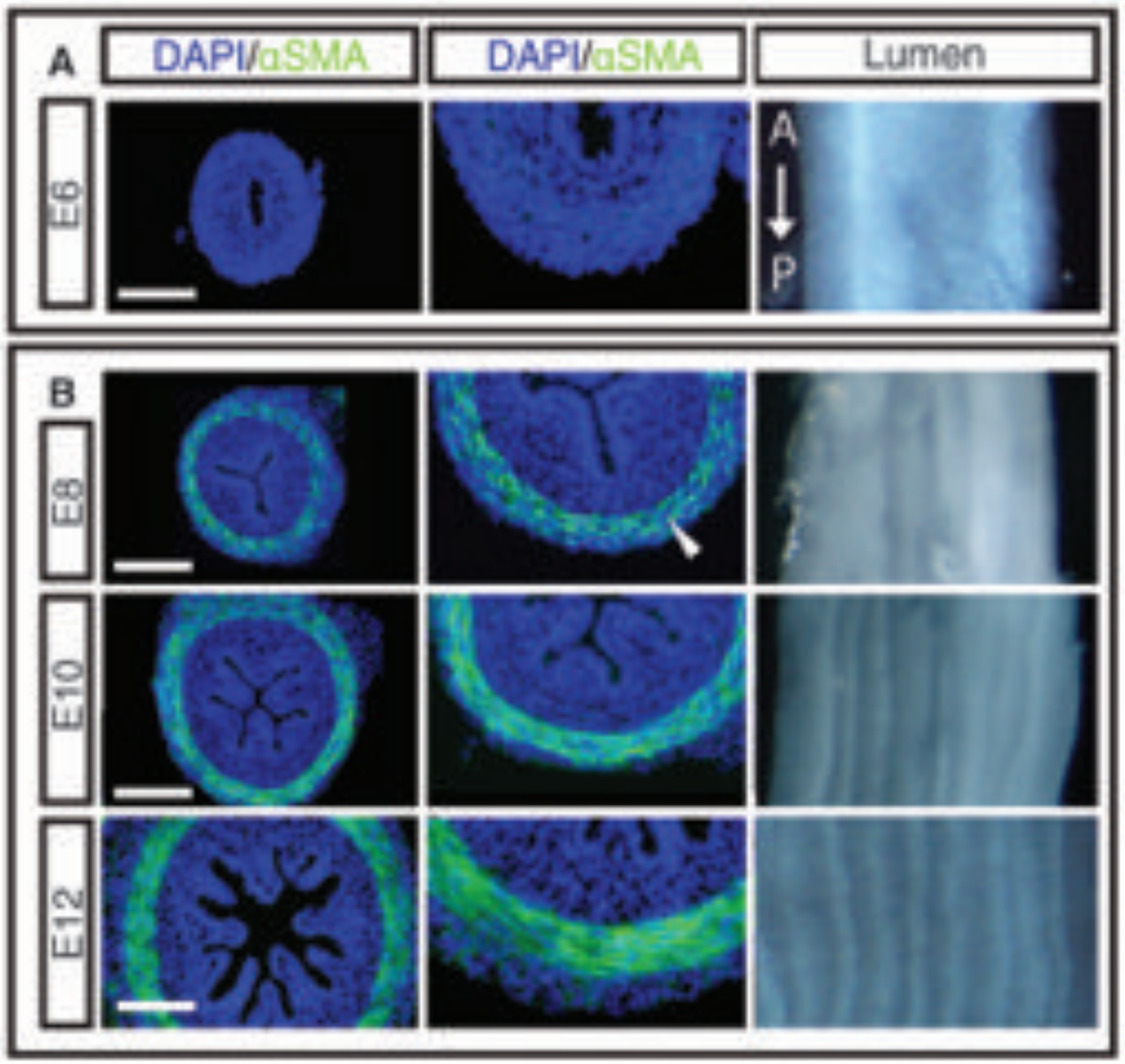
Lumen patterns in chick embryo

*E*6

*E*8

*E*10

*E*12



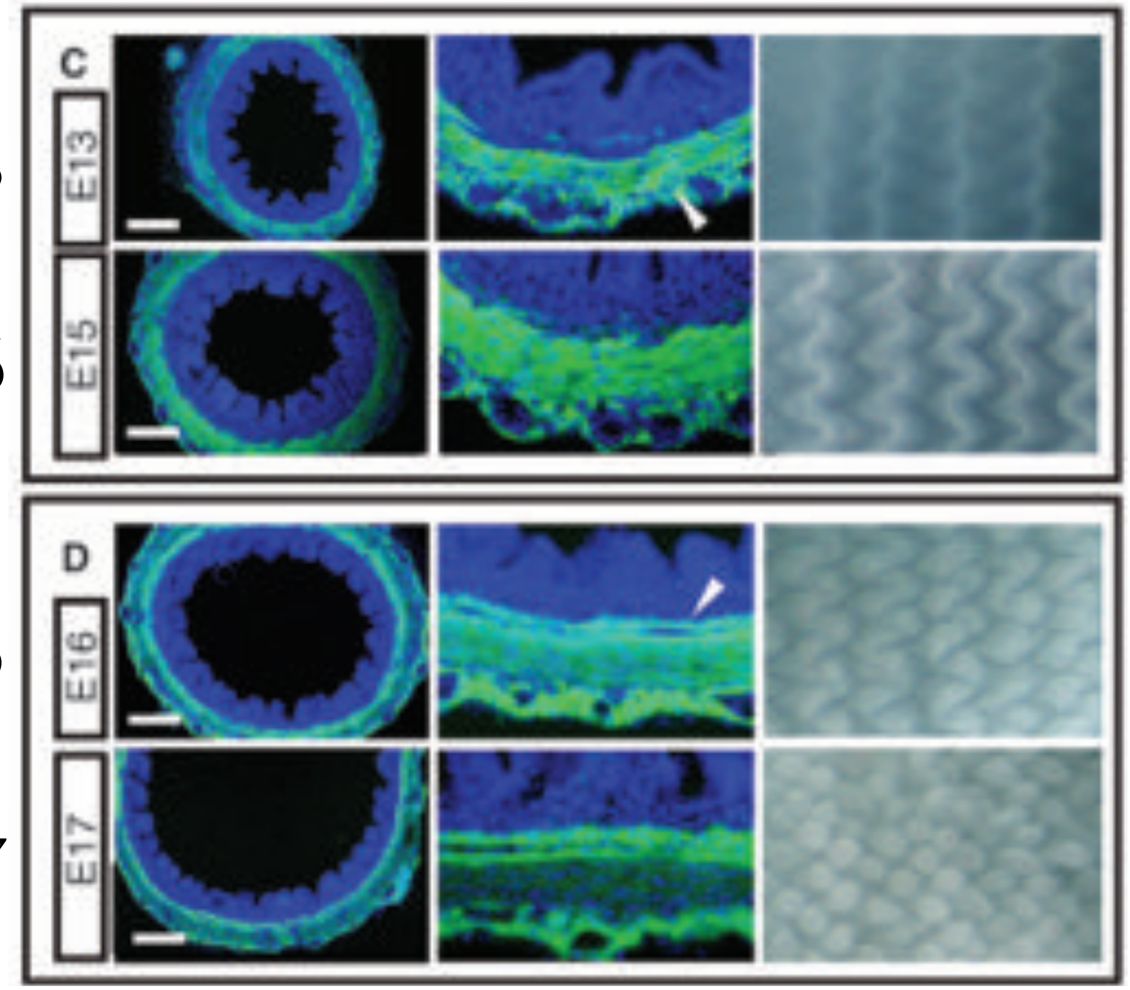
Formation of longitudinal muscles at E13 produces longitudinal compression

*E*13

*E*15

*E*16

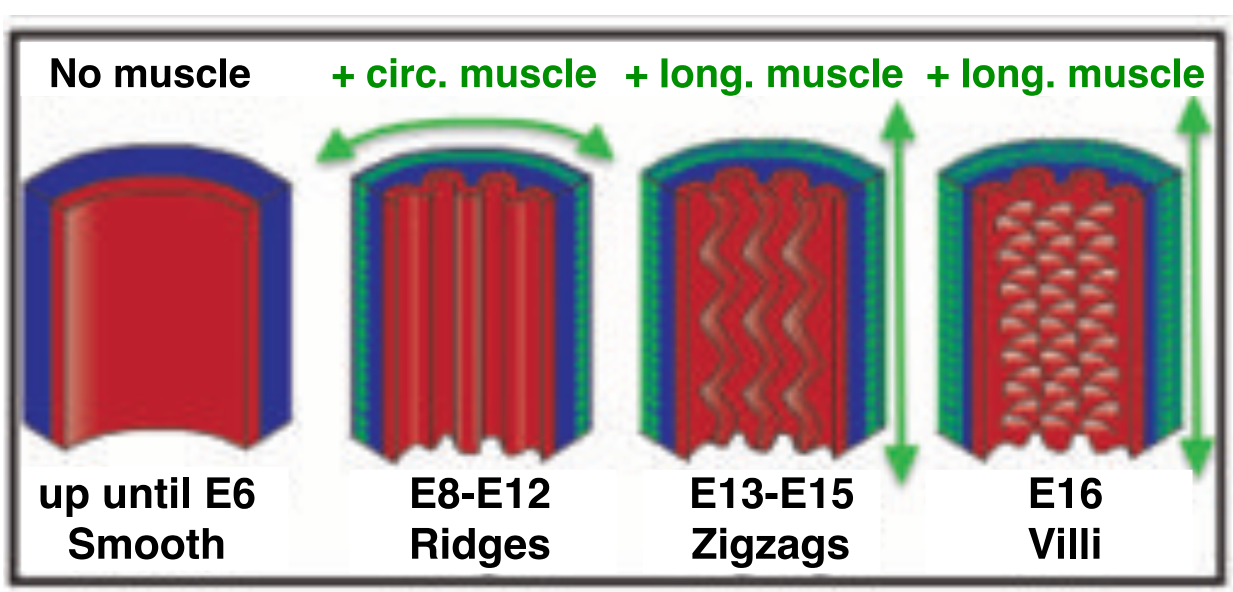
*E*17



endoderm

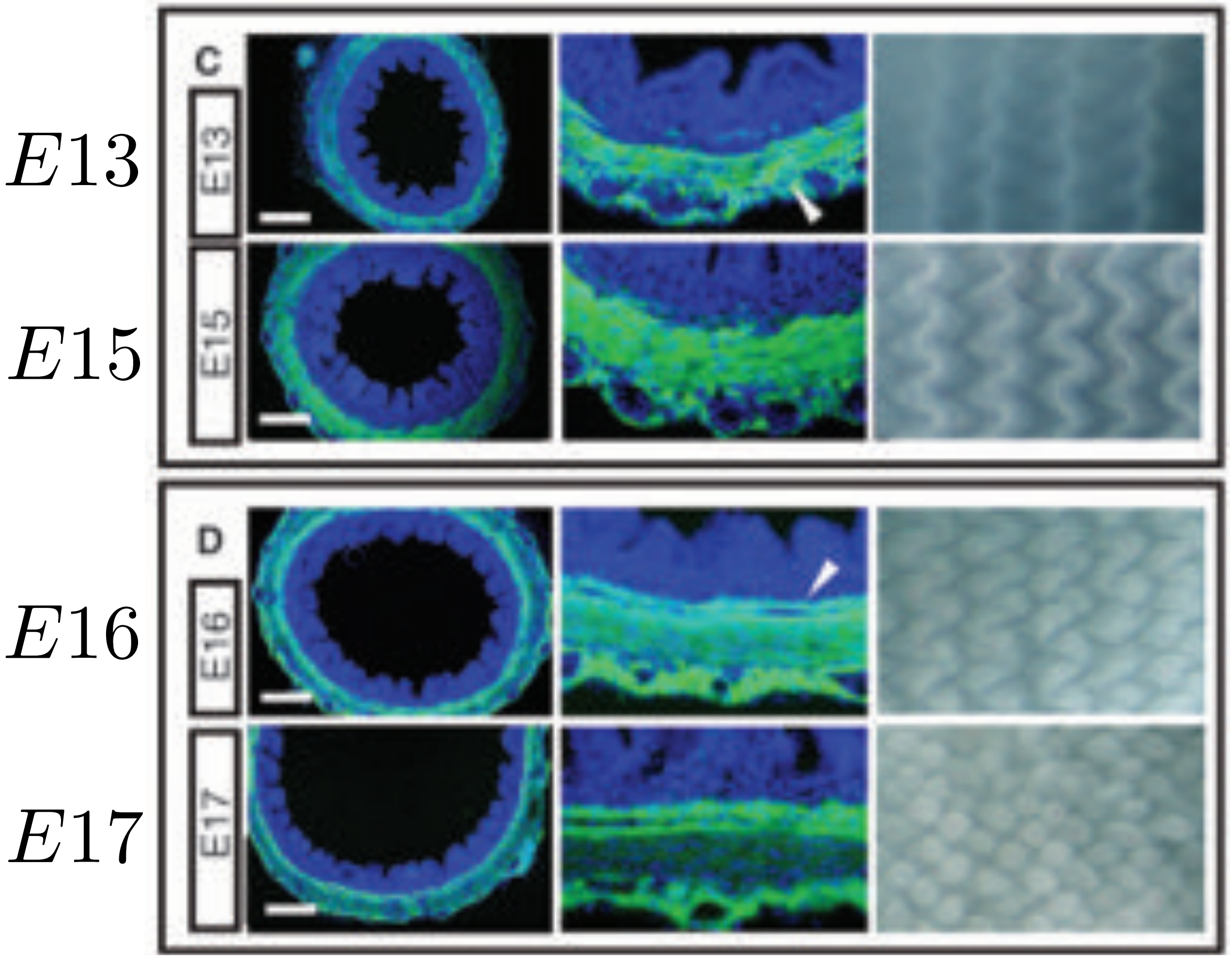
mesenchyme

muscle

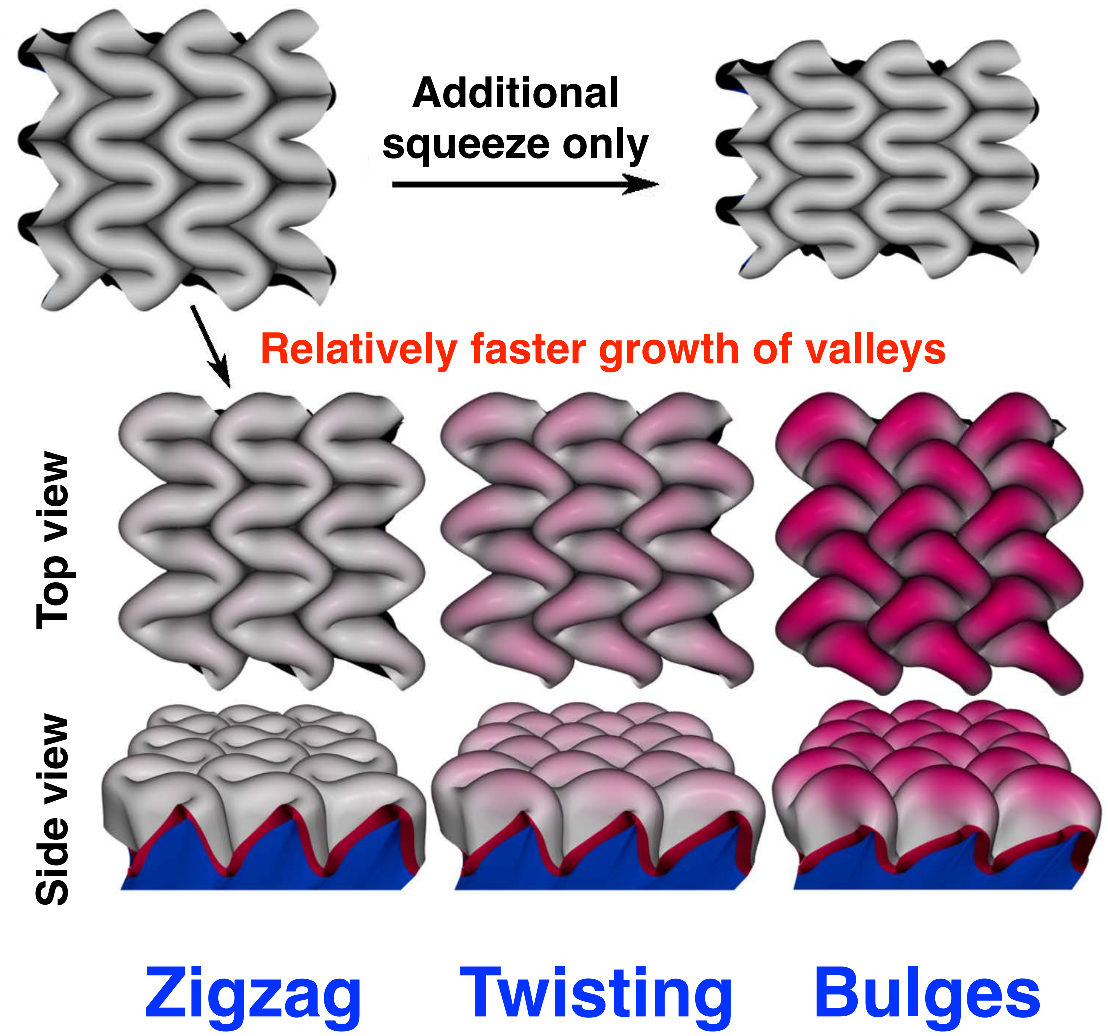


A. Shyer et al., Science **342**, 212 (2013)

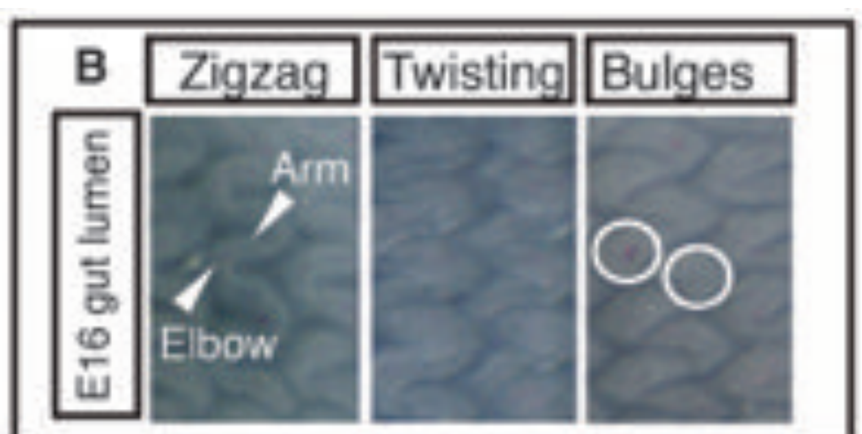
Lumen patterns in chick embryo

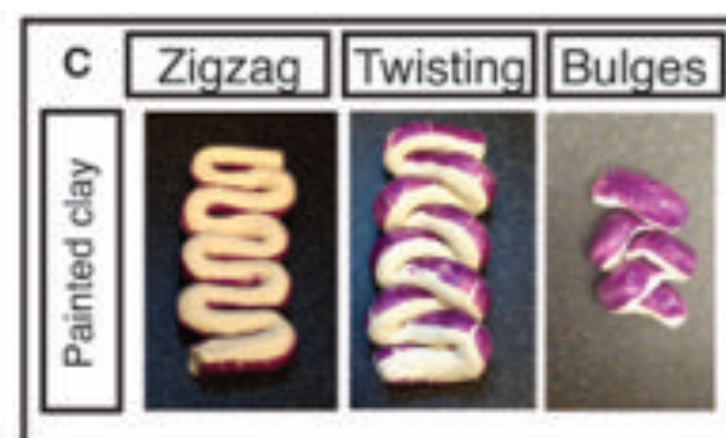


Villi start forming at E16 because of the faster growth in valleys



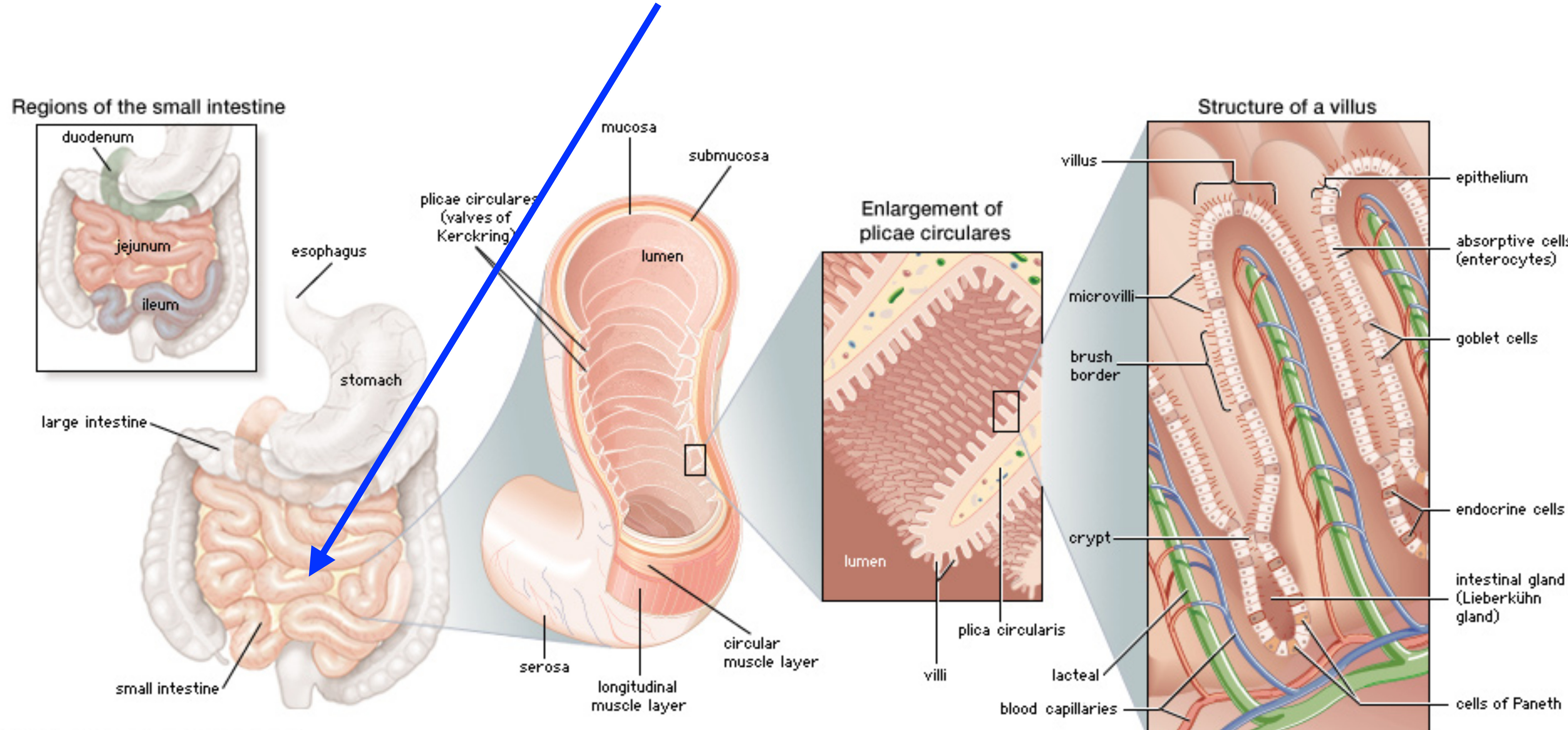
The same mechanism for villi formation also works in other organisms!





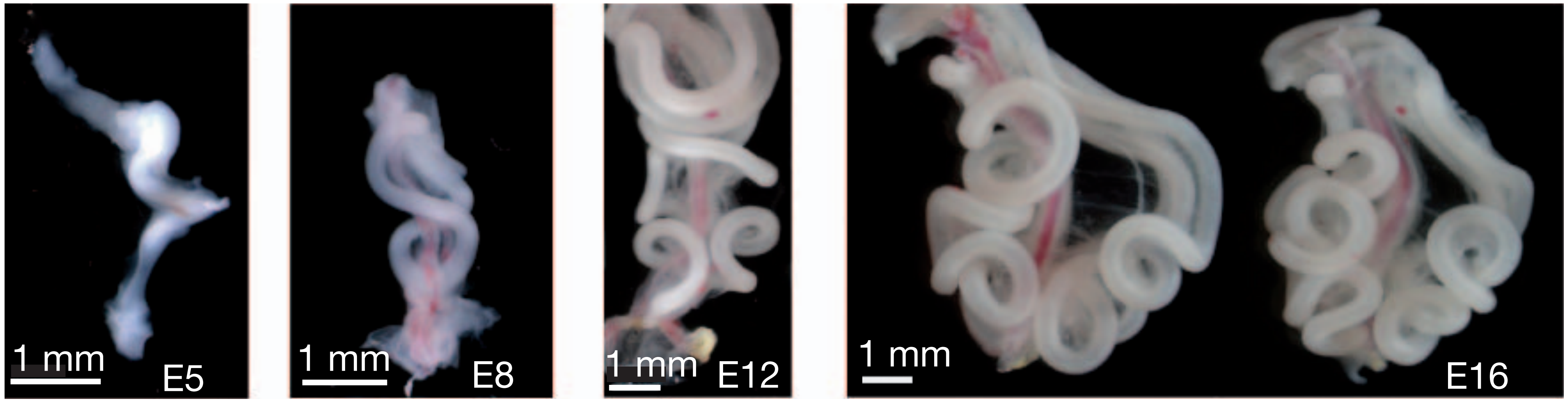
A. Shyer et al., Science **342**, 212 (2013)

Why are guts shaped like that?



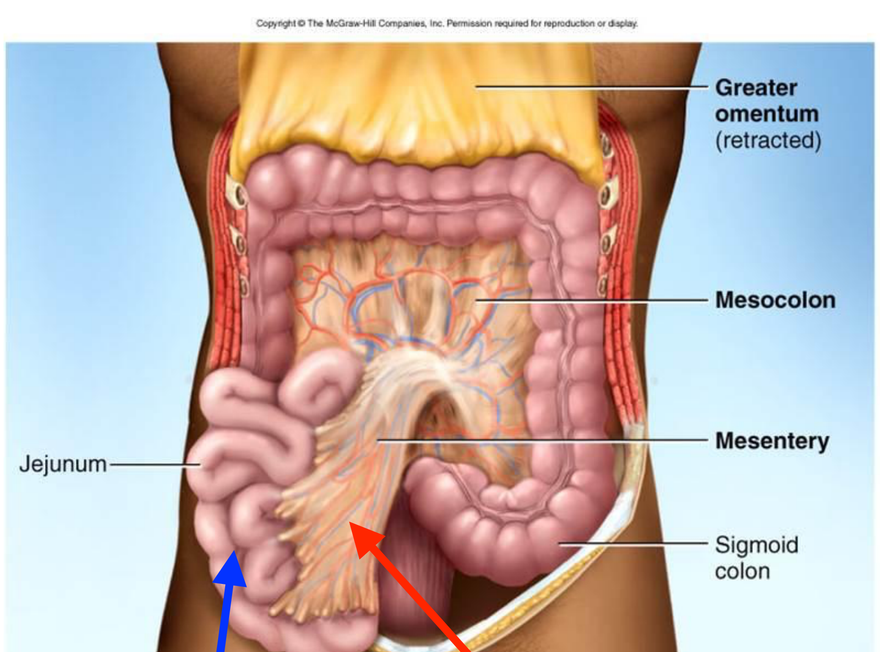
Guts in chick embryo

Surgically removed guts from chick embryo

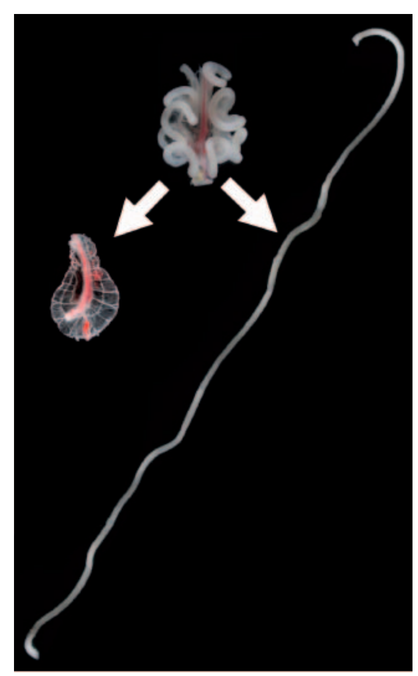


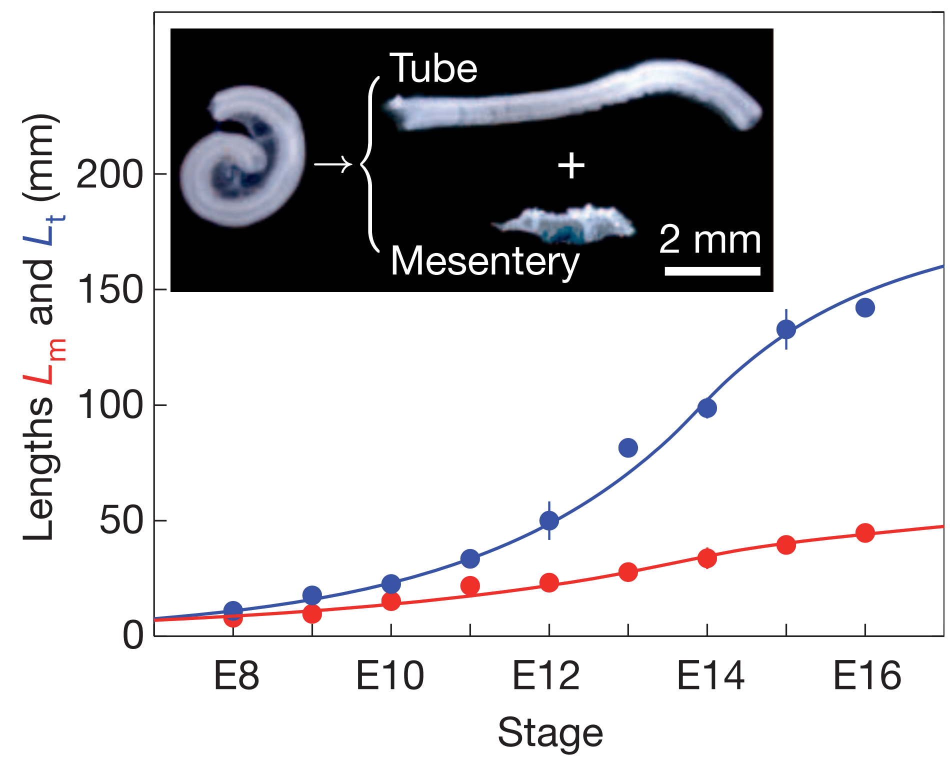
Tube straightens after separation from mesentery

Tube grows faster than mesentery sheet!



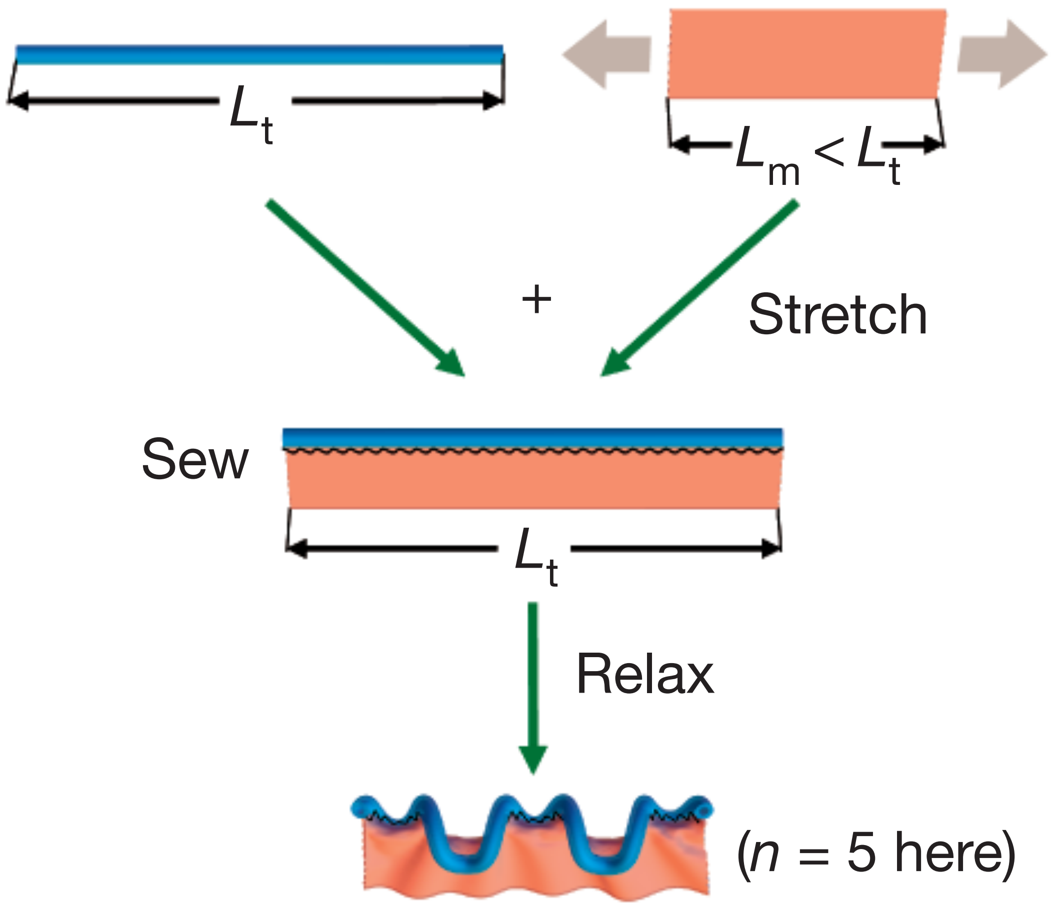
tube **mesentery**



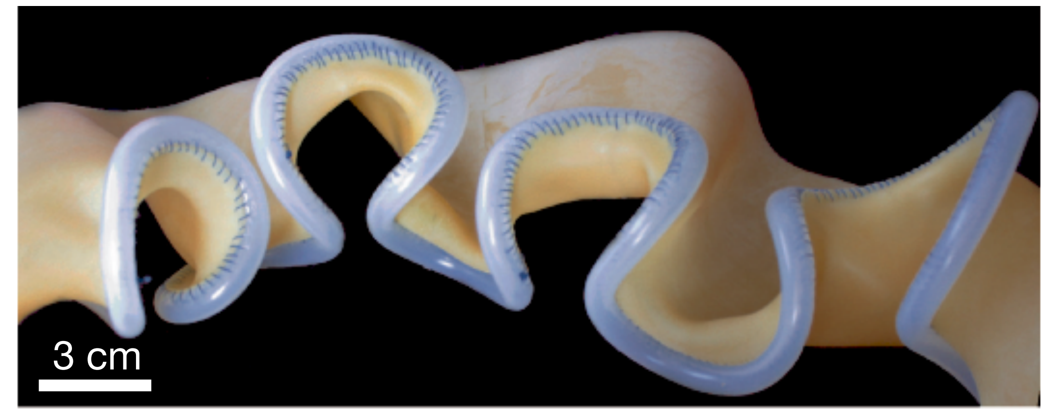


T. Savin et al., Nature **476**, 57 (2011)

Synthetic analog of guts



Rubber model of guts

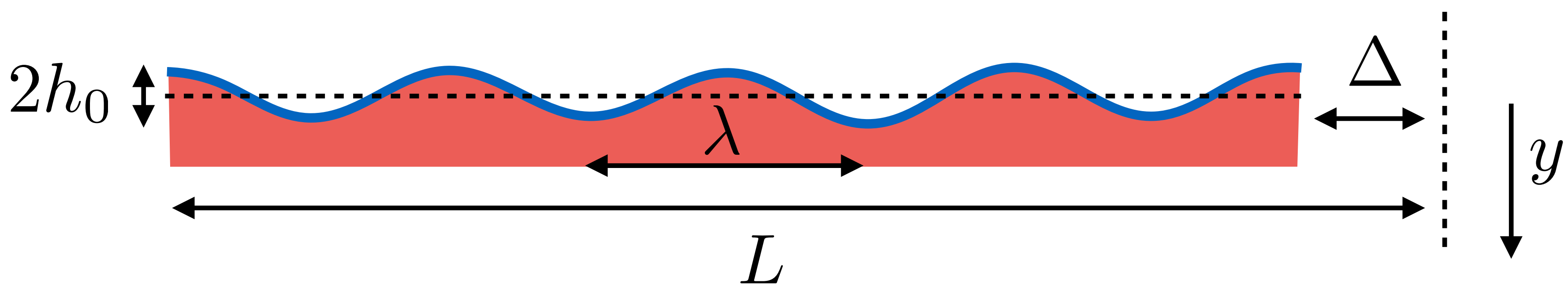


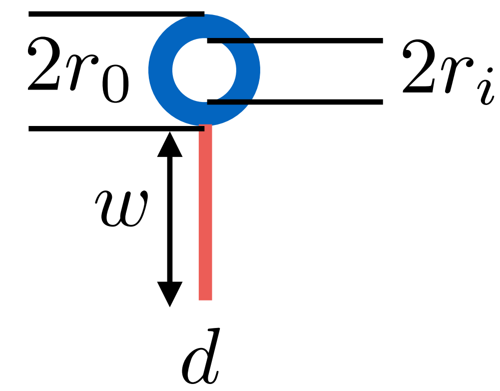
Chick guts at E12

What is the wavelength of this oscillations?

T. Savin et al., Nature **476**, 57 (2011)

Compression of stiff tube on soft elastic mesentery sheet





assumed profile $h(s) = h_0 \cos(2\pi s/\lambda)$

amplitude of wrinkles

$$h_0 = \frac{\lambda}{\pi}\sqrt{\frac{\Delta}{L}} = \frac{\lambda\sqrt{\epsilon}}{\pi}$$

deformation of the soft mesentery decays exponentially away from the surface

$$h(s,y) \approx h_0 \cos(2\pi s/\lambda) e^{-2\pi y/\lambda}$$

bending energy of stiff tube

$$U_b \sim L \times \kappa_t \times \frac{1}{R^2} \sim L \times E_t I_t \times \frac{h_0^2}{\lambda^4} \sim \frac{L E_t I_t \epsilon}{\lambda^2}$$

deformation energy of soft mesentery

$$U_m \sim A \times E_m d \times \epsilon_m^2 \sim L\lambda \times E_m d \times \frac{h_0^2}{\lambda^2} \sim L E_m d \lambda \epsilon$$

minimize total energy (U_b+U_m) with respect to λ

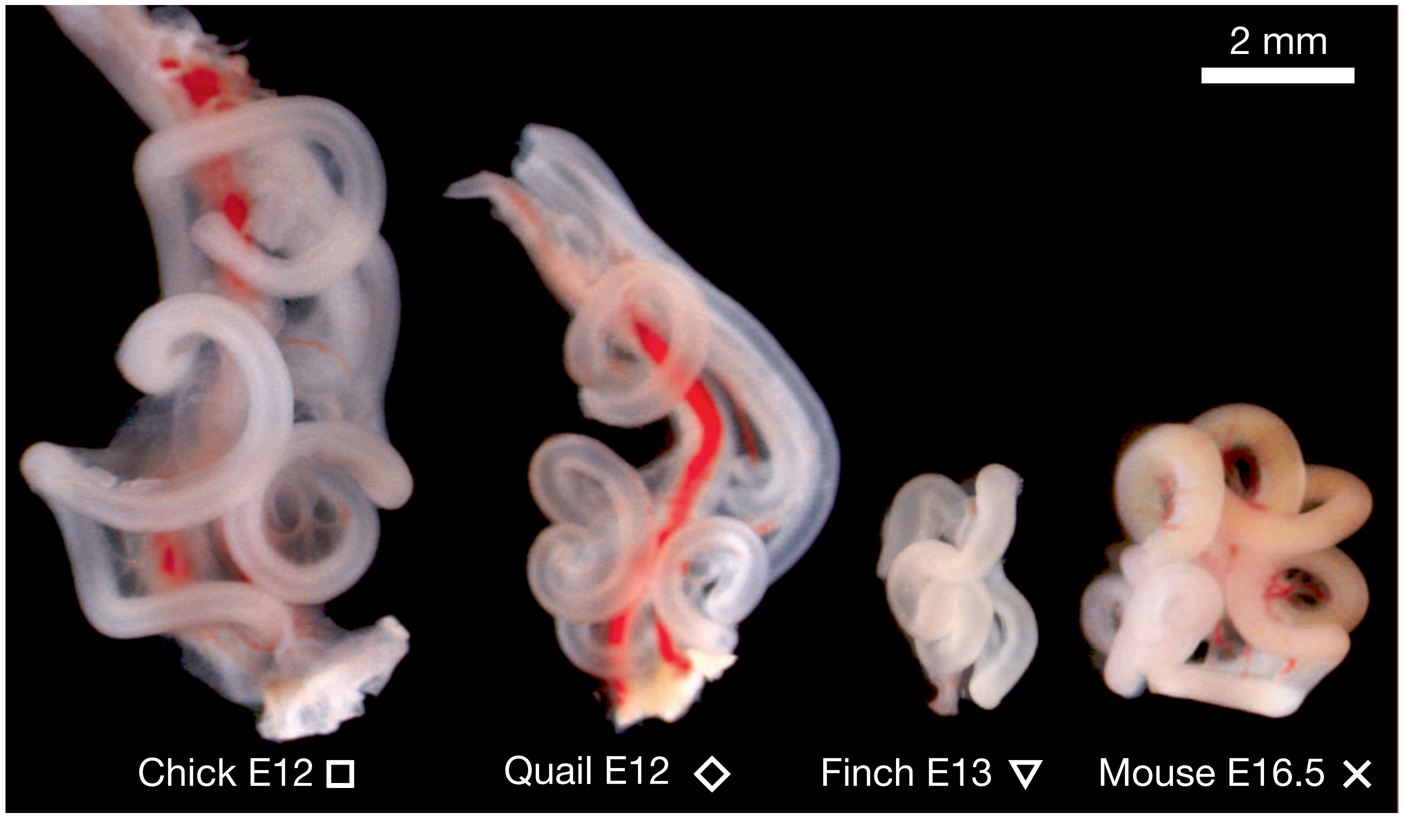
$$\lambda \sim \left(\frac{E_t I_t}{E_m d}\right)^{1/3}$$

bending stiffness of tube

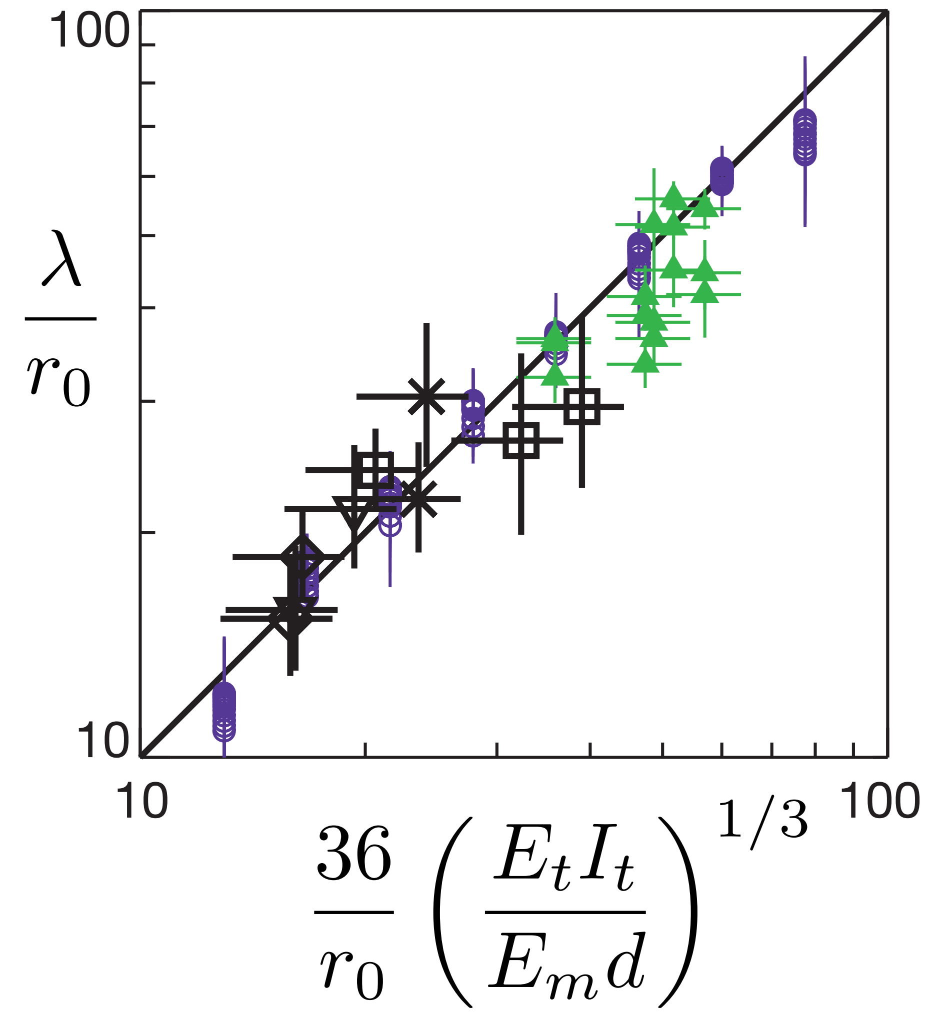
$$\kappa_t = E_t I_t$$

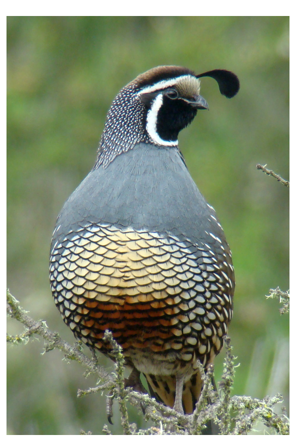
$$\kappa_t \propto E_t (r_0^4 - r_i^4)$$

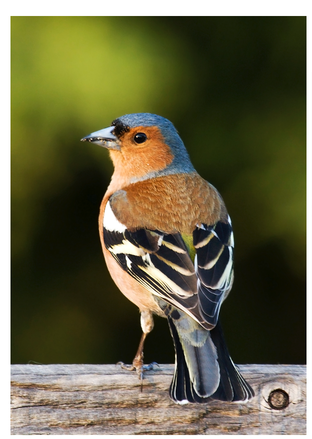
Wavelength of oscillations in guts



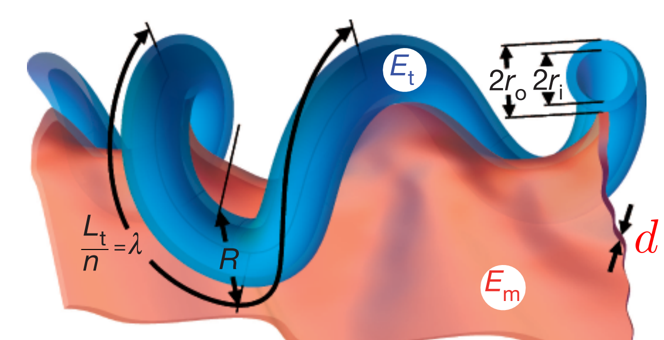
animal data, rubber model, computer simulations







chick **quail** **finch** **mouse**

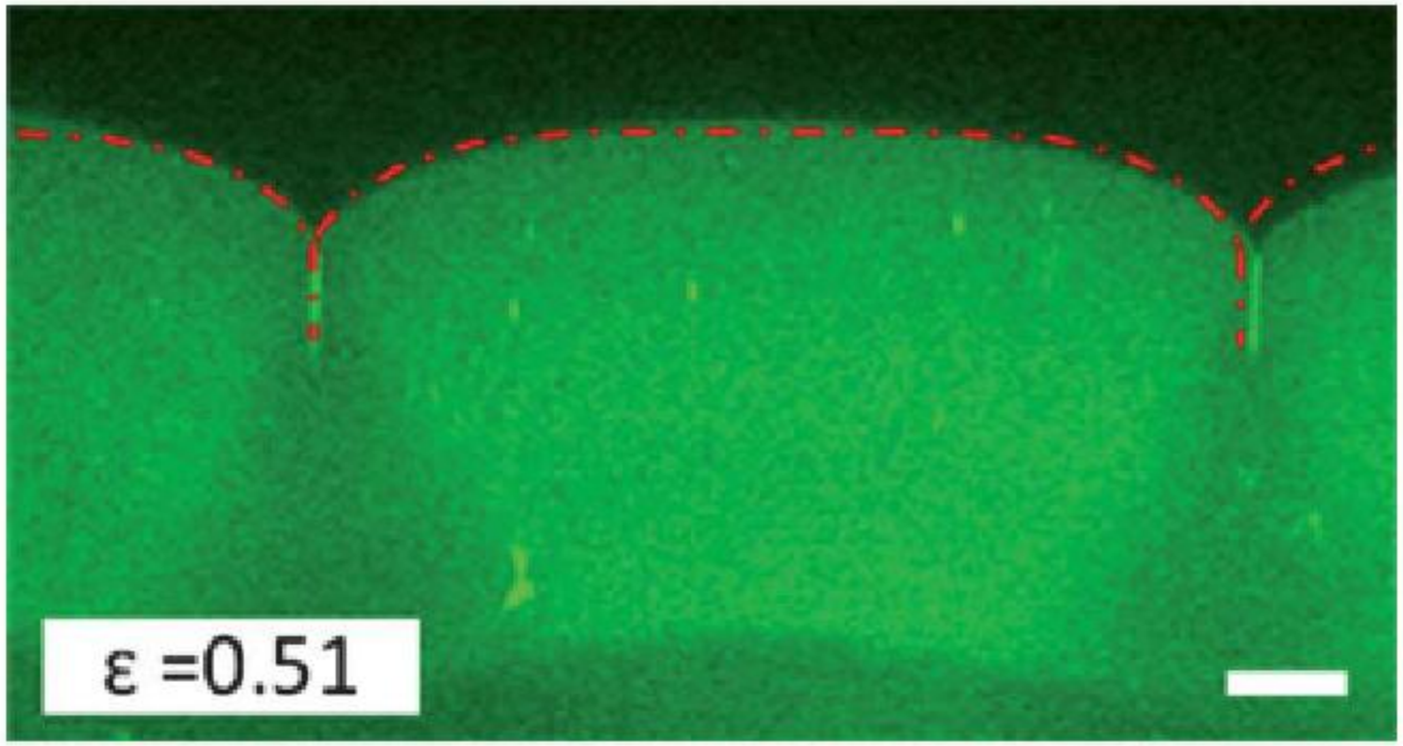


T. Savin et al., Nature **476**, 57 (2011)

Compression of soft elastic material

When soft elastic material is compressed by more than 35% surface forms sharp creases. This is effect of nonlinear elasticity!

swollen gel
on a stiff substrate

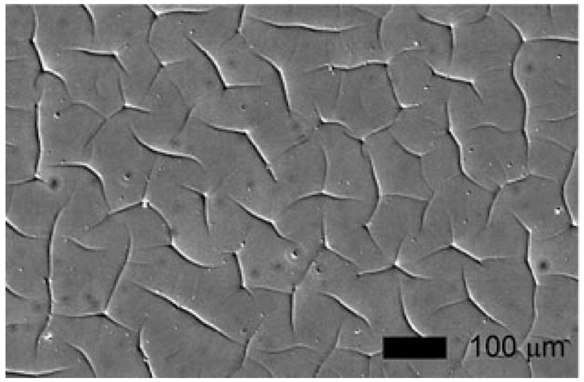


arm of
an infant

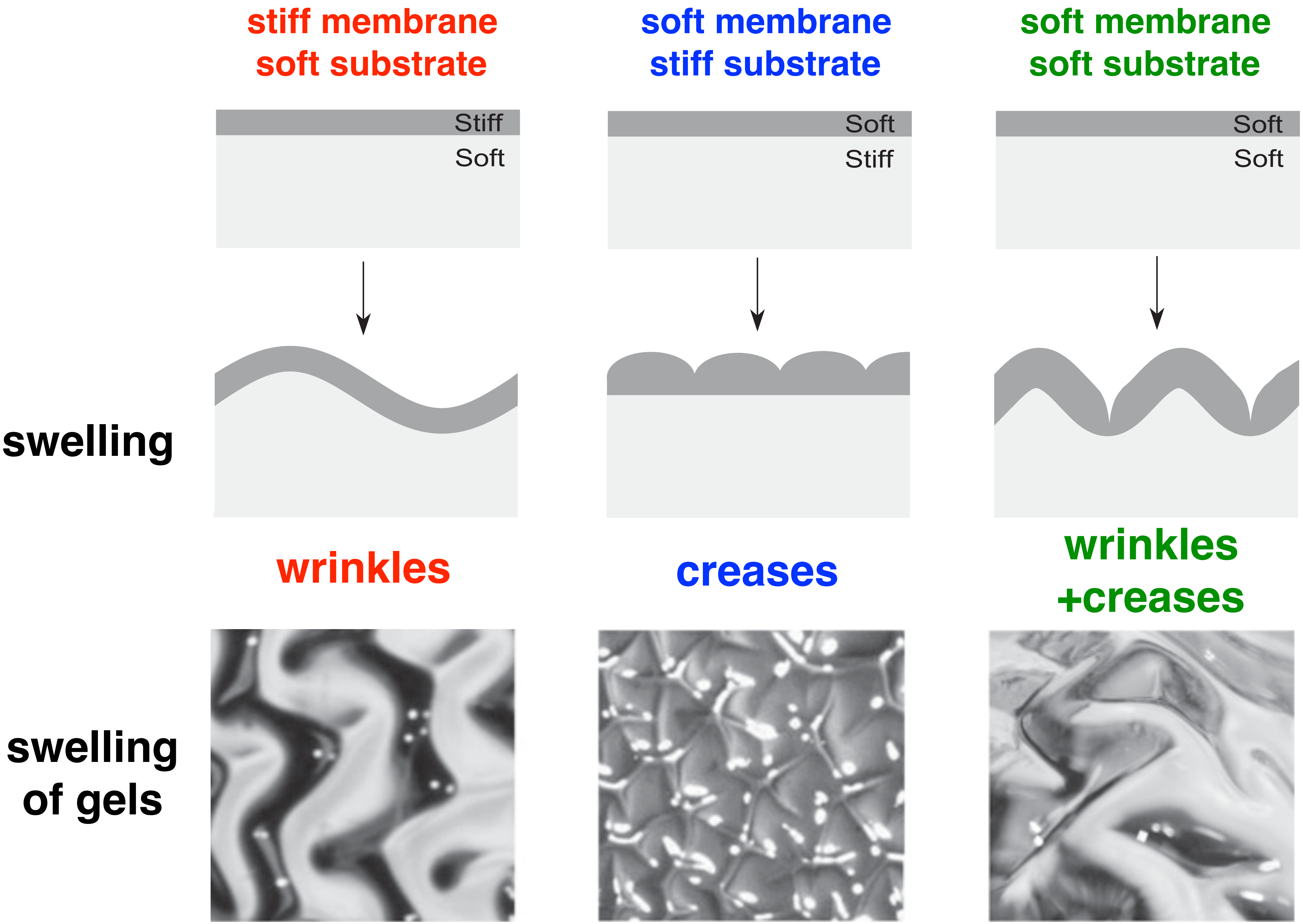
rising
dough

Liangfen
(starch jelly)

swollen gel
on a stiff substrate

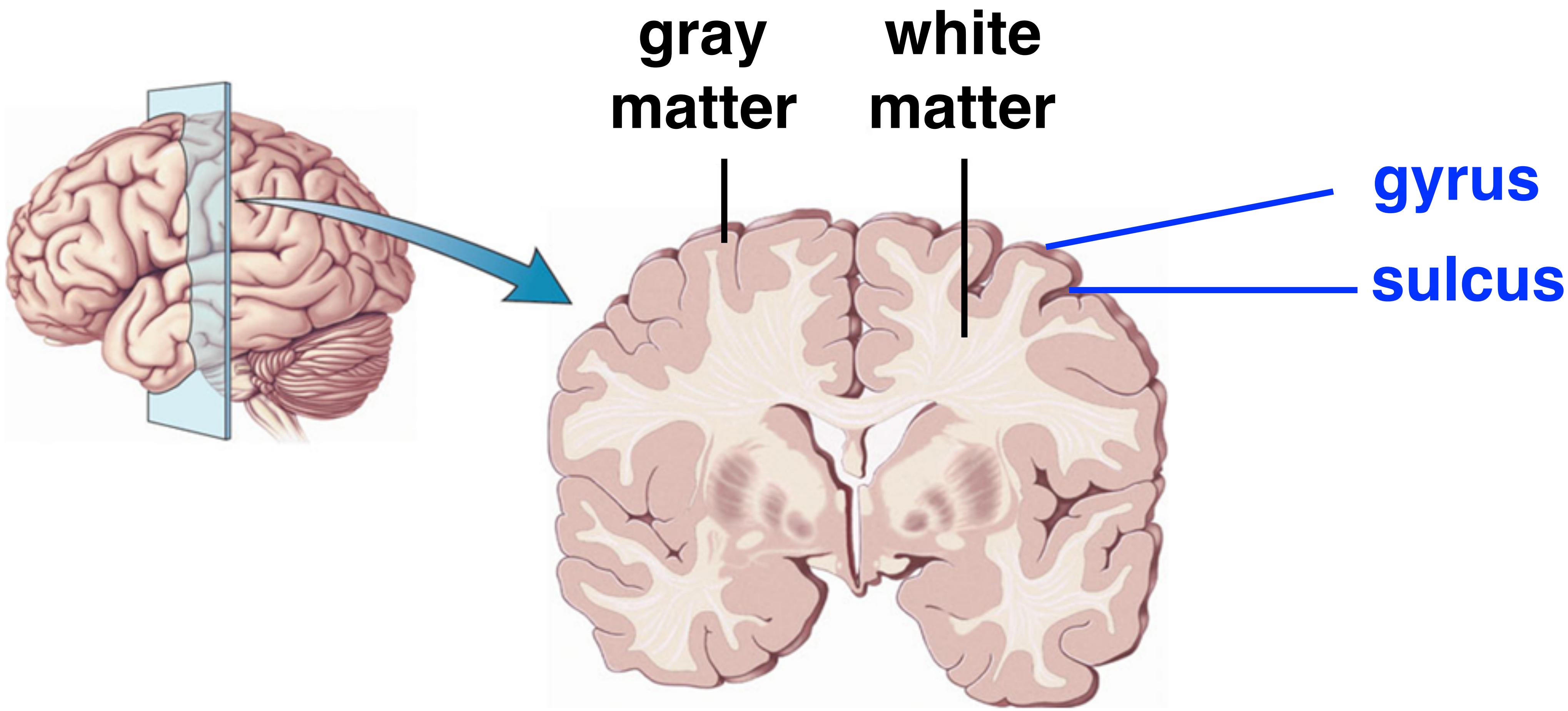


Swelling of thin membranes on elastic substrates

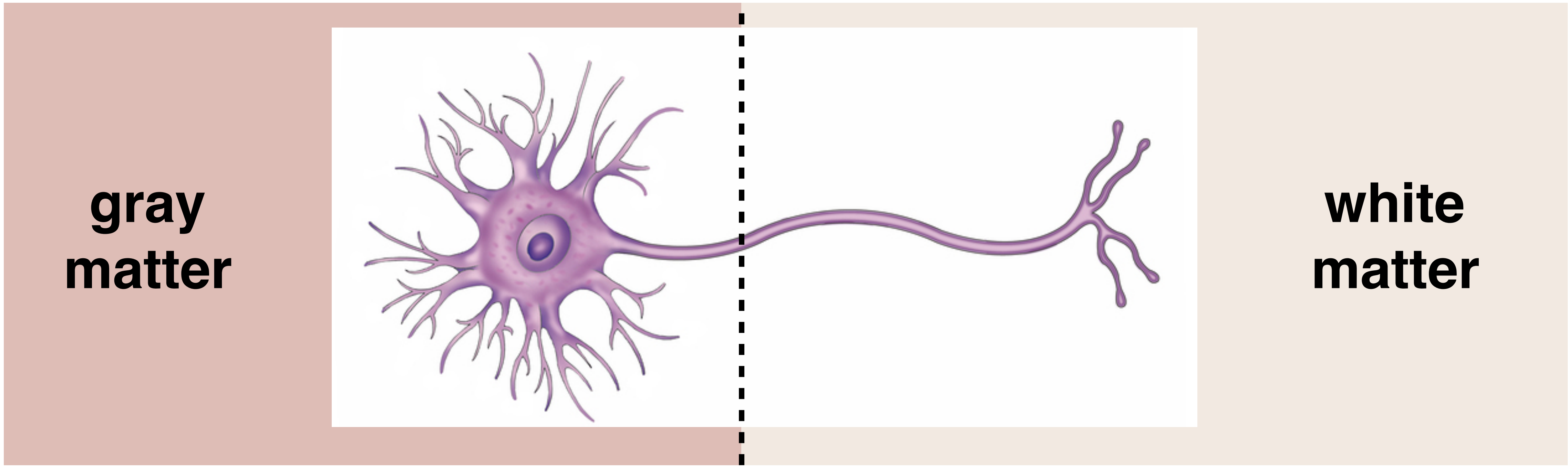


T. Tallinen et al., PNAS **111**, 12667 (2014)

Cortical convolutions in brains

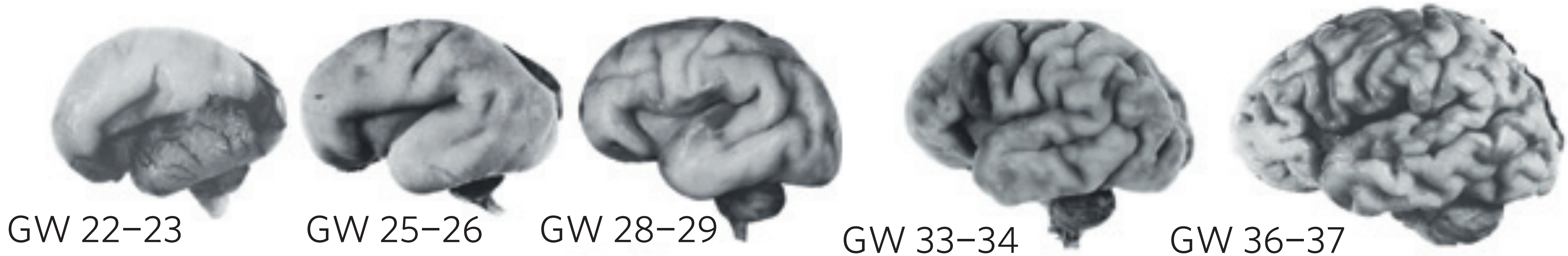


Migration of neurons to the cortex leads to "swelling" of gray matter!



Formation of cortical convolutions in developing brains

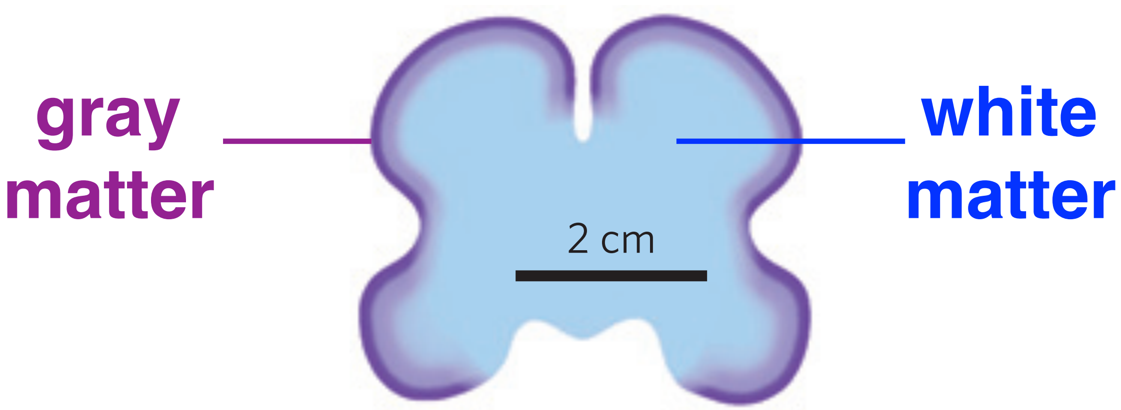
Magnetic resonance images (MRI) of fetal brains



gestational week (GW): age of fetus in weeks

Numerical simulations of developing brain

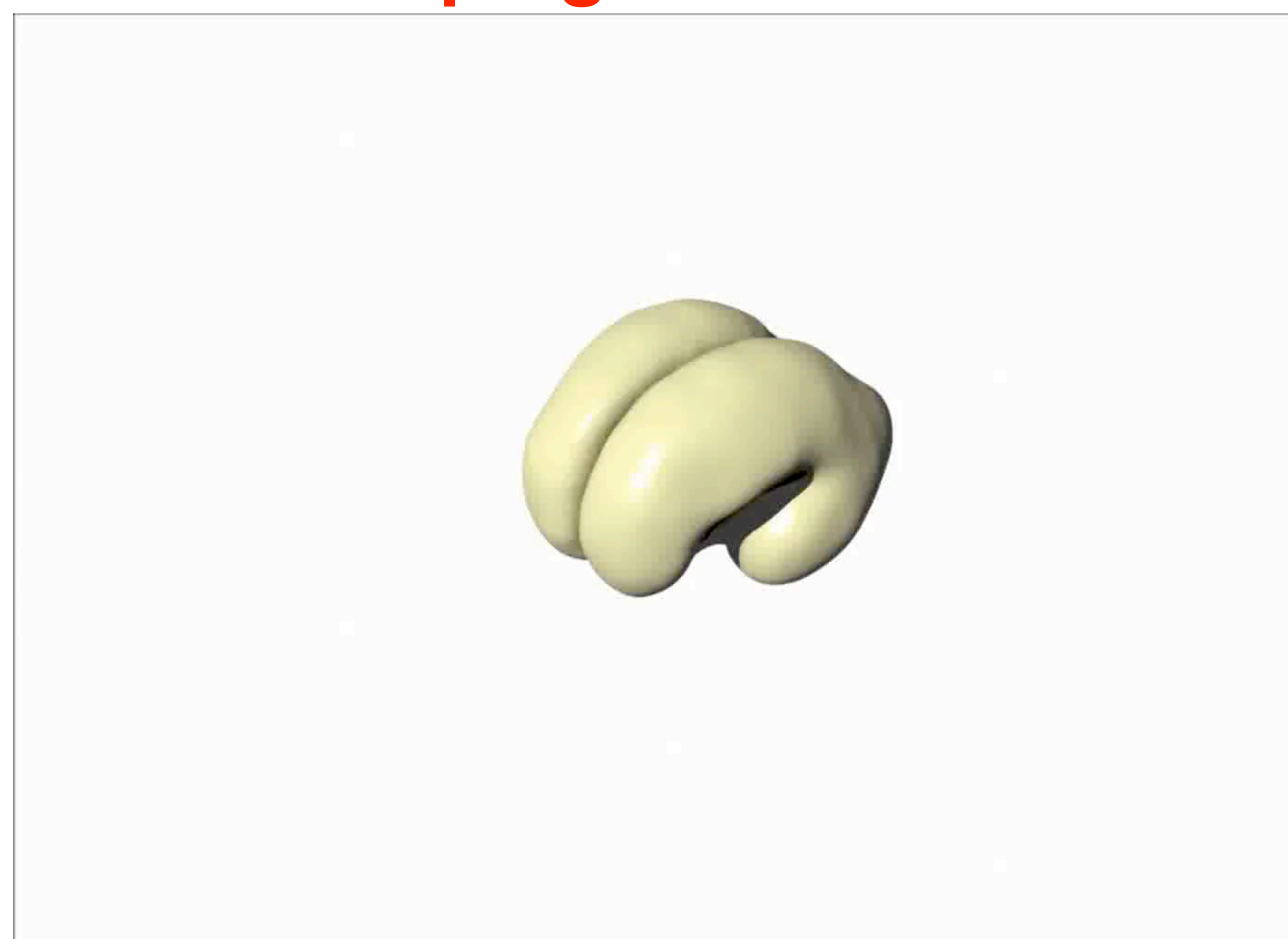
Initial condition: shape from MRI image of fetal brain at GW 22.



Assumptions:

isotropic expansion of white matter

additional tangential expansion of gray matter



T. Tallinen et al., Nature Physics **12**, 588 (2016)

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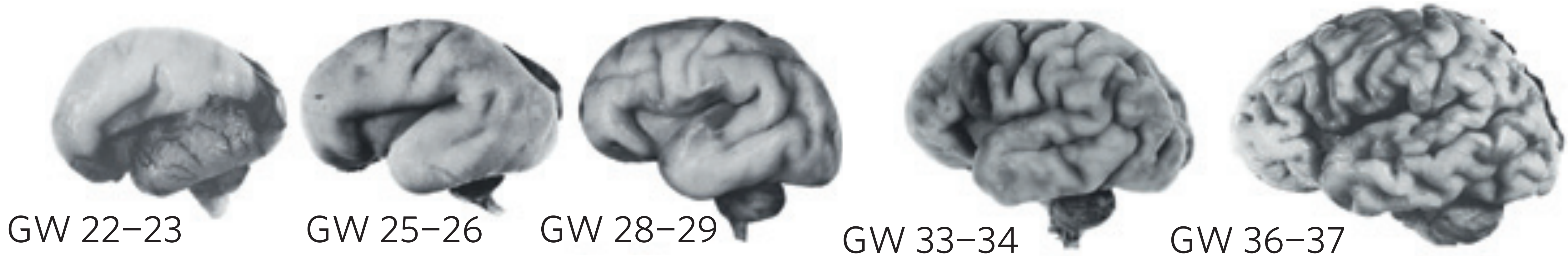

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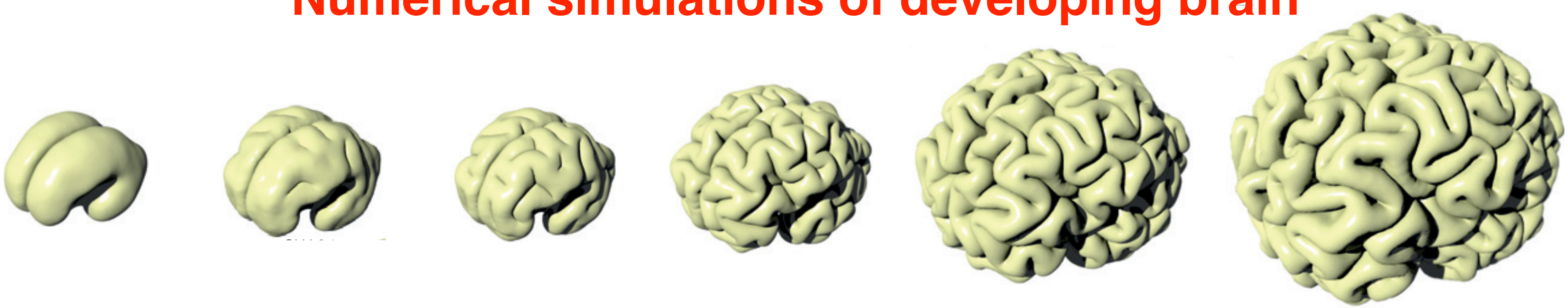


GW 22 **GW 26** **GW 29** **GW 34** **GW 40** **adult**

From GW 22 to adult stage:

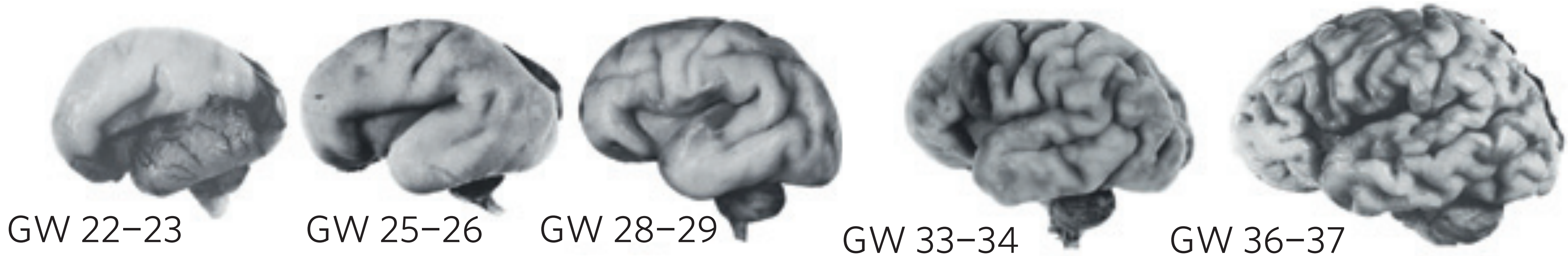
brain volume increases 20-fold from 60 ml to 1,200 ml

cortical area increases 30-fold from 80 cm^2 to 2,400 cm^2

T. Tallinen et al., Nature Physics **12**, 588 (2016)

Formation of cortical convolutions in developing brains

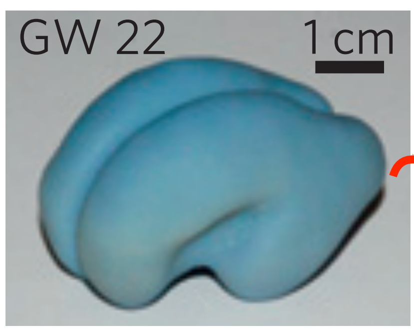
Magnetic resonance images (MRI) of fetal brains



gestational week (GW): age of fetus in weeks

Swelling of gel models of brain

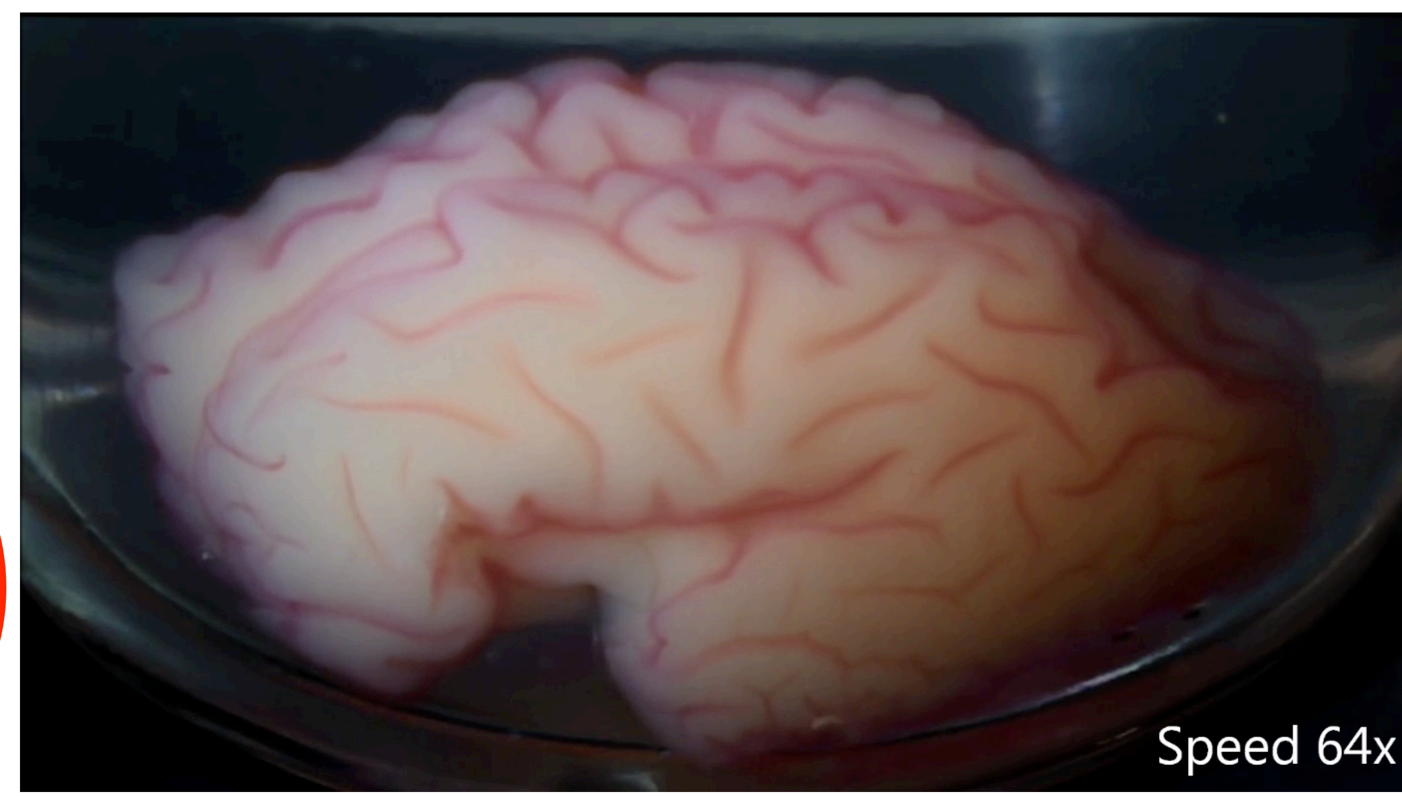
3D-printed brain model



master mould

replicated gel-brain

gel-brain coated with thin layer



In experiments only the thin coated layer swells by absorbing a liquid!

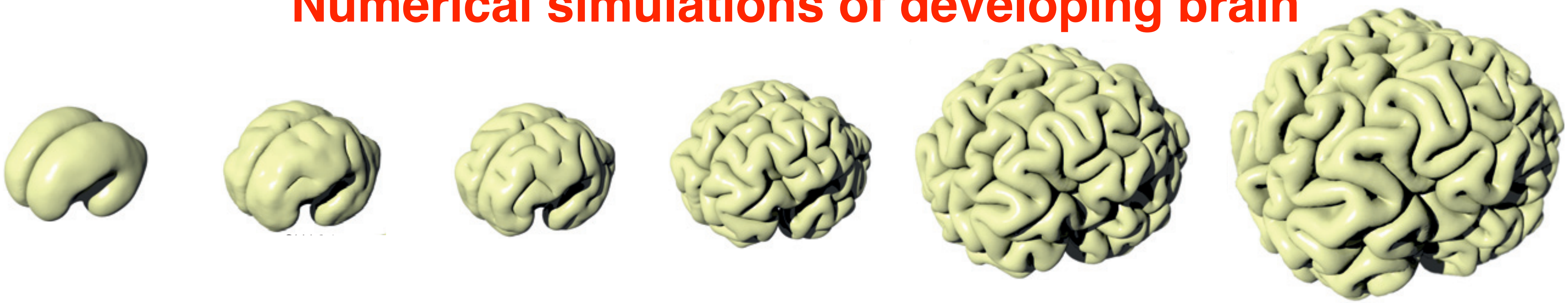
T. Tallinen et al., Nature Physics **12**, 588 (2016)

Formation of cortical convolutions in developing brains

Magnetic resonance images (MRI) of fetal brains

gestational week (GW): age of fetus in weeks

Numerical simulations of developing brain



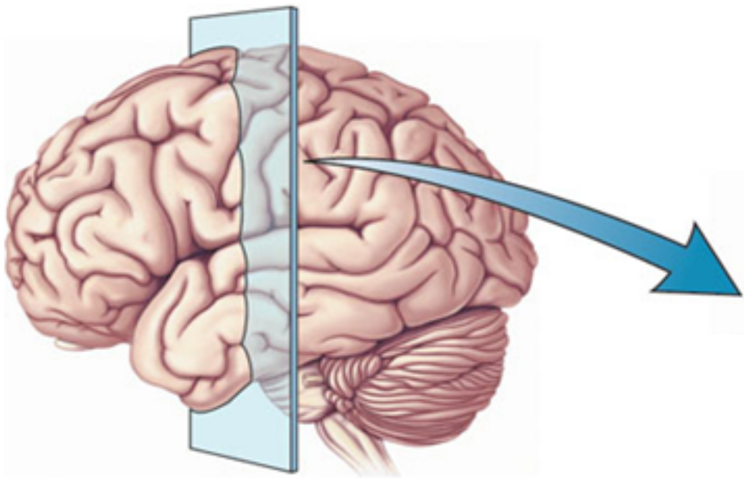
GW 22 **GW 26** **GW 29** **GW 34** **GW 40** **adult**

Swelling of gel models of brain

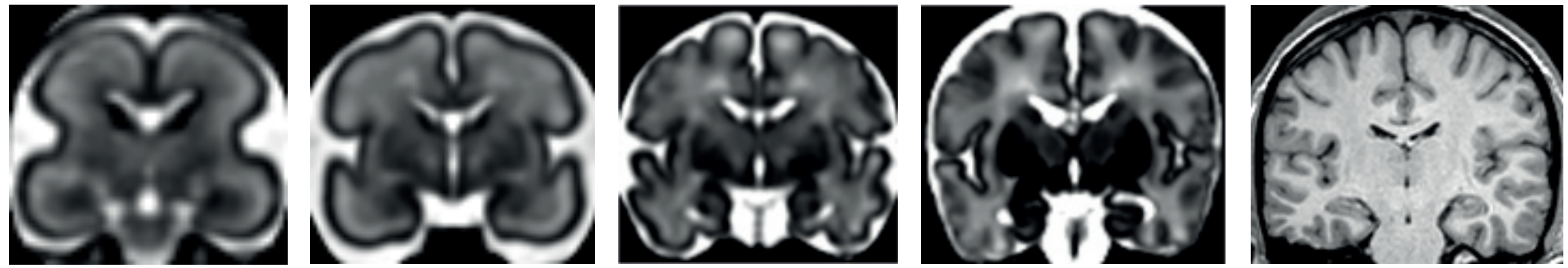
GW 22 (t=0) **GW 26 (t=4 min)** **GW 29 (t=9 min)** **GW 34 (t=16 min)**

T. Tallinen et al., Nature Physics **12**, 588 (2016)

Formation of cortical convolutions in developing brains

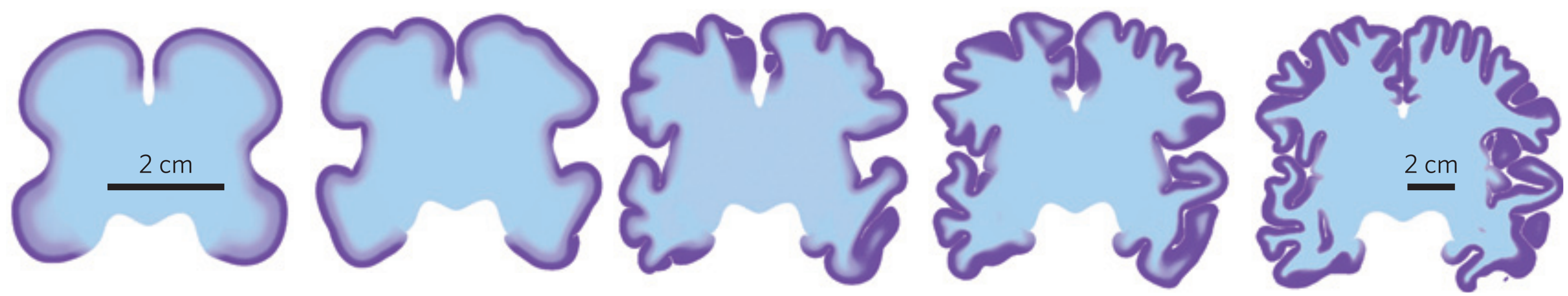


Magnetic resonance images (MRI) of brains



GW 22 | GW 29 | GW 34 | GW 40 | adult

Numerical simulations of developing brain



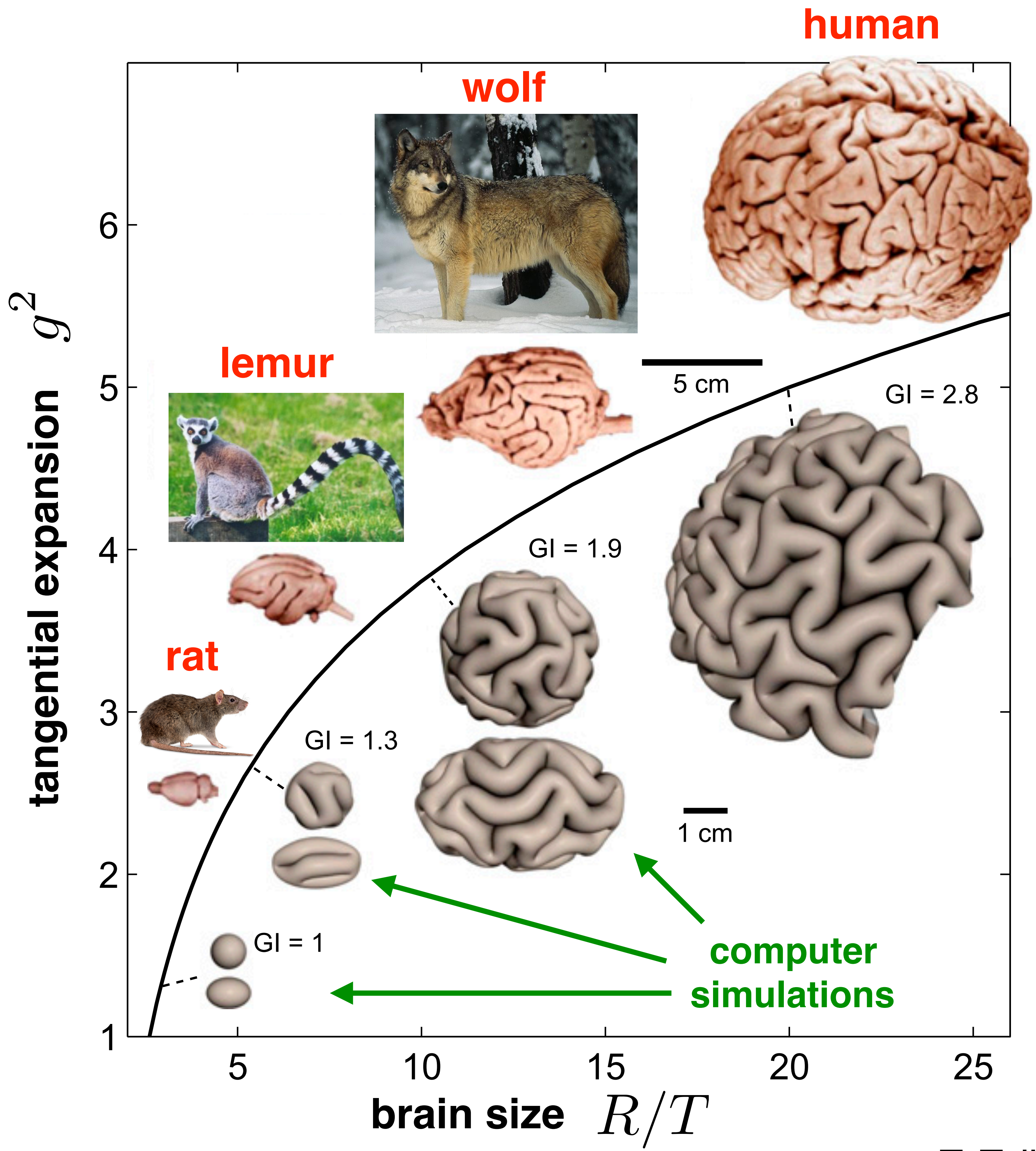
GW 22 | GW 29 | GW 34 | GW 40 | adult

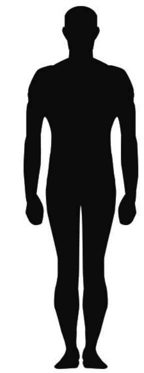
Swelling of gel models of brain

GW 22 (t=0) | GW 29 (t=9 min) | GW 34 (t=16 min)

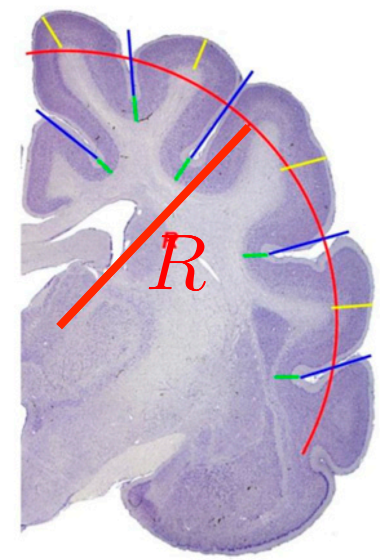
T. Tallinen et al., Nature Physics **12**, 588 (2016)

Brains for various organisms





measurements of brain parameters



***R*: brain size**

***T*: thickness of gray matter**

tangential expansion

$$g = \frac{\text{contour length of gray matter}}{\text{length of circular section}}$$

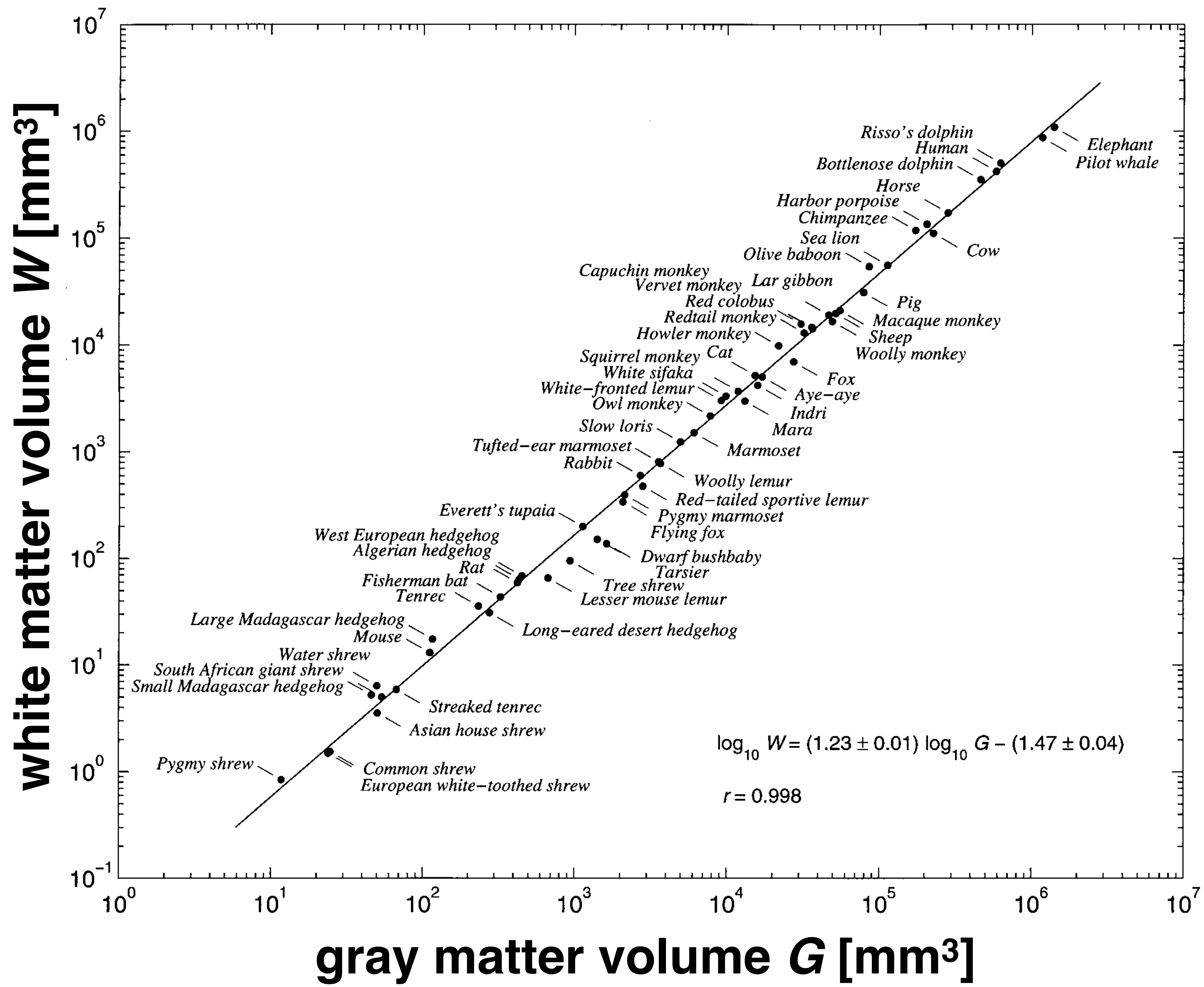
gyrification index

$$\mathrm{GI} = \frac{\text{area of brain surface}}{\text{area of convex hull}}$$

T. Tallinen et al., PNAS **111**, 12667 (2014)

Power law scaling for the brain size of various organisms

$$\text{white matter volume} \propto \left(\text{gray matter volume}\right)^{1.23}$$



another scaling relation

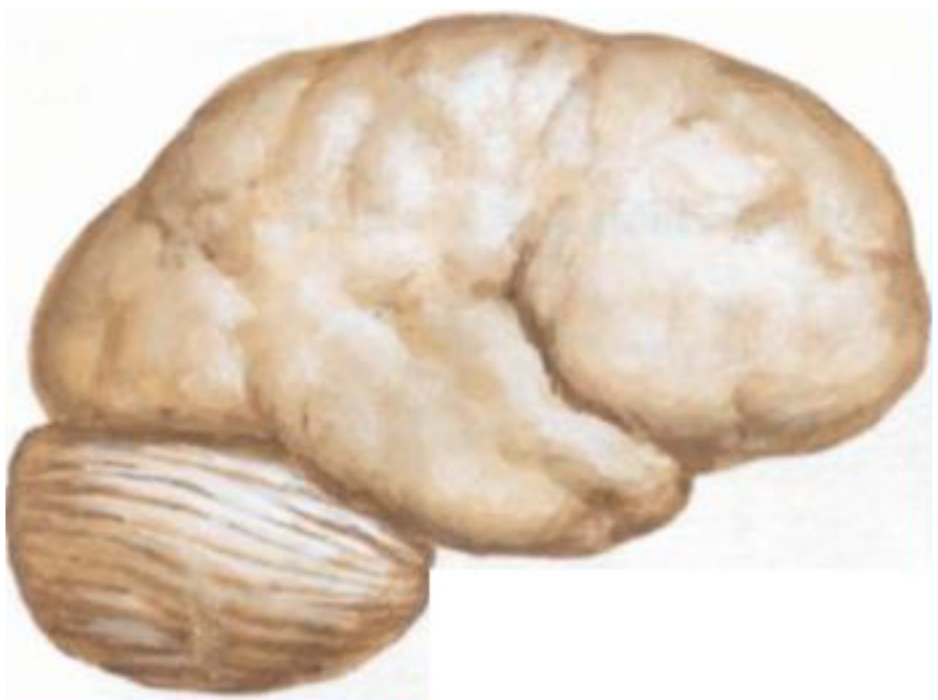
$$\text{gray matter thickness} \propto \left(\text{gray matter volume}\right)^{0.10}$$

Note: power law scalings of various quantities among organisms are very common!

K. Zhang and T.J. Sejnowski, PNAS **97**, 5621 (2000)

Brain malformations

lissencephaly
pachygyria
(small number of larger gyri)



Reduced neuronal migration to cortex

↓

Gray matter is thicker and it swells less!

polymicrogyria

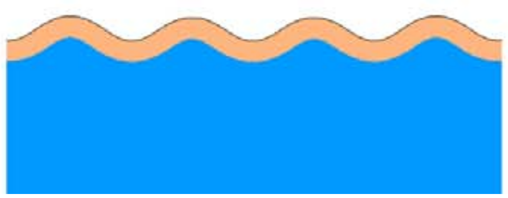
(large number of smaller gyri)

Typically gray matter has only four rather than six layers in the affected areas.

Compression of thin membranes on elastic substrates with finite adhesion

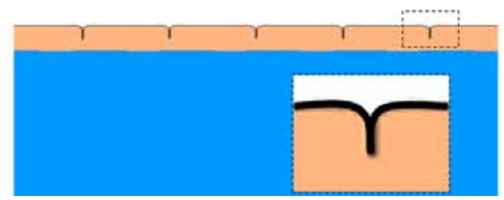
Strong adhesion between membrane and substrate

stiff membrane
soft substrate



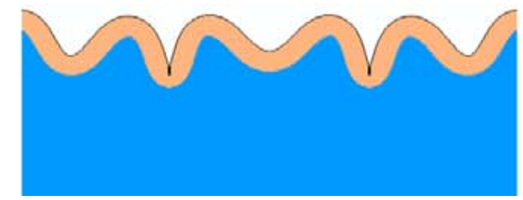
wrinkles

soft membrane
stiff substrate



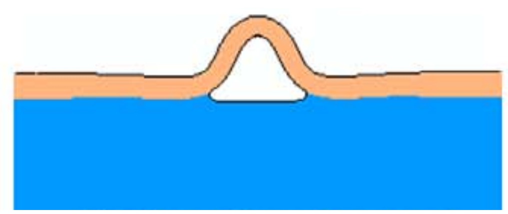
creases

soft membrane
soft substrate



folds

Weak adhesion between membrane and substrate



thin membrane
delaminates/buckles!

The morphology of compressed structures can be obtained by minimizing the total energy

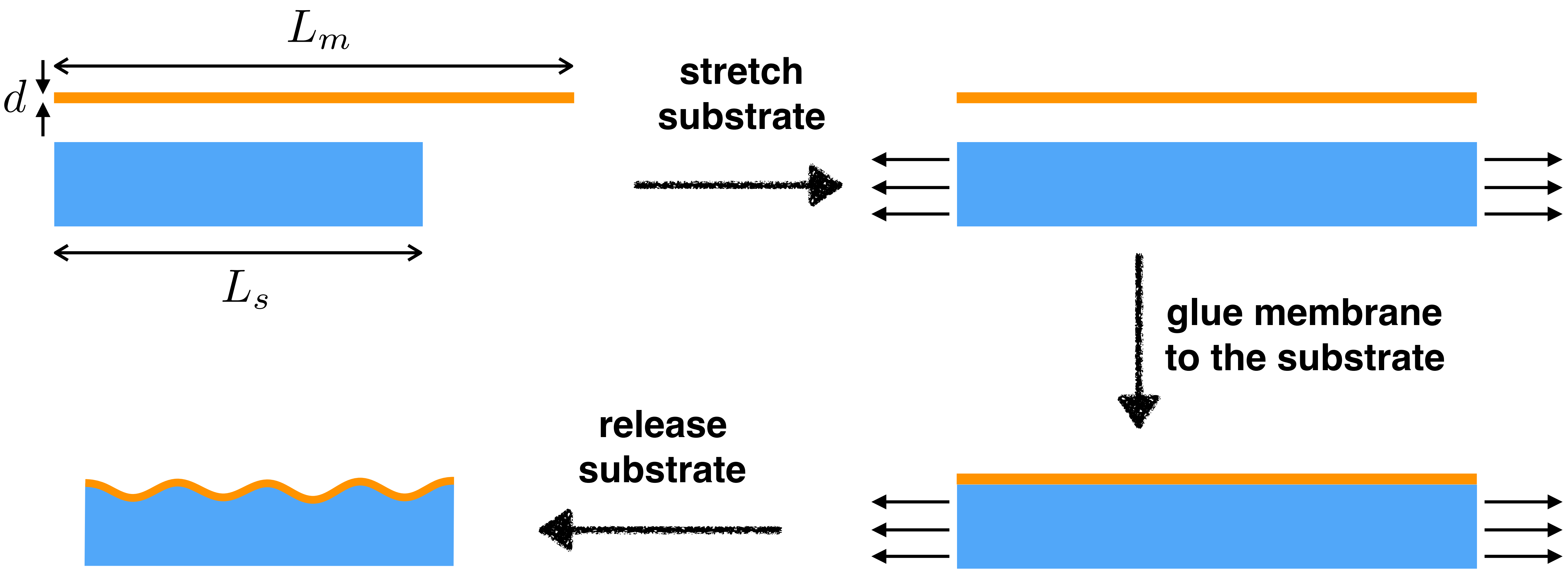
$$U_{\mathrm{total}} = U_{\mathrm{substrate}} + U_{\mathrm{membrane}} + U_{\mathrm{adhesion}}$$

- $U_{\mathrm{substrate}}$: elastic energy of deformed substrate
- U_{membrane}: elastic energy of deformed membrane
- U_{adhesion}: adhesion energy between membrane and substrate → $U = -\Gamma A$

Γ adhesion constant
A contact area

Compression of thin membranes on elastic substrates with finite adhesion

Experimental protocol



$$\epsilon = \frac{L_m - L_s}{L_m}$$ compressive strain

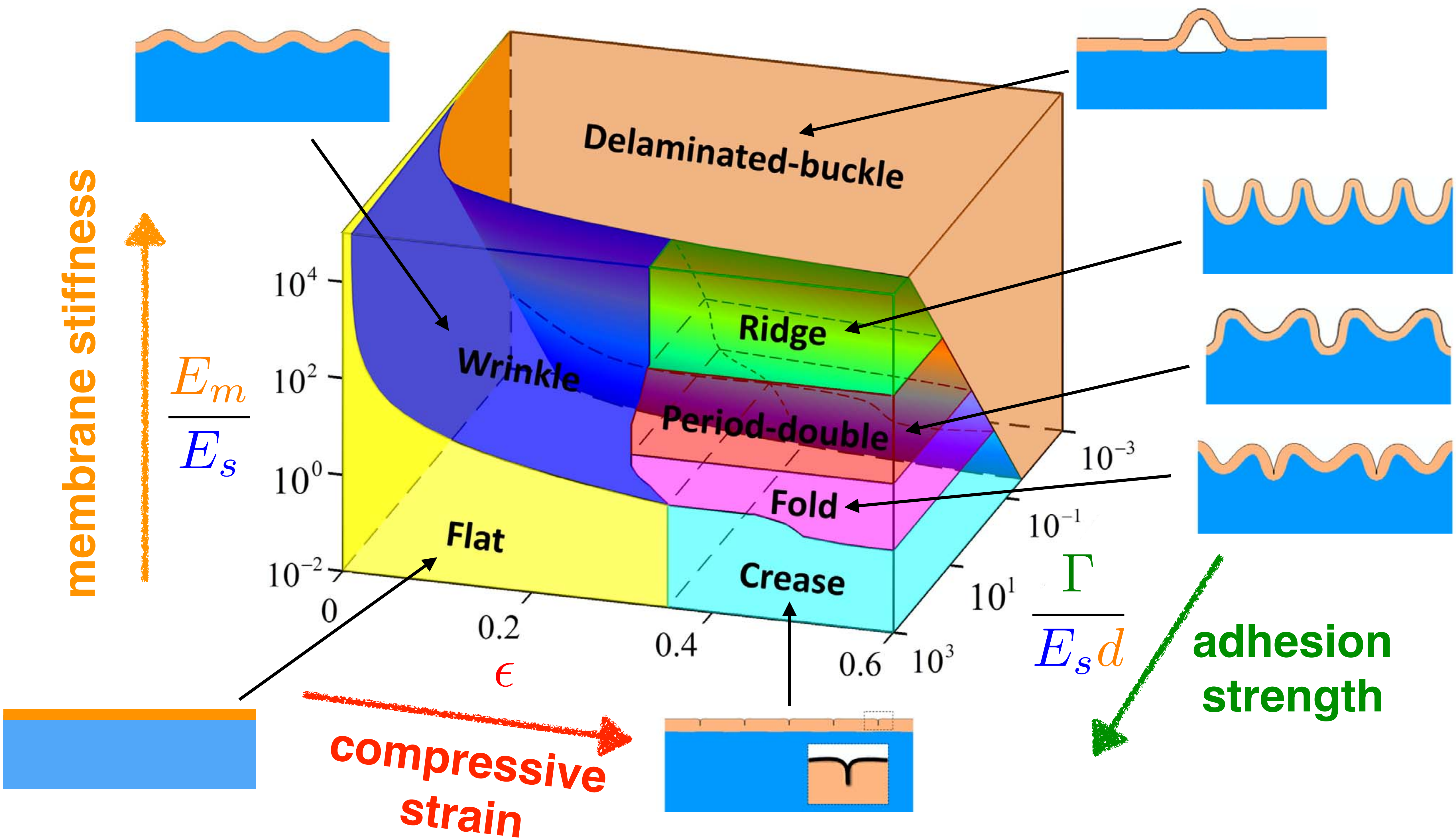
Γ adhesion constant (strength of glue)

E_m membrane Young's modulus

E_s substrate Young's modulus

Compression of thin membranes on elastic substrates with finite adhesion

Computationally predicted phase diagram



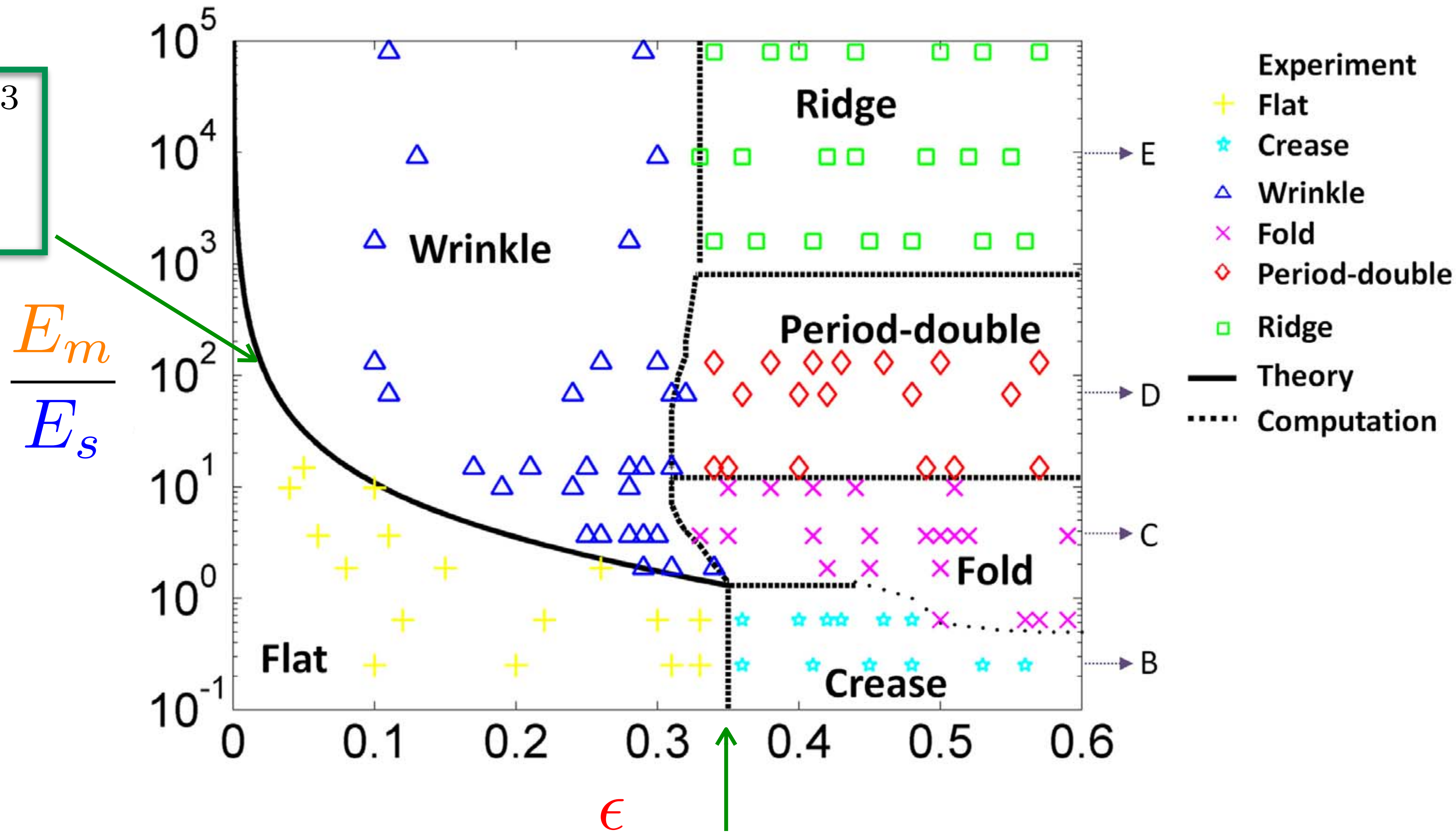
 Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

Compression of thin membranes on elastic substrates with finite adhesion

Very strong adhesion ($\Gamma/(E_s d) \gg 1$)

wrinkling transition

$$\epsilon_c \sim \left(\frac{E_s}{E_m}\right)^{2/3}$$



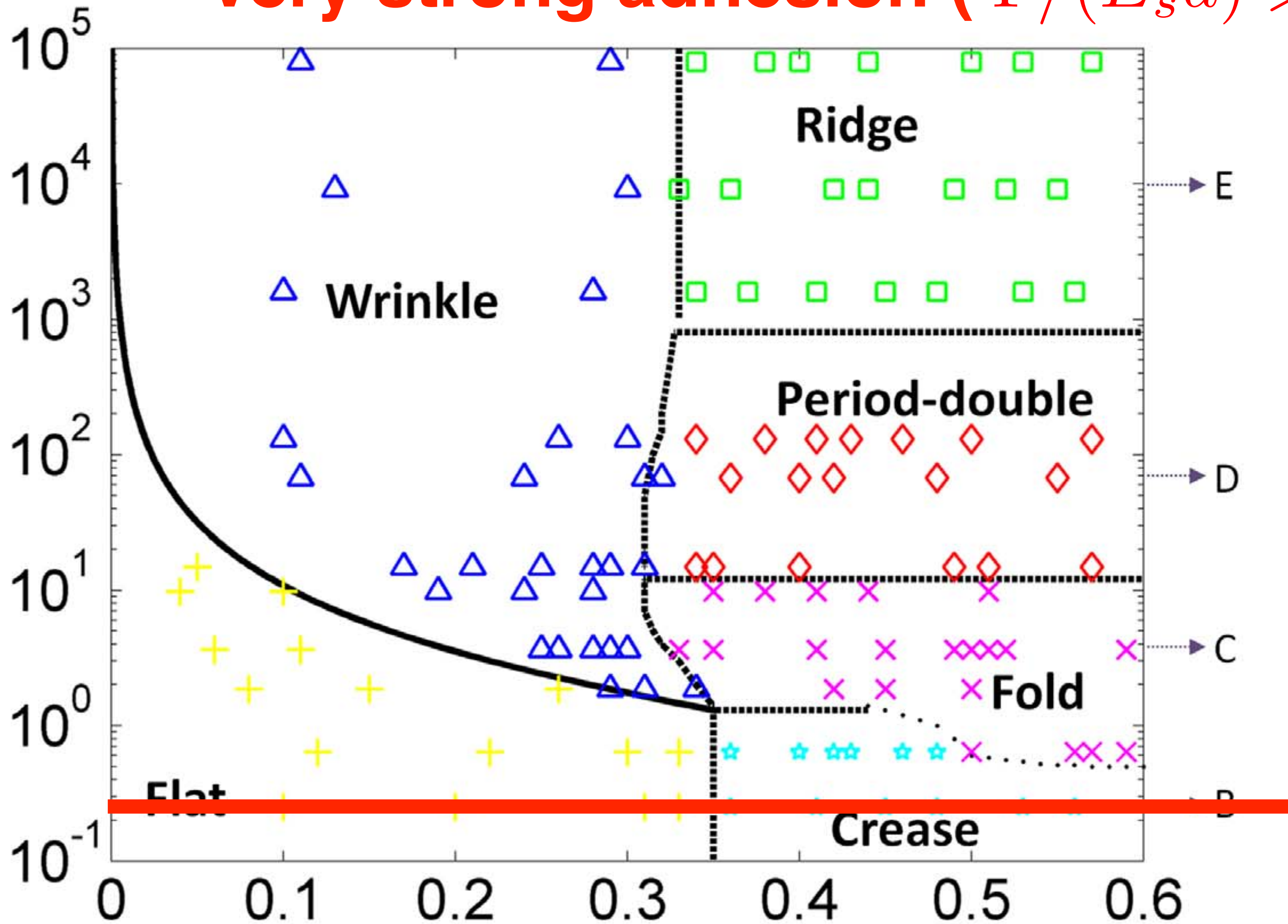
crease transition
$\epsilon_c = 0.35$

Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

Compression of thin membranes on elastic substrates with finite adhesion

Very strong adhesion ($\Gamma/(E_s d) \gg 1$)

$\frac{E_m}{E_s}$



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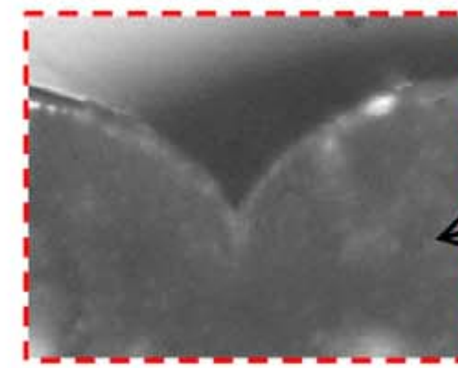
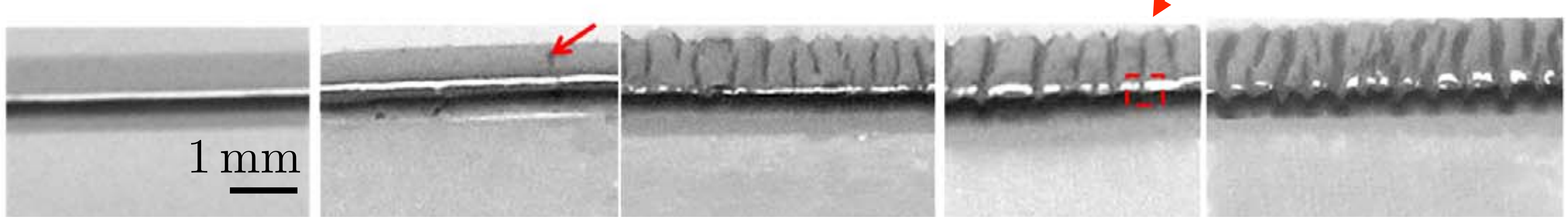


$\frac{E_m}{E_s} = 0.3$



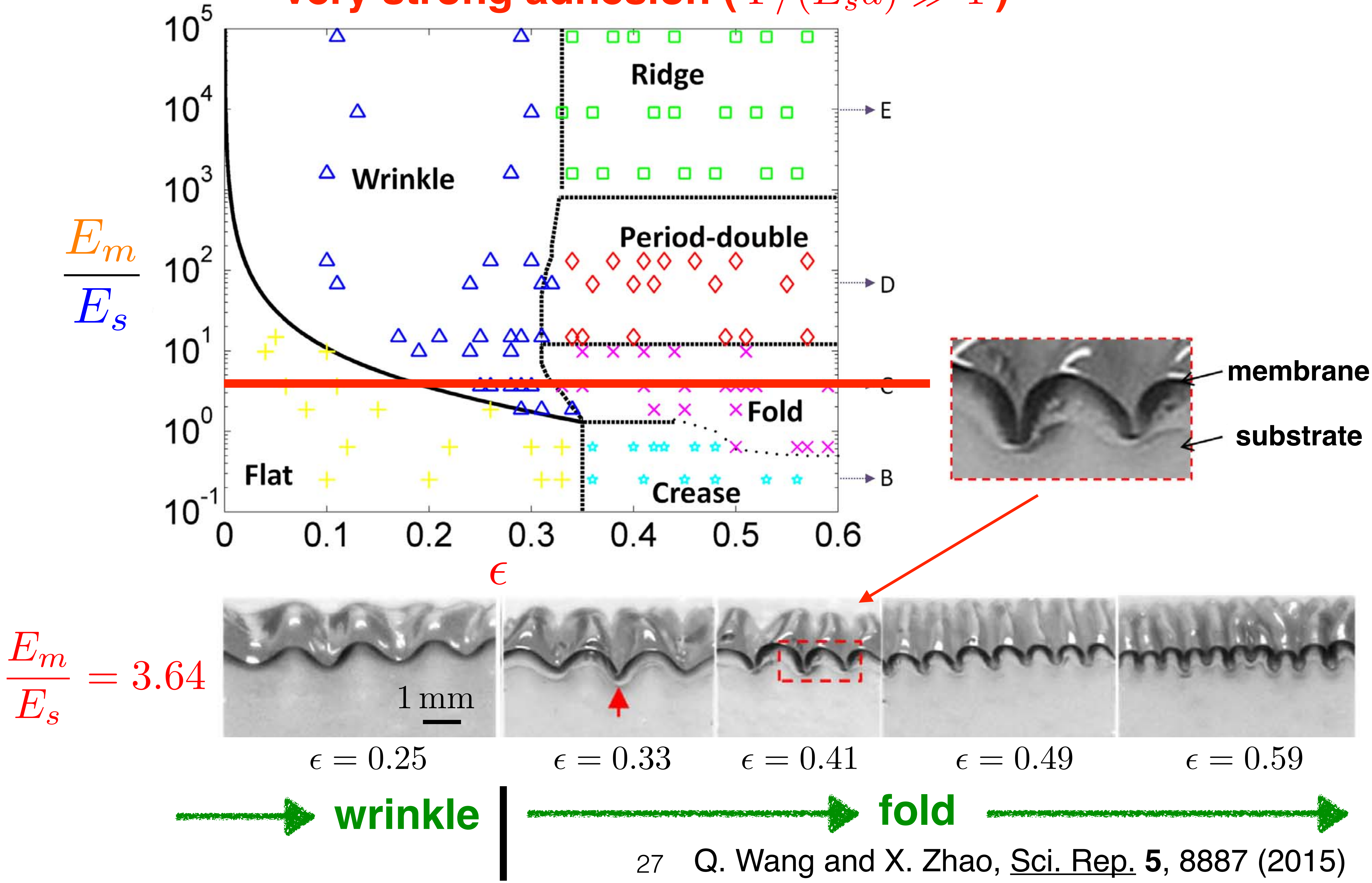
$\epsilon = 0.31$ $\epsilon = 0.36$ $\epsilon = 0.41$ $\epsilon = 0.48$ $\epsilon = 0.56$

flat | crease

Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

Compression of thin membranes on elastic substrates with finite adhesion

Very strong adhesion ($\Gamma/(E_s d) \gg 1$)



Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

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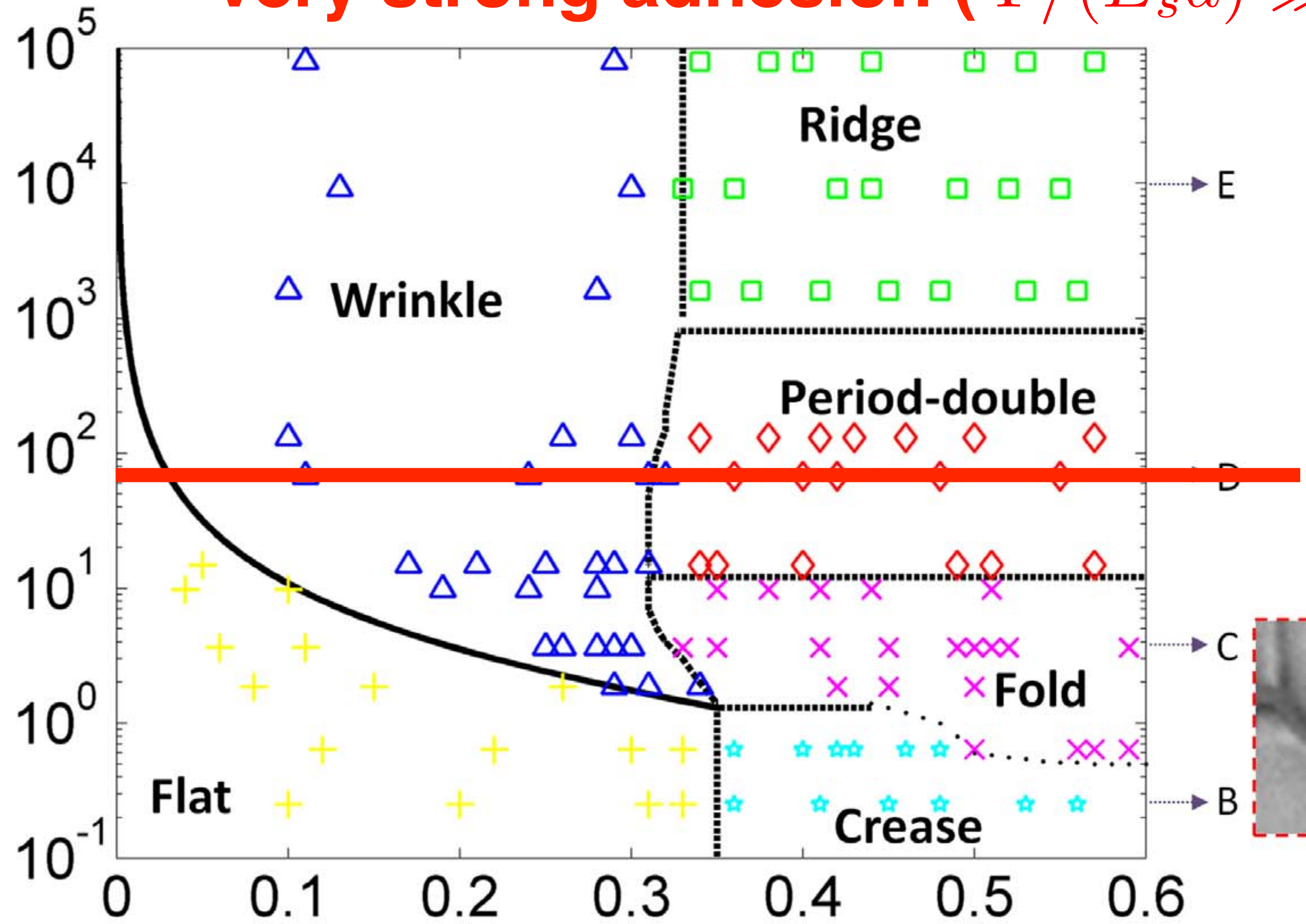
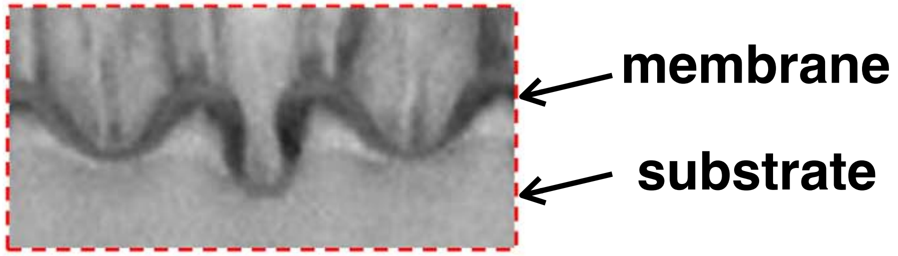
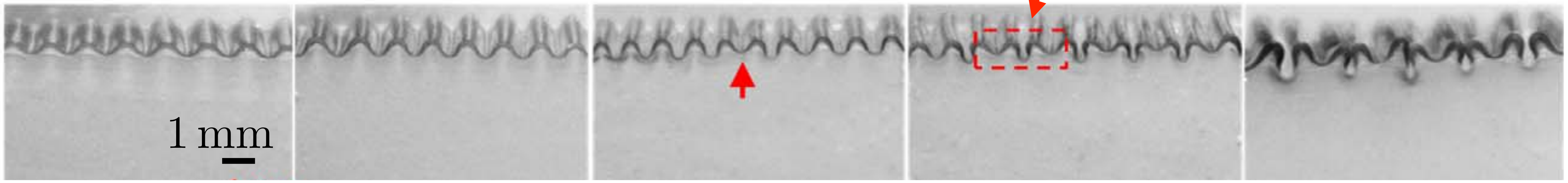




$$\frac{E_m}{E_s} = 67.24$$



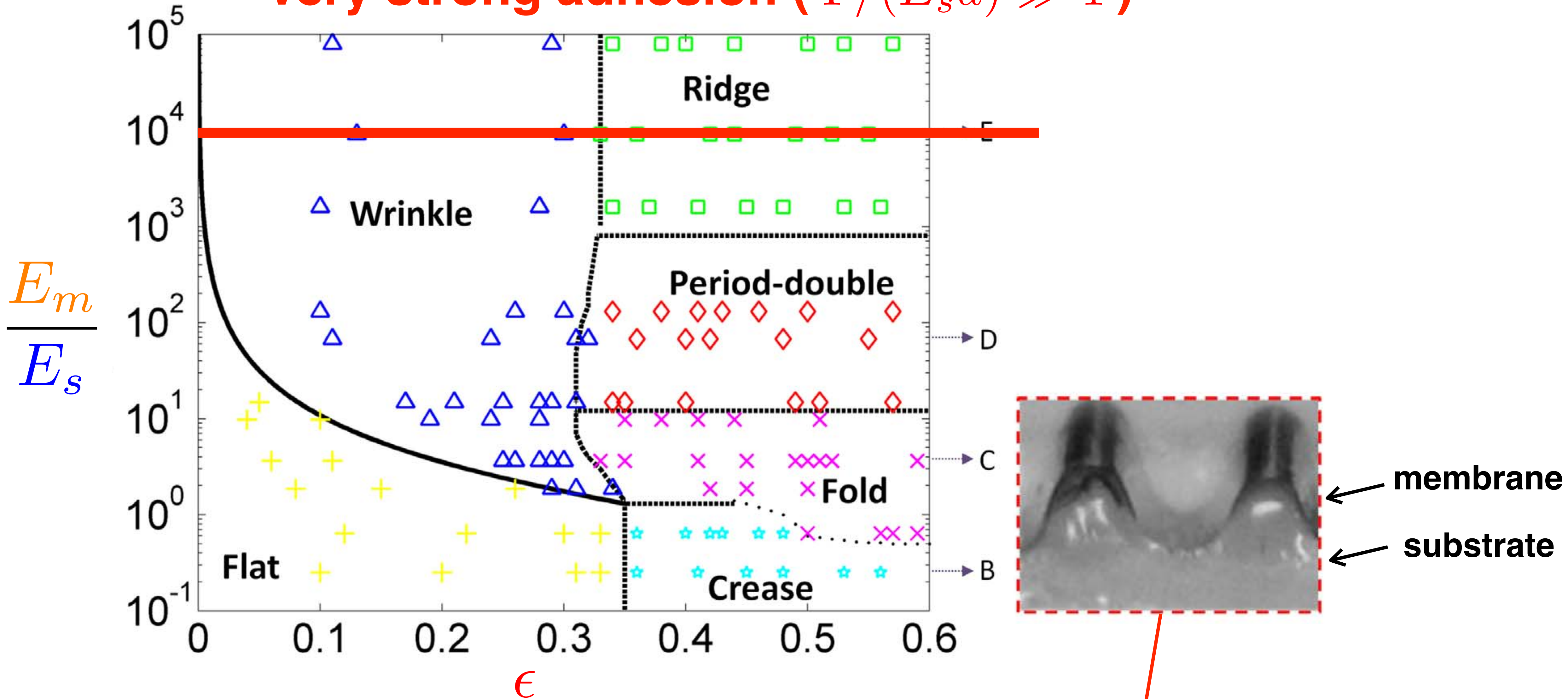
$\epsilon = 0.11$ $\epsilon = 0.24$ $\epsilon = 0.31$ $\epsilon = 0.36$ $\epsilon = 0.48$

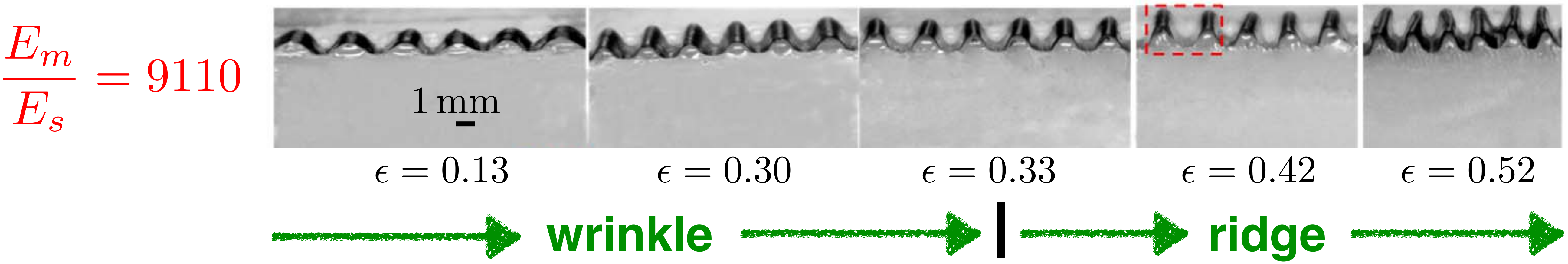
wrinkle | **period-double**

Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

Compression of thin membranes on elastic substrates with finite adhesion

Very strong adhesion ($\Gamma/(E_s d) \gg 1$)





Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

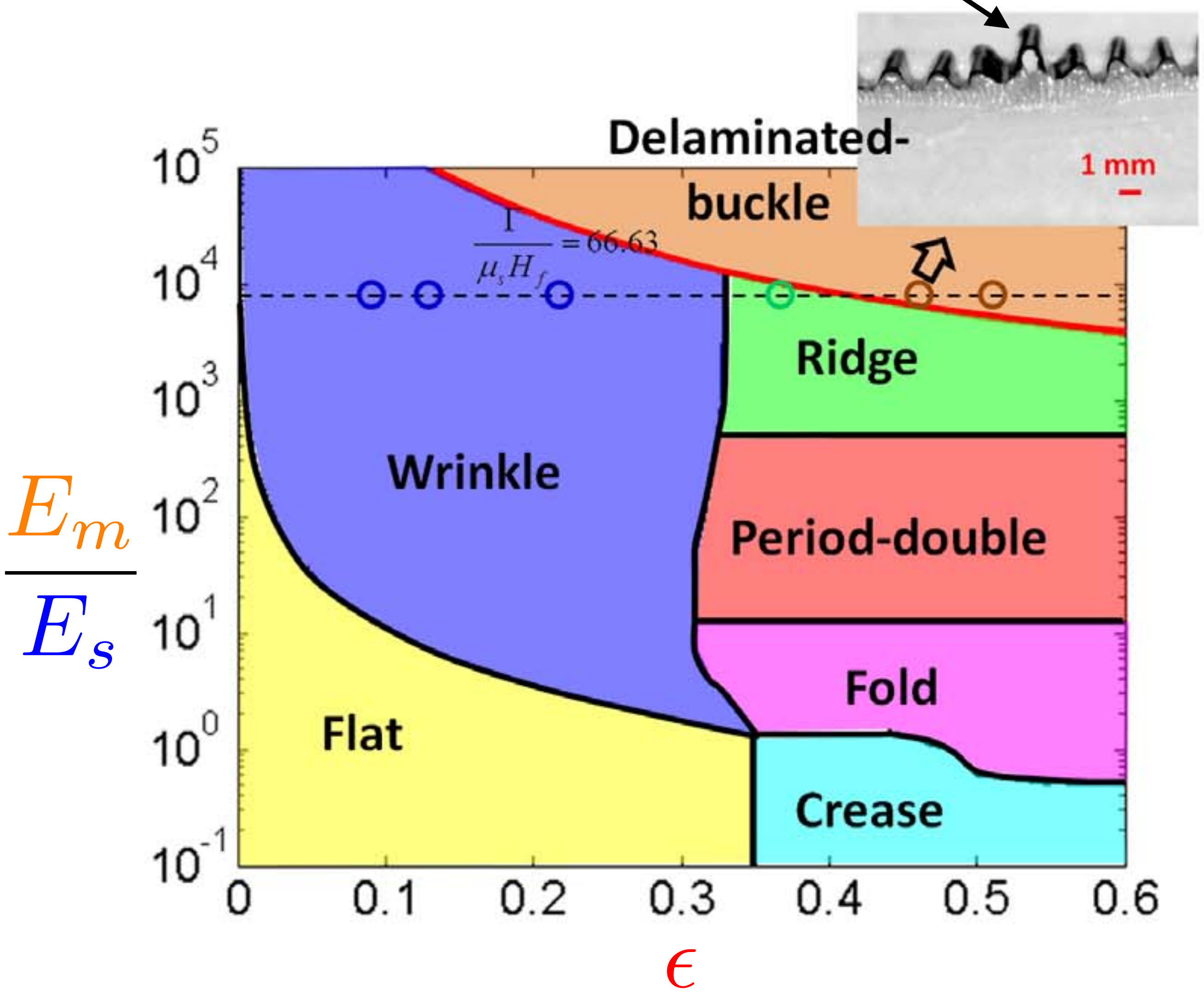
Compression of thin membranes on elastic substrates with finite adhesion

Strong adhesion

$$\frac{\Gamma}{E_s d} = 66.63$$

all phases are present

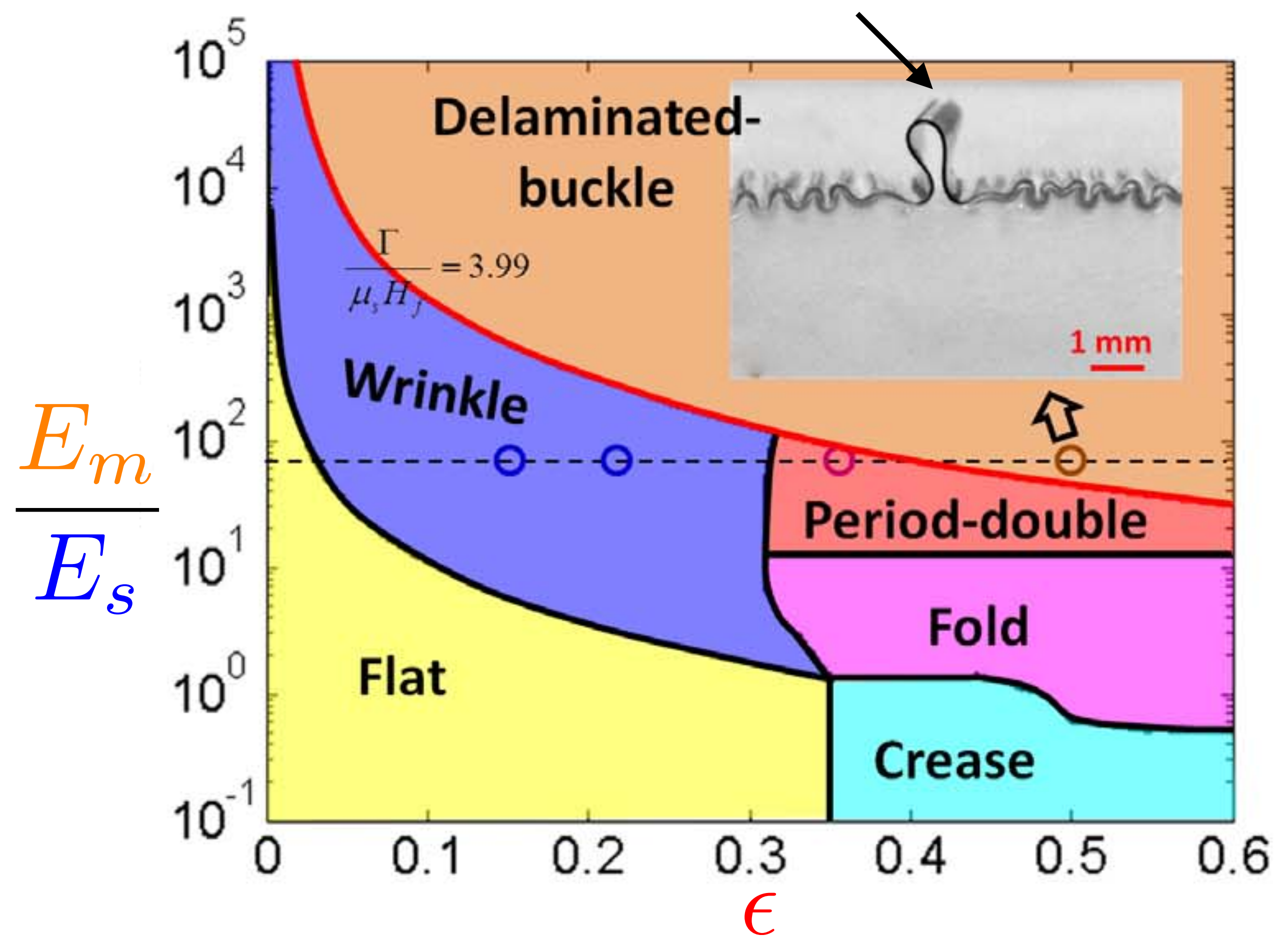
delamination/buckling of ridges



$$\frac{\Gamma}{E_s d} = 3.99$$

the region of "Delaminated-buckle" phase expands, the "Ridge" phase disappears

delamination/buckling of wrinkles



Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

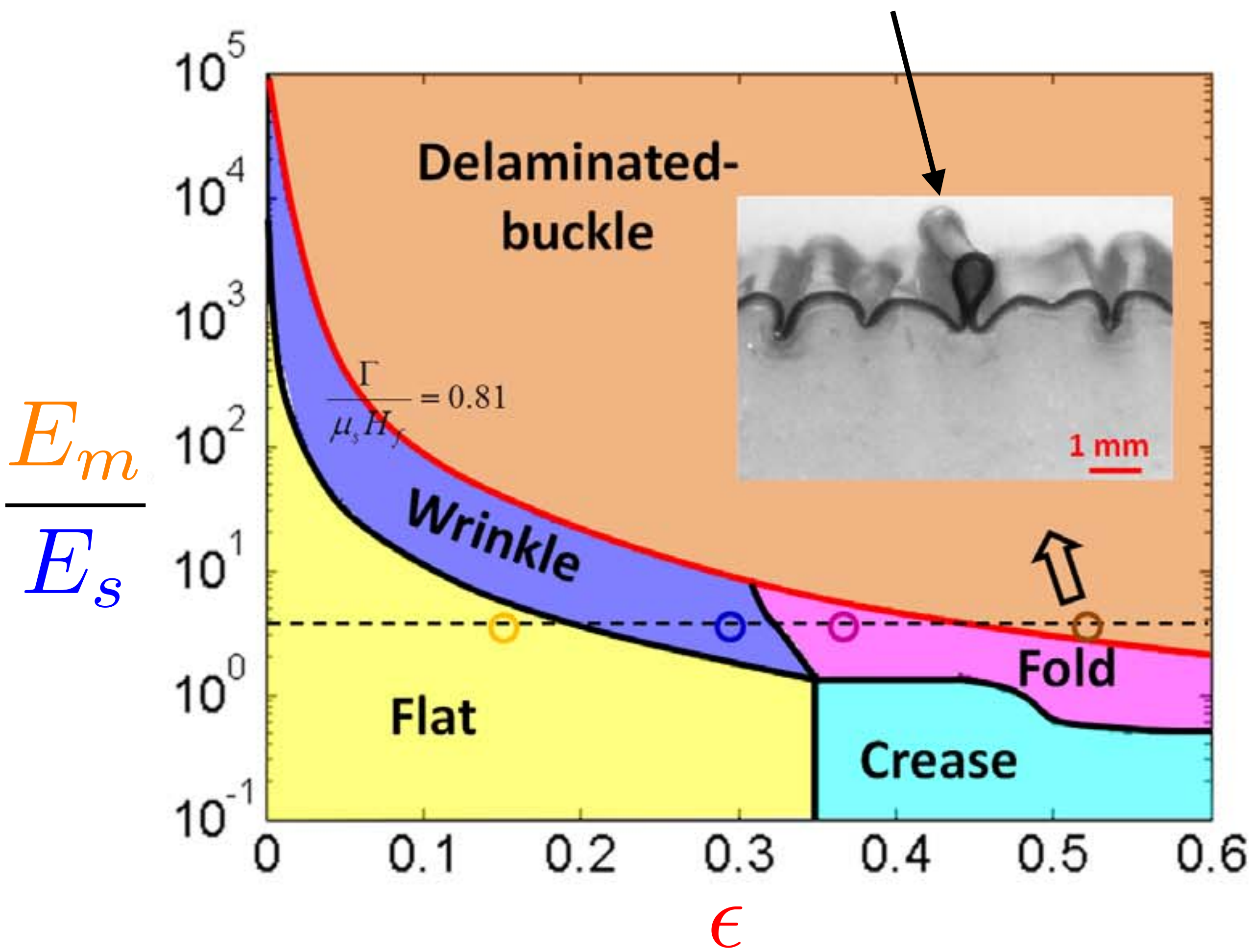
Compression of thin membranes on elastic substrates with finite adhesion

Moderate adhesion

$$\frac{\Gamma}{E_s d} = 0.81$$

"Ridge" and "Period-double" phases disappear

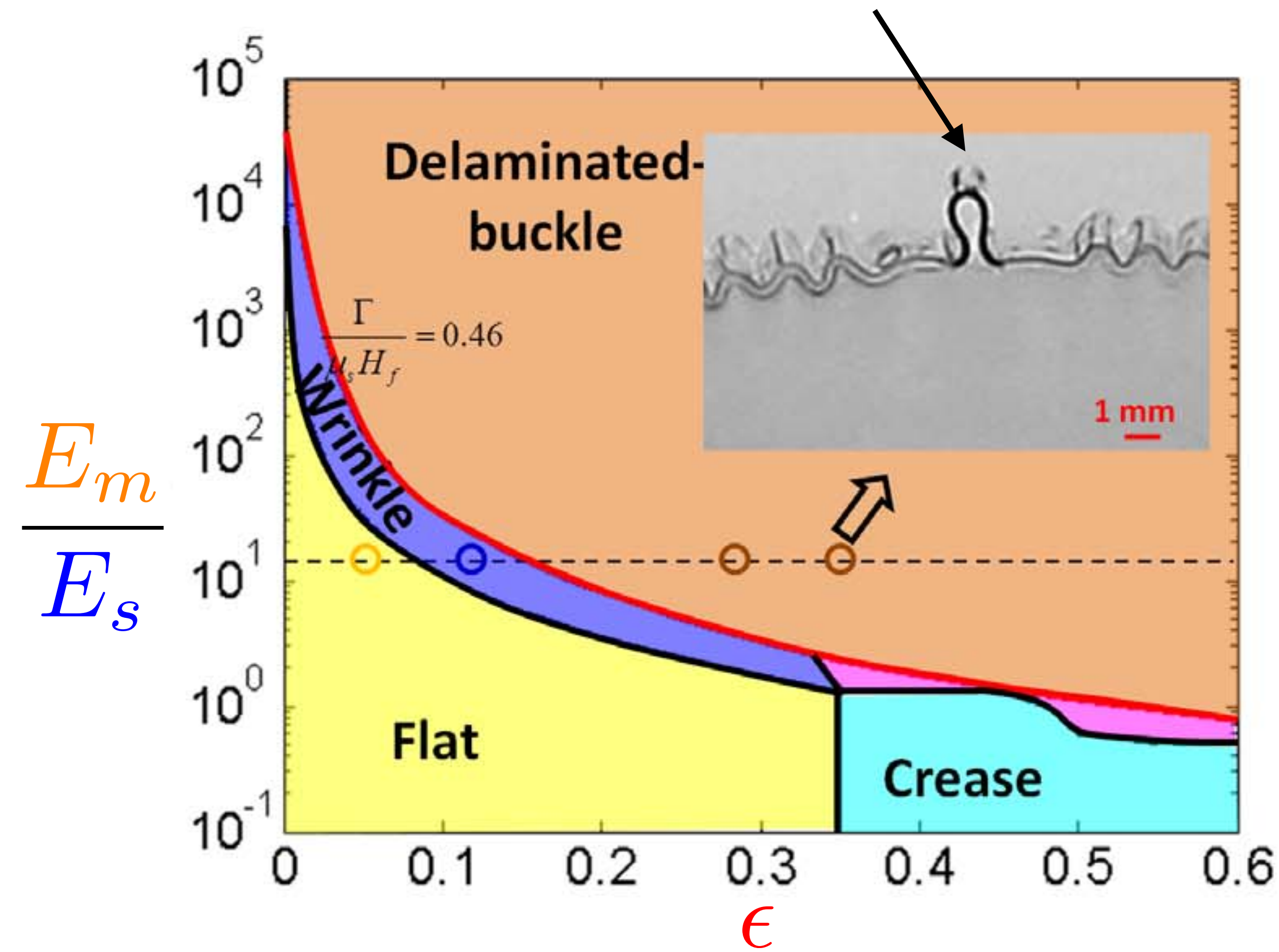
delamination/buckling of folds



$$\frac{\Gamma}{E_s d} = 0.46$$

"Ridge" and "Period-double" phases disappear

delamination/buckling of wrinkles



Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

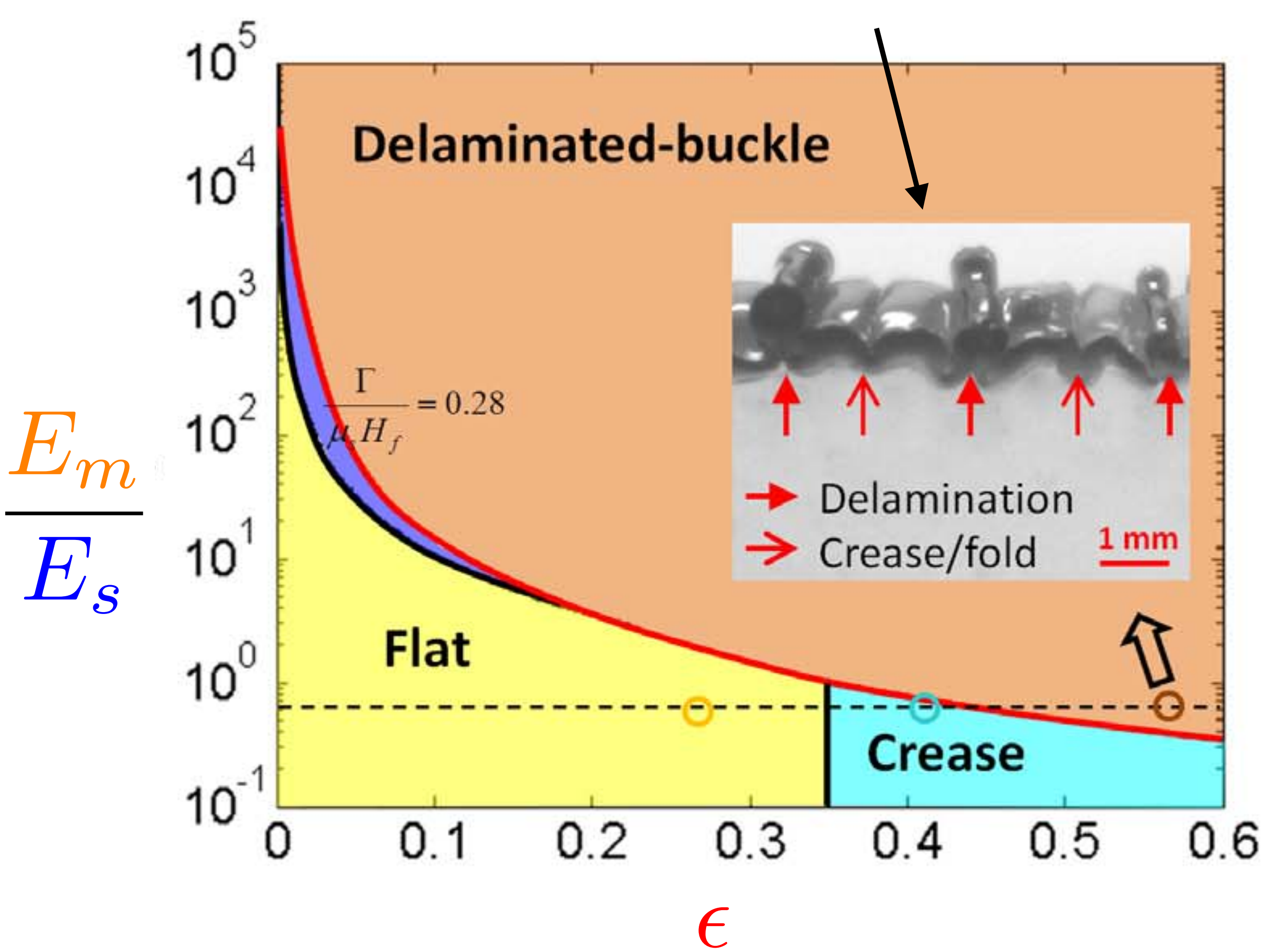
Compression of thin membranes on elastic substrates with finite adhesion

Weak adhesion

$$\frac{\Gamma}{E_s d} = 0.28$$

"Ridge", "Period-double" and "Fold" phases disappear

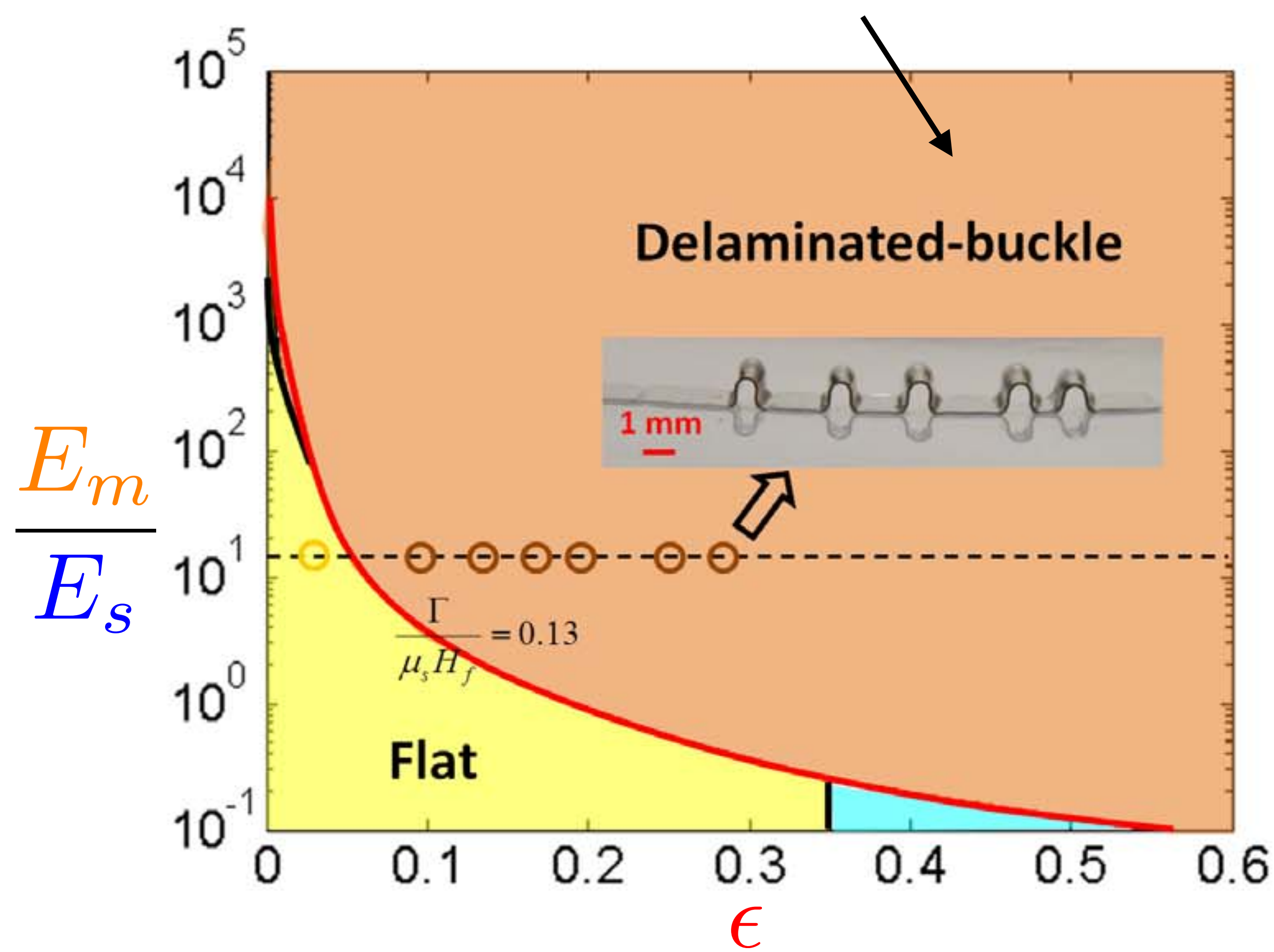
delamination/buckling of creases



$$\frac{\Gamma}{E_s d} = 0.13$$

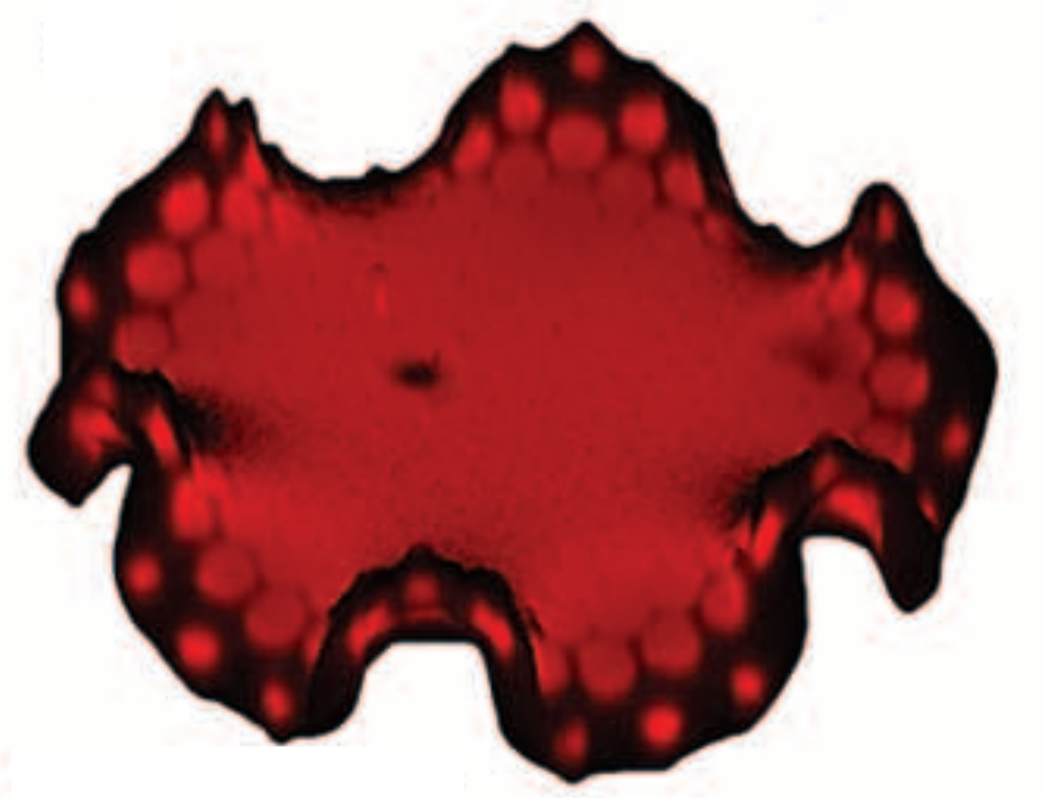
delaminatied/buckled phase almost completely takes over the other phases

delamination/buckling of flat phase



Q. Wang and X. Zhao, Sci. Rep. **5**, 8887 (2015)

Shapes of growing/swelling sheets and coiling of rods



Shapes of flowers and leaves

saddles

wrinkled edges

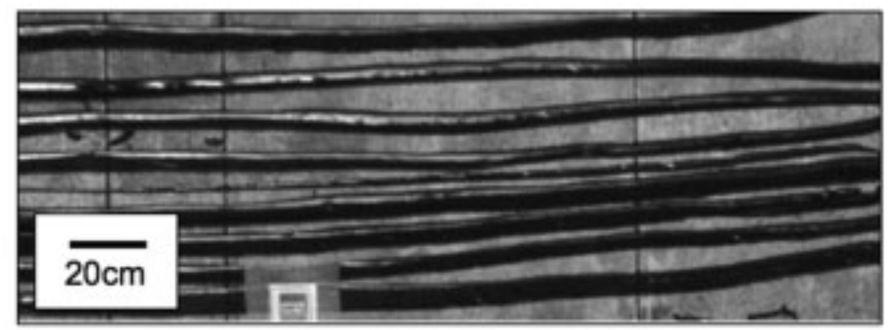
helices

Wrinkled and straight blades in macroalgae

bull kelp (seaweed)

Slow water flow environment (v~0.5 m/s)

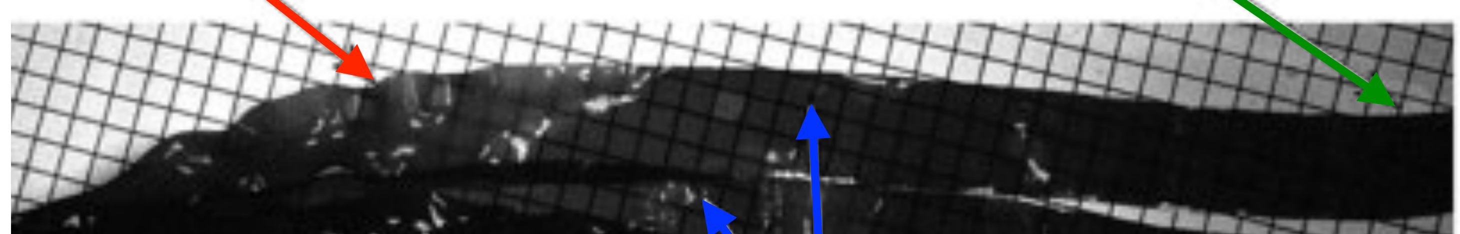
Fast water flow environment (v~1.5 m/s)



new growth after transplantation (wrinkled)

old growth before transplanted (flat)

Transplantation of blade from one environment to the other changes morphology!

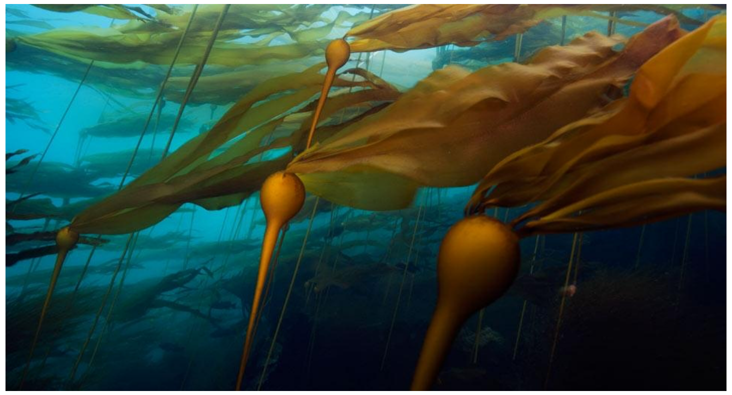


blades

M. Koehl et al., Integ. Comp. Biol. 48, 834 (2008)

Wrinkled and straight blades in macroalgae

bull kelp
(seaweed)



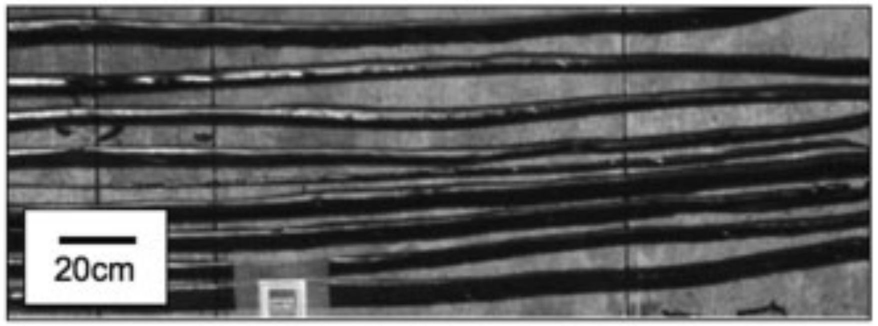
Slow water flow environment (v~0.5 m/s)

increased drag

blades flap like flags

flapping prevents bundling of blades, which can thus receive more sunlight (photosynthesis)

Fast water flow environment (v~1.5 m/s)



reduced drag to prevent detachment from base (=death)

minimal flapping

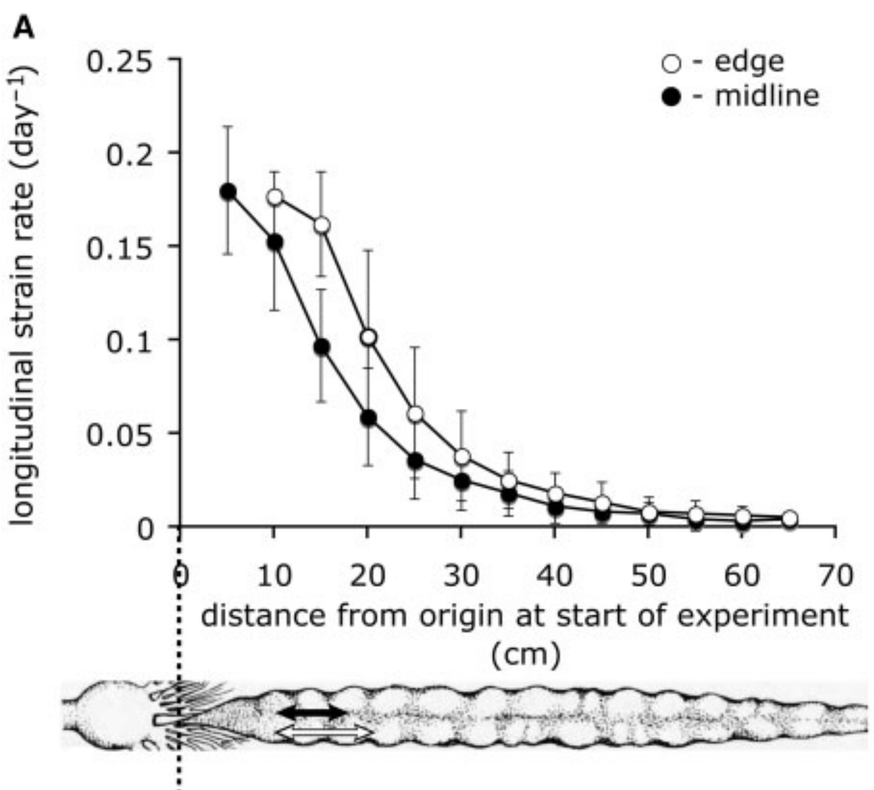
blades bundle together and some blades on the bottom receive less sunlight

M. Koehl et al., Integ. Comp. Biol. 48, 834 (2008)

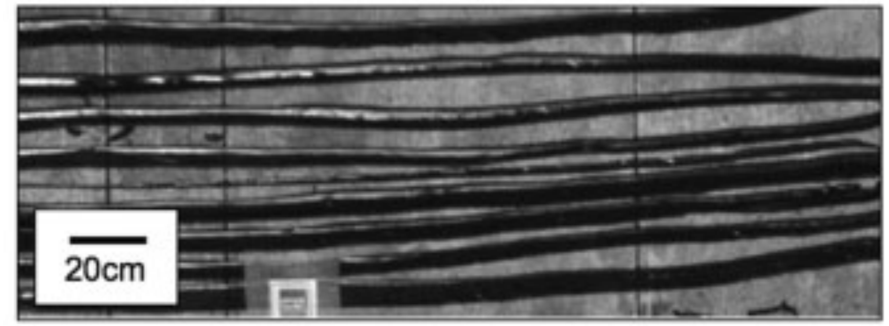
Wrinkled and straight blades in macroalgae

Slow water flow environment (v~0.5 m/s)

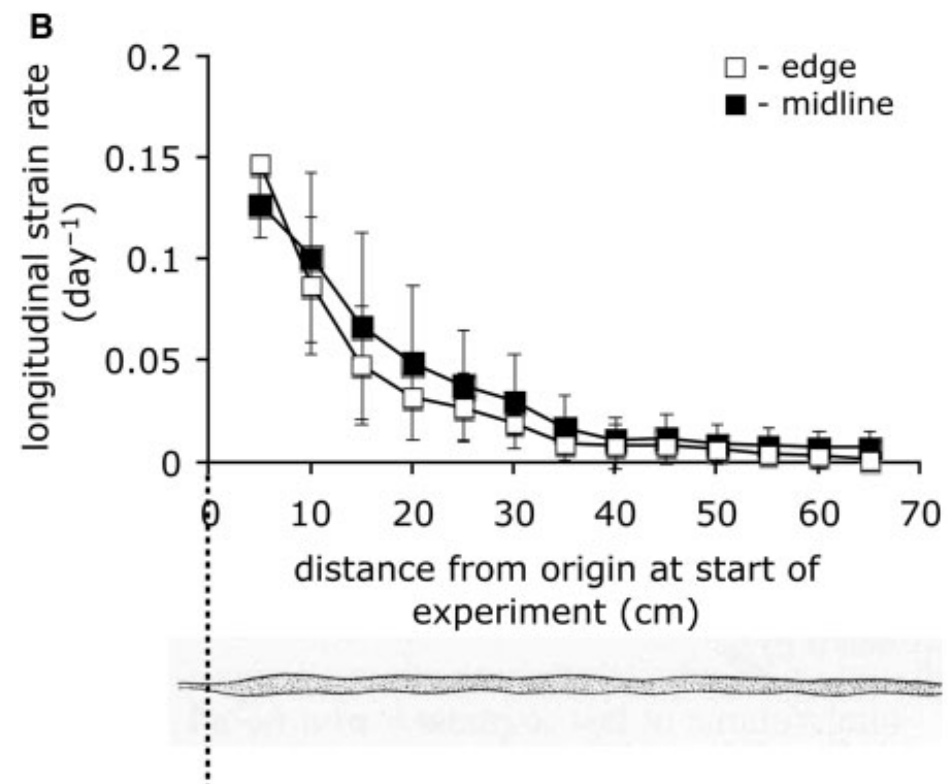
edges of blades grow faster than the midline



Fast water flow environment (v~1.5 m/s)



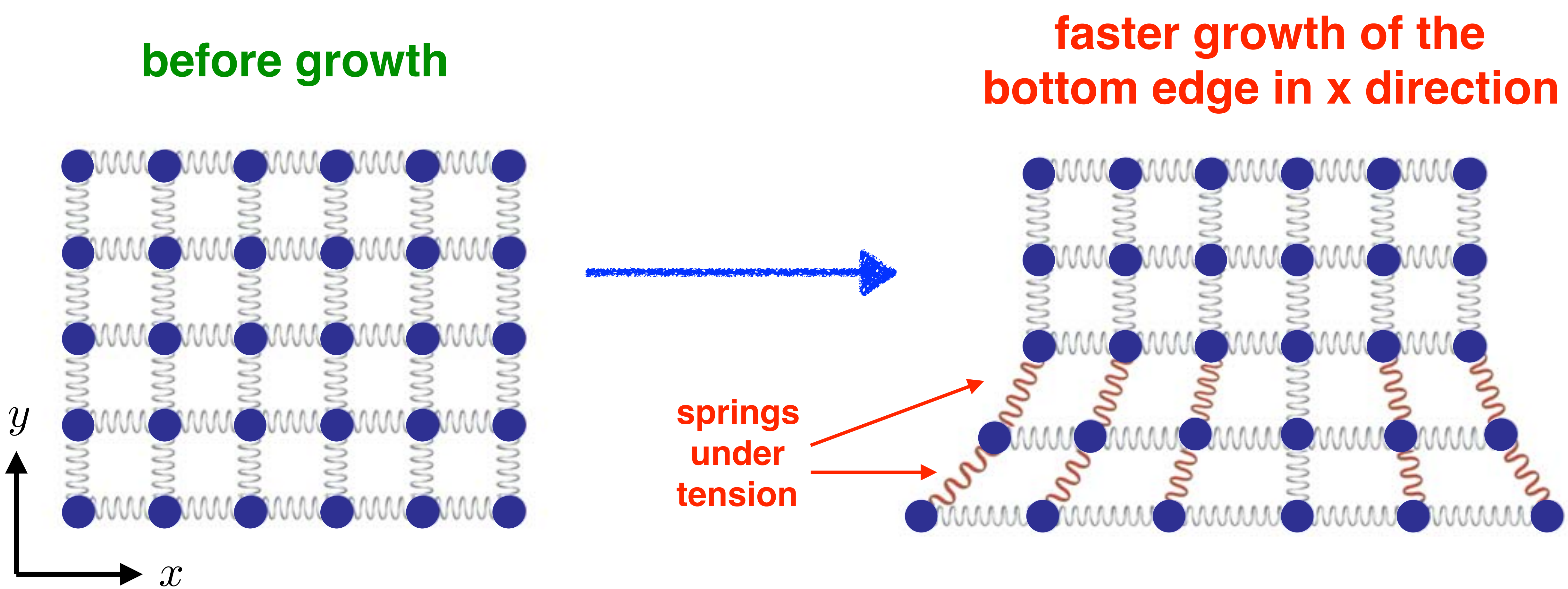
edges of blades grow at the same speed as the midline



What is the effect of differential growth rate between the edge and the midline of the blade?

M. Koehl et al., Integ. Comp. Biol. 48, 834 (2008)

Differential growth produces internal stress



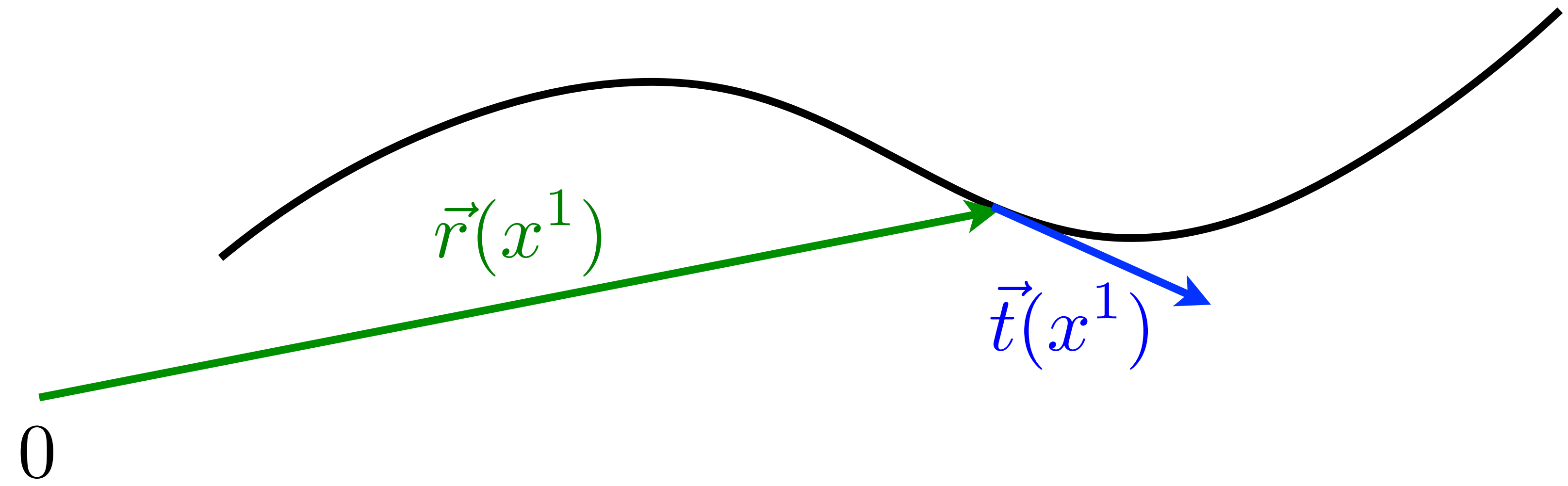
Differential growth produces internal stresses, which can be partially released via bending!

Next: Short detour to differential geometry.

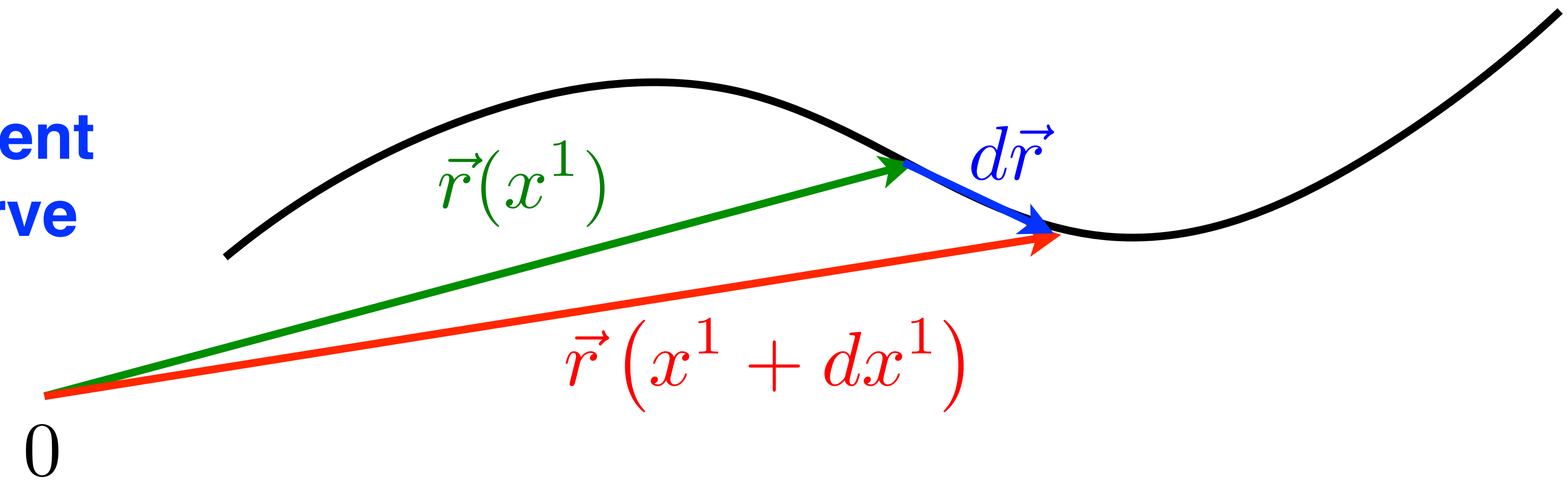
Metric for measuring distances along curves

x^1 **parameter describing position along the curve**

$\vec{r}(x^1)$ **function describing shape of the curve**



$$\vec{t}(x^1) = \frac{d\vec{r}(x^1)}{dx^1}$$ **local tangent to the curve**



metric for measuring lengths

$$d\ell^2 = d\vec{r}^{\,2} = \vec{t}^{\,2}\left(dx^1\right)^2 = g\ \left(dx^1\right)^2$$

$$g = \vec{t}^{\,2}$$

$$d\ell = \sqrt{g}dx^1$$

Natural parametrization corresponds to $g \equiv 1$, where x^1 measures distance along the beam.

Metric for measuring distances along curves

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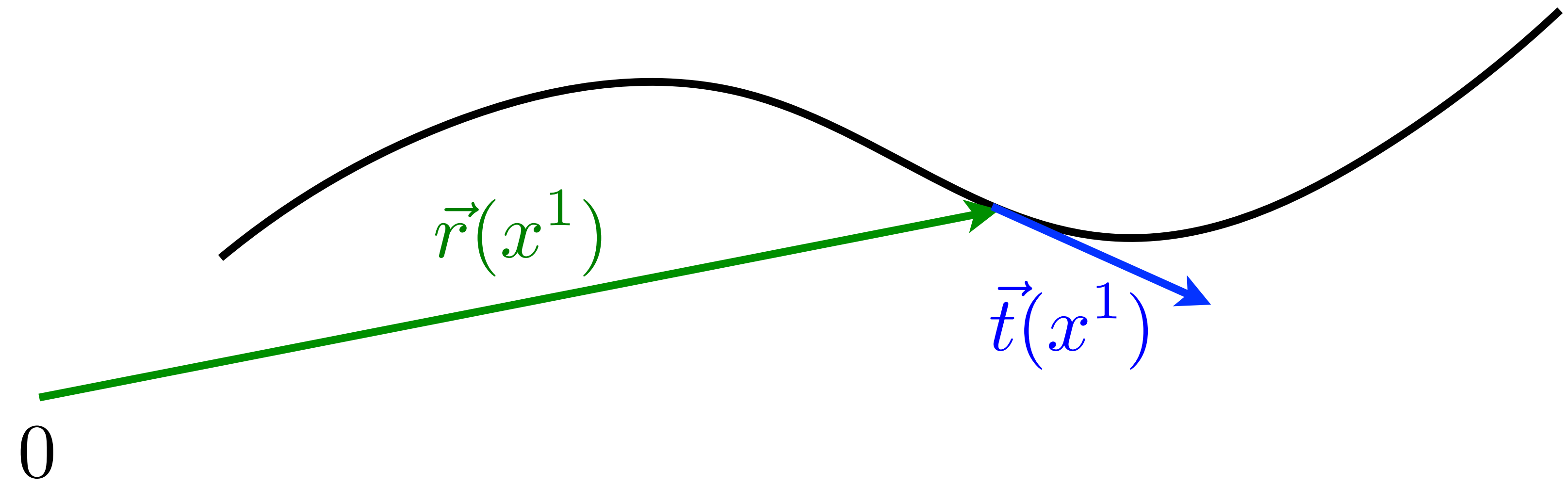
$\vec{r}(x^1)$ **function describing shape of the curve**



$$\vec{t}(x^1) = \frac{d\vec{r}(x^1)}{dx^1}$$ **local tangent to the curve**

metric for measuring lengths

$$d\ell^2 = d\vec{r}^{\,2} = \vec{t}^{\,2}\,(dx^1)^2 = g\,(dx^1)^2$$

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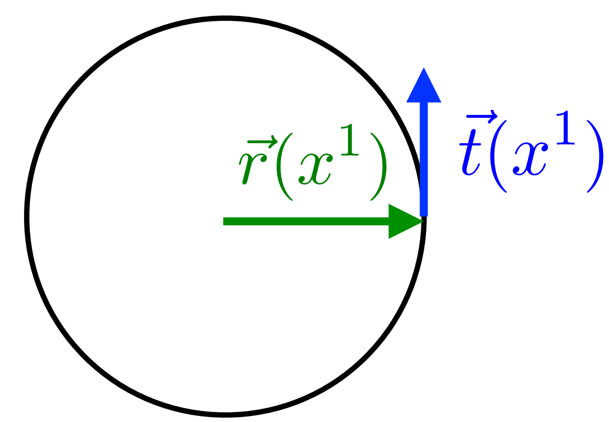
$$d\ell = \sqrt{g}\,dx^1$$

Natural parametrization corresponds to $g \equiv 1$, where x^1, measures distance along the beam.

Example



$$\vec{r}(x^1) = R\big(\cos(\omega x^1), \sin(\omega x^1)\big)$$

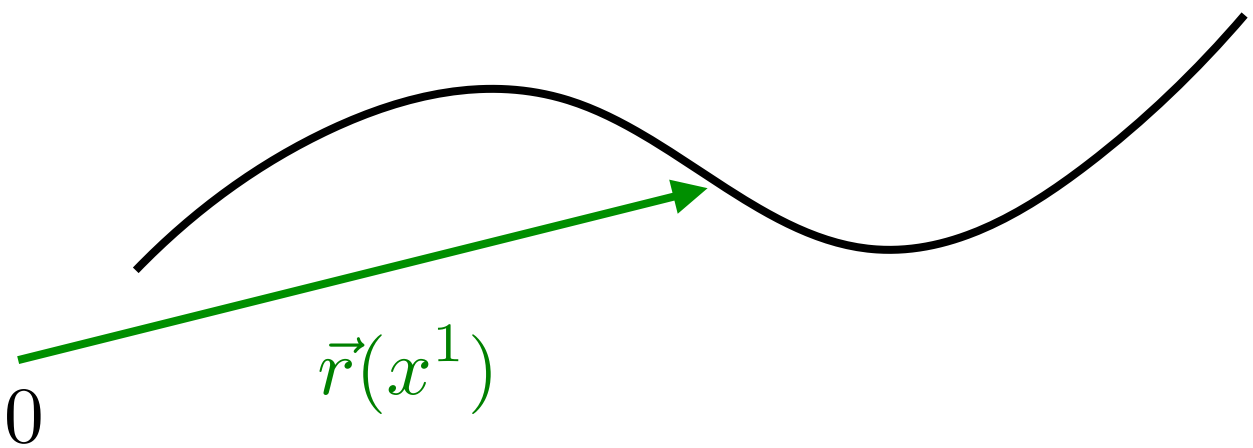
$$\vec{t}(x^1) = R\omega\big(-\sin(\omega x^1), \cos(\omega x^1)\big)$$

$$g(x^1) = R^2\omega^2$$

$$d\ell = R\omega\, dx^1$$

Strain and energy of beam deformations

undeformed beam



$$g = \left(d\vec{r}/dx^1\right)^2$$

$$d\ell = \sqrt{g}dx^1$$

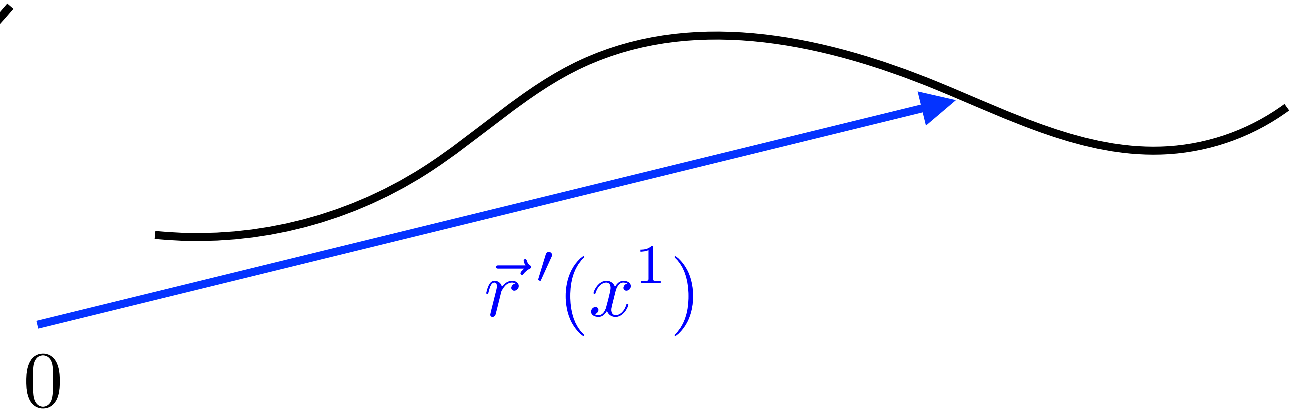
strain

$$d\ell'^2 - d\ell^2 = (2\epsilon + \epsilon^2)d\ell^2 \approx 2\epsilon\, d\ell^2$$

$$\epsilon = \frac{d\ell'^2 - d\ell^2}{2d\ell^2} = \frac{1}{2}g^{-1}\left(g' - g\right)$$

strain measures the difference of metric g' for deformed beam from the preferred metric g !

deformed beam



$$g' = \left(d\vec{r}'/dx^1\right)^2$$

$$d\ell' = \sqrt{g'}dx^1 = d\ell(1+\epsilon)$$

Energy cost for stretching/compressing

$$U = \int \left(\sqrt{g}dx^1\right)\, \frac{1}{2}k\epsilon^2$$

$$k = EA$$

E **- 3D Young’s modulus**

A **- beam cross-section area**

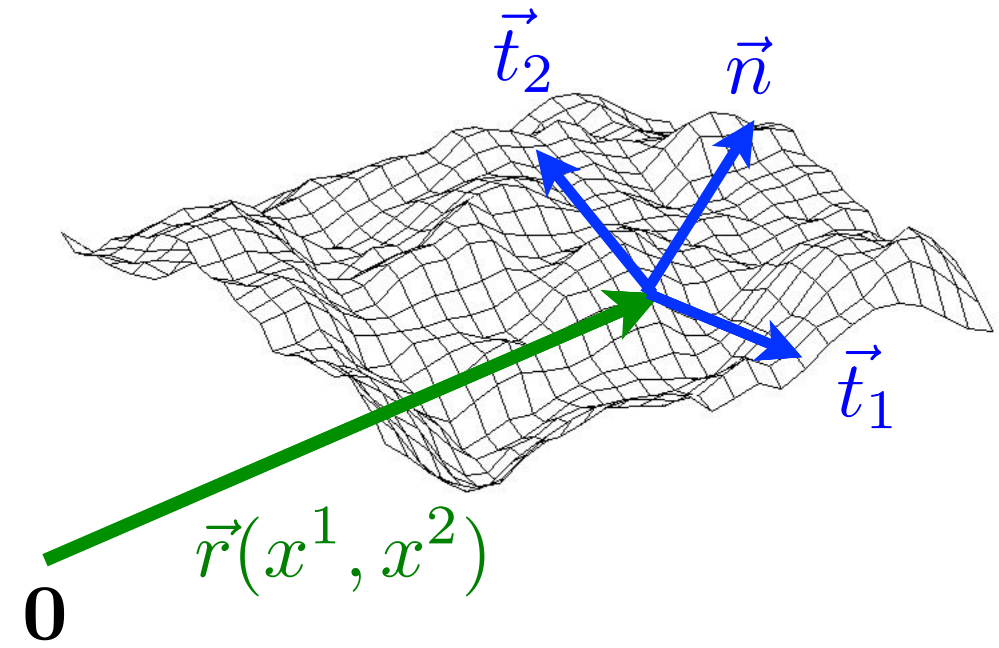
Metric tensor for measuring distances on surfaces

x^1, x^2 **parameters describing position along the surface**

$\vec{r}(x^1, x^2)$ **function describing shape of the surface**

$\vec{t}_i = \dfrac{\partial \vec{r}}{\partial x^i}$ **local tangent vectors to the surface**

$\vec{n} = \dfrac{\vec{t}_1 \times \vec{t}_2}{|\vec{t}_1 \times \vec{t}_2|}$ **unit normal vector of the surface**



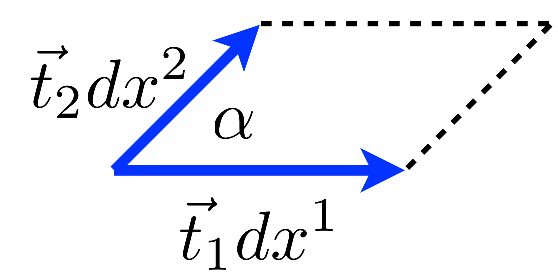
metric tensor for measuring lengths

$$d\ell^2 = d\vec{r}^{\,2} = \sum_{i,j} \vec{t}_i \cdot \vec{t}_j dx^i dx^j = \sum_{i,j} g_{ij} dx^i dx^j$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} \vec{t}_1 \cdot \vec{t}_1, & \vec{t}_1 \cdot \vec{t}_2 \\ \vec{t}_2 \cdot \vec{t}_1 & \vec{t}_2 \cdot \vec{t}_2 \end{pmatrix}$$

$$g = \det(g_{ij}) = |\vec{t}_1 \times \vec{t}_2|^2$$

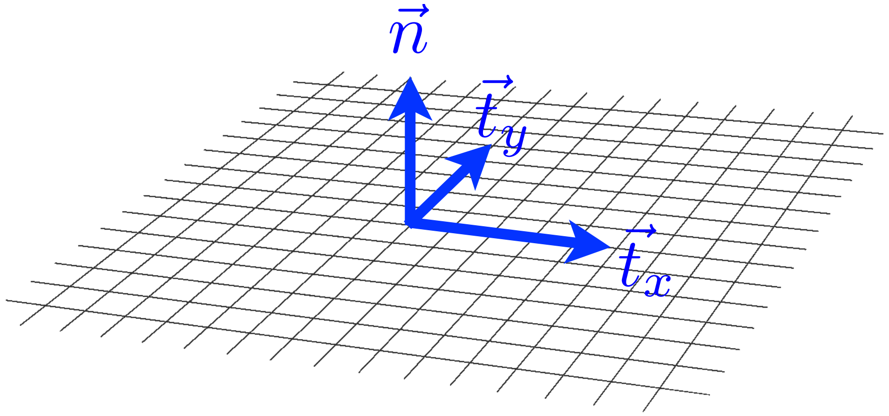
area element



$$dA = |\vec{t}_1||\vec{t}_2| \sin\alpha \, dx^1 dx^2$$

$$dA = \sqrt{g}\, dx^1 dx^2$$

Examples



$$\vec{r}(x,y) = (x,y,0)$$

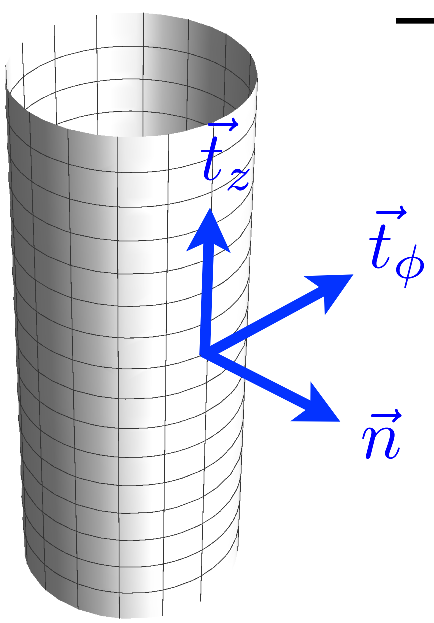
$$\vec{t}_x = \frac{\partial \vec{r}}{\partial x} = (1,0,0)$$

$$\vec{t}_y = \frac{\partial \vec{r}}{\partial y} = (0,1,0)$$

$$\vec{n} = \frac{\vec{t}_x \times \vec{t}_y}{|\vec{t}_x \times \vec{t}_y|} = (0,0,1)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

$$dA = dxdy$$



$$\vec{r}(\phi,z) = (R\cos\phi, R\sin\phi, z)$$

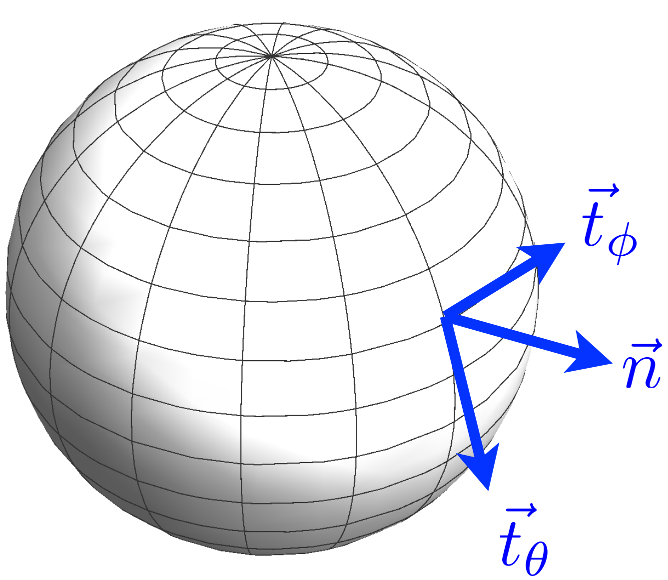
$$\vec{t}_\phi = \frac{\partial \vec{r}}{\partial \phi} = R(-\sin\phi, \cos\phi, 0)$$

$$\vec{t}_z = \frac{\partial \vec{r}}{\partial z} = (0,0,1)$$

$$\vec{n} = \frac{\vec{t}_\phi \times \vec{t}_z}{|\vec{t}_\phi \times \vec{t}_z|} = (\cos\phi, \sin\phi, 0)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} R^2, & 0 \\ 0, & 1 \end{pmatrix}$$

$$dA = R\, d\phi dz$$



$$\vec{r}(\theta,\phi) = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{t}_\theta = \frac{\partial \vec{r}}{\partial \theta} = R(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\vec{t}_\phi = \frac{\partial \vec{r}}{\partial \phi} = R\sin\theta(-\sin\phi, \cos\phi, 0)$$

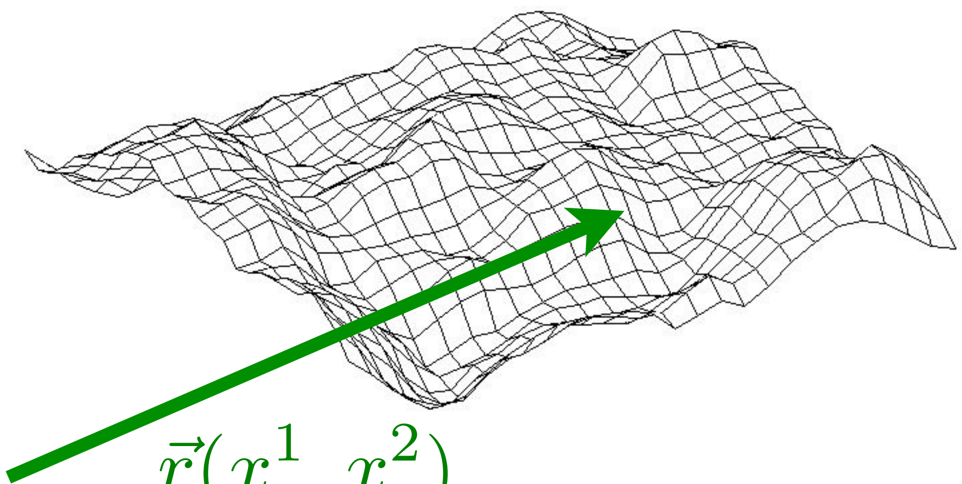
$$\vec{n} = \frac{\vec{t}_\theta \times \vec{t}_\phi}{|\vec{t}_\theta \times \vec{t}_\phi|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} R^2, & 0 \\ 0, & R^2\sin^2\theta \end{pmatrix}$$

$$dA = R^2 \sin\theta\, d\theta d\phi$$

Strain tensor and energy of shell deformations

undeformed shell



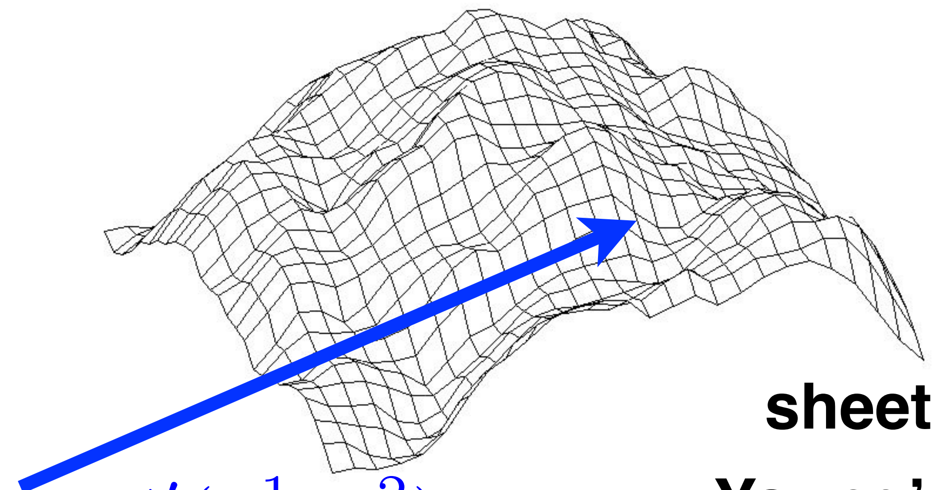
$\vec{r}(x^1, x^2)$

$\mathbf{0}$

$$g_{ij} = \frac{\partial \vec{r}}{\partial x^i} \cdot \frac{\partial \vec{r}}{\partial x^j}$$

$$d\ell^2 = \sum_{i,j} g_{ij} dx^i dx^j$$

deformed shell



$\mathbf{0}$

$\vec{r}\,'(x^1, x^2)$

sheet thickness d

Young's modulus E

Poisson's ratio ν

$$g'_{ij} = \frac{\partial \vec{r}\,'}{\partial x^i} \cdot \frac{\partial \vec{r}\,'}{\partial x^j}$$

$$d\ell'^2 = \sum_{i,j} g'_{ij} dx^i dx^j$$

strain tensor

$$u_{ij} = \frac{1}{2} \sum_k (g^{-1})_{ik} (g'_{kj} - g_{kj})$$

inverse metric tensor

$$\sum_k (g^{-1})_{ik} g_{kj} = \sum_k g_{ik} (g^{-1})_{kj} = \delta_{ij}$$

Energy cost for stretching, compressing and shearing

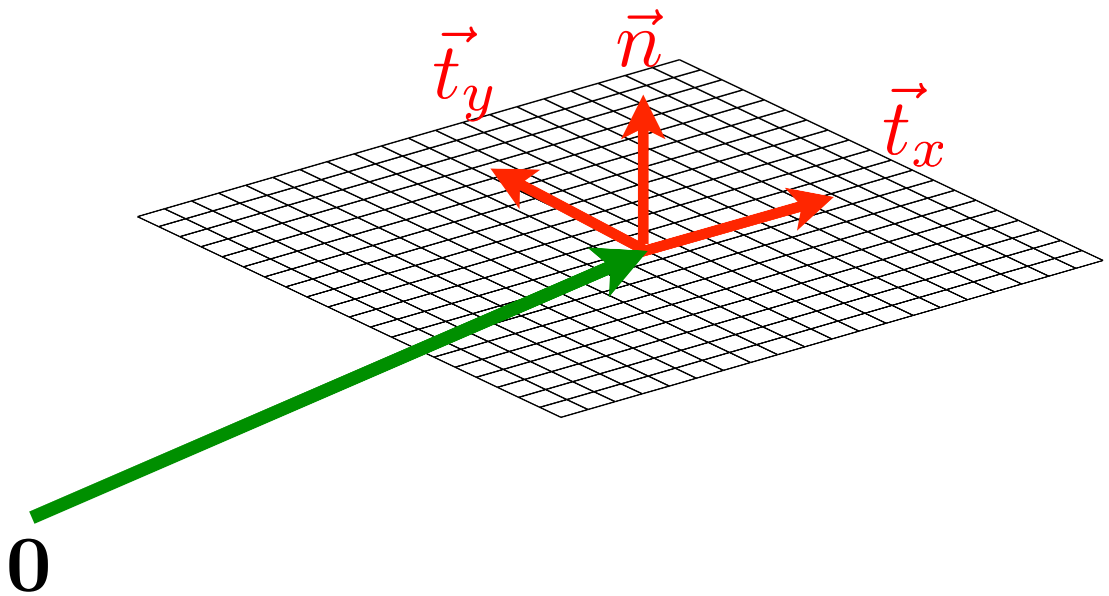
$$U = \int (\sqrt{g} dx^1 dx^2) \; \frac{1}{2} \left[\lambda \left(\sum_i u_{ii} \right)^2 + 2\mu \sum_{i,j} u_{ij} u_{ji} \right]$$

Lame constants

$$g = \det(g_{ij}) \qquad \lambda = \frac{E\nu d}{(1-\nu^2)} \quad \mu = \frac{Ed}{2(1+\nu)}$$

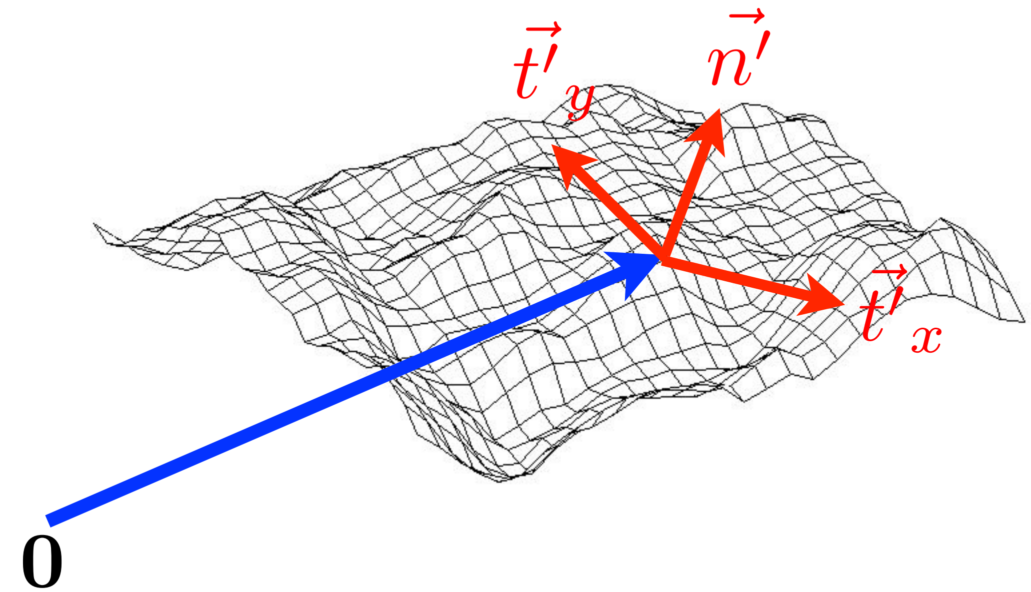
Strain tensor for deformation of flat plates

undeformed plate



$$\vec{r}(x,y) = x\vec{e}_x + y\vec{e}_y$$

deformed plate



$$\vec{r'}(x,y) = \vec{r}(x,y) + u_x(x,y)\vec{e}_x + u_y(x,y)\vec{e}_y + h(x,y)\vec{e}_z$$

$$i,j,k \in \{x,y\}$$

local tangents

$$\vec{t}_i = \partial_i \vec{r} \equiv \frac{\partial \vec{r}}{\partial i} = \vec{e}_i$$

metric tensor

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \delta_{ij} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

local tangents

$$\vec{t'}_i = \partial_i \vec{r'} = \vec{e}_i + \sum_k (\partial_i u_k)\vec{e}_k + (\partial_i h)\vec{e}_z$$

strain tensor

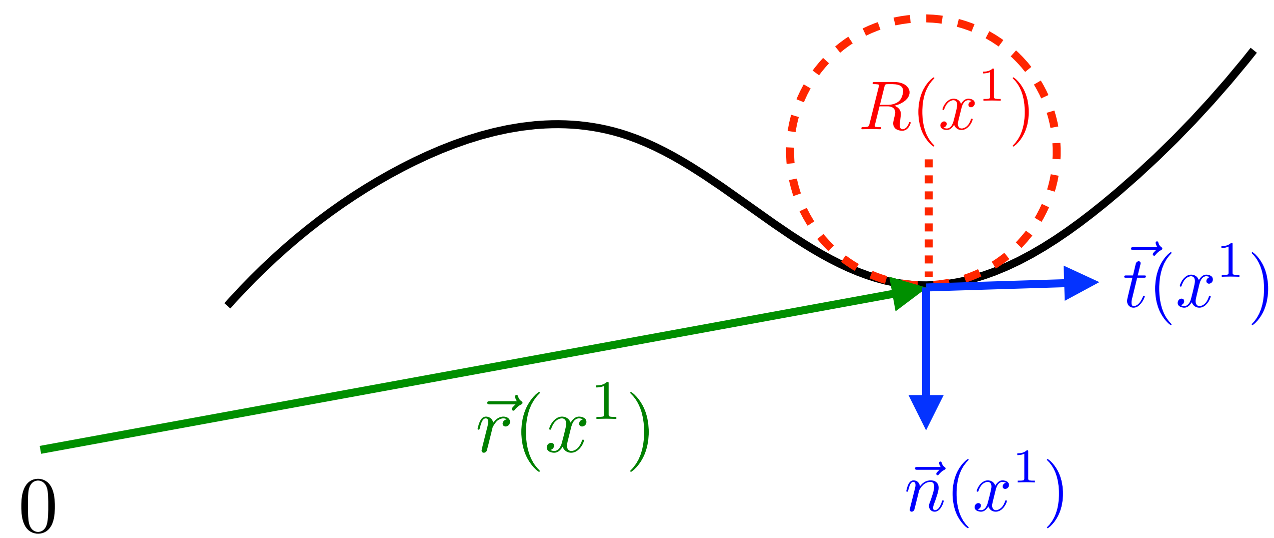
$$u_{ij} = \frac{1}{2}\left(g'_{ij} - \delta_{ij}\right)$$

$$2u_{ij} = (\partial_i u_j + \partial_j u_i) + \sum_k \partial_i u_k \partial_j u_k + \partial_i h \partial_j h$$

Curvature of curves

x^1 **parameter describing position along the curve**

$\vec{r}(x^1)$ **function describing shape of the curve**



$$\vec{t}(x^1) = \frac{d\vec{r}(x^1)}{dx^1}$$ **local tangent to the curve**

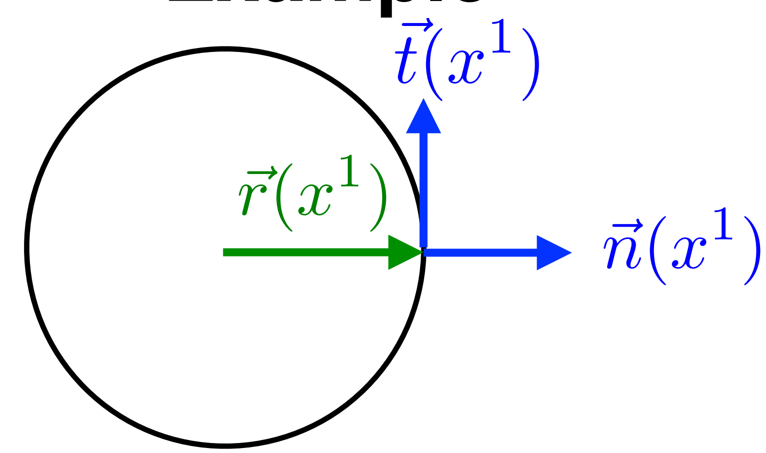
$\vec{n}(x^1)$ **local unit normal vector to the curve**

$g = \vec{t}^{\,2}$ **metric for measuring lengths**

curvature of curve

$$\frac{1}{R} = K = \frac{1}{g}\left(\vec{n}\cdot\frac{d^2\vec{r}}{d(x^1)^2}\right)$$

Example



$$\vec{r}(x^1) = R\big(\cos(\omega x^1), \sin(\omega x^1)\big)$$

$$\vec{n}(x^1) = \big(\cos(\omega x^1), \sin(\omega x^1)\big)$$

$$g(x^1) = R^2\omega^2$$

$$K = -\frac{1}{R}$$

Curvature tensor for surfaces

x^1, x^2 **parameters describing position along the surface**

$\vec{r}(x^1, x^2)$ **function describing shape of the surface**

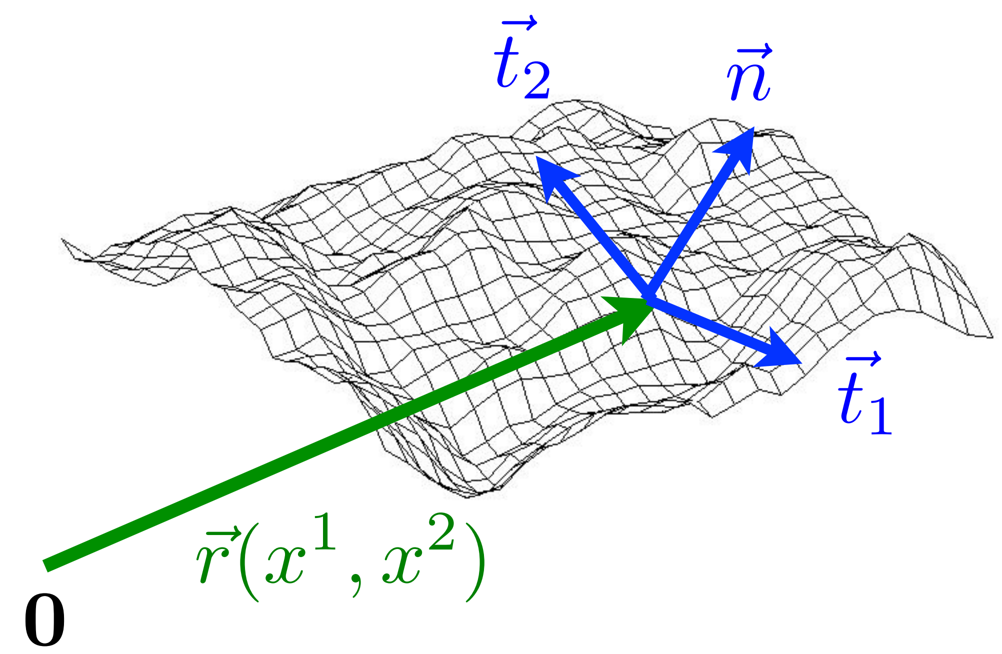
$\vec{t}_i = \frac{\partial \vec{r}}{\partial x^i}$ **local tangent vectors to the surface**

$\vec{n} = \frac{\vec{t}_1 \times \vec{t}_2}{|\vec{t}_1 \times \vec{t}_2|}$ **unit normal vector of the surface**

$g_{ij} = \vec{t}_i \cdot \vec{t}_j$ **metric tensor for measuring lengths**

curvature tensor for surfaces

$$K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)$$



principal curvatures correspond to the eigenvalues of curvature tensor

$$\frac{1}{R_1}, \frac{1}{R_2}$$

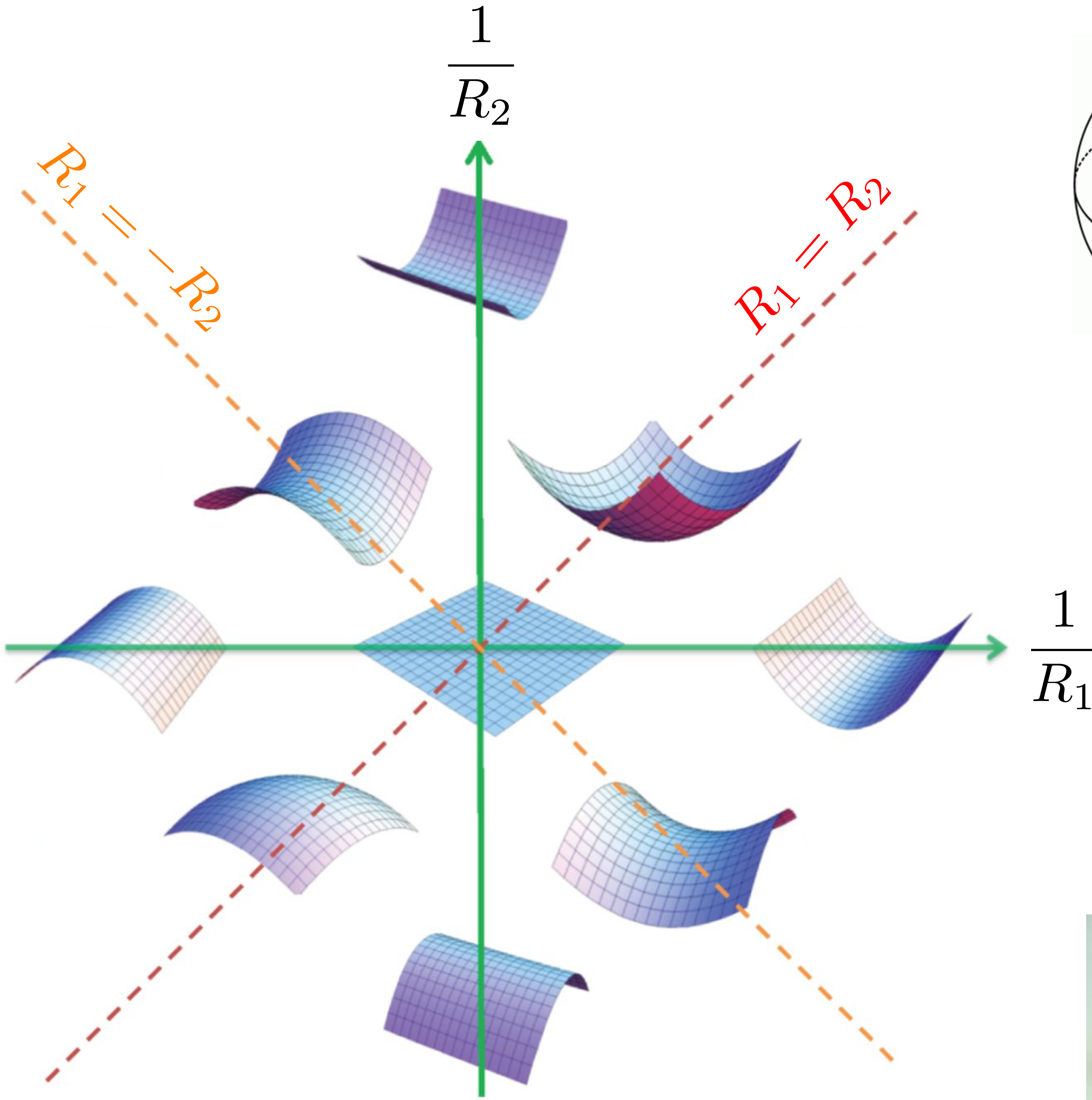
mean curvature

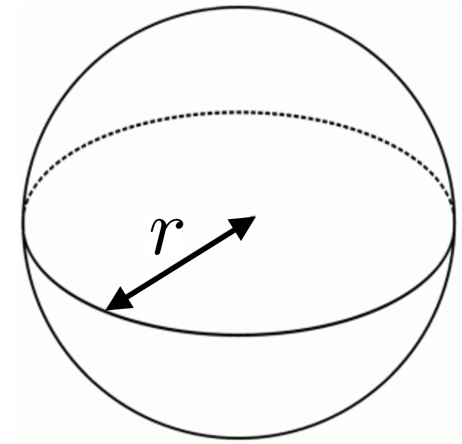
$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\sum_i K_{ii} = \frac{1}{2}\mathrm{tr}(K_{ij})$$

Gaussian curvature

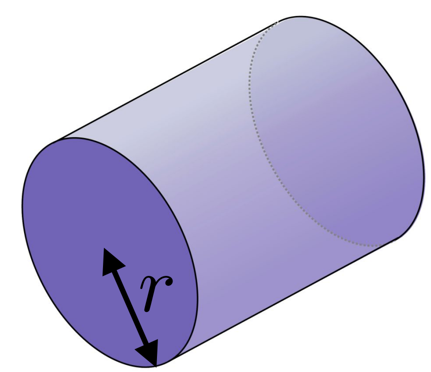
$$\frac{1}{R_1 R_2} = \det(K_{ij})$$

Surfaces of various principal curvatures





$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{r}$$



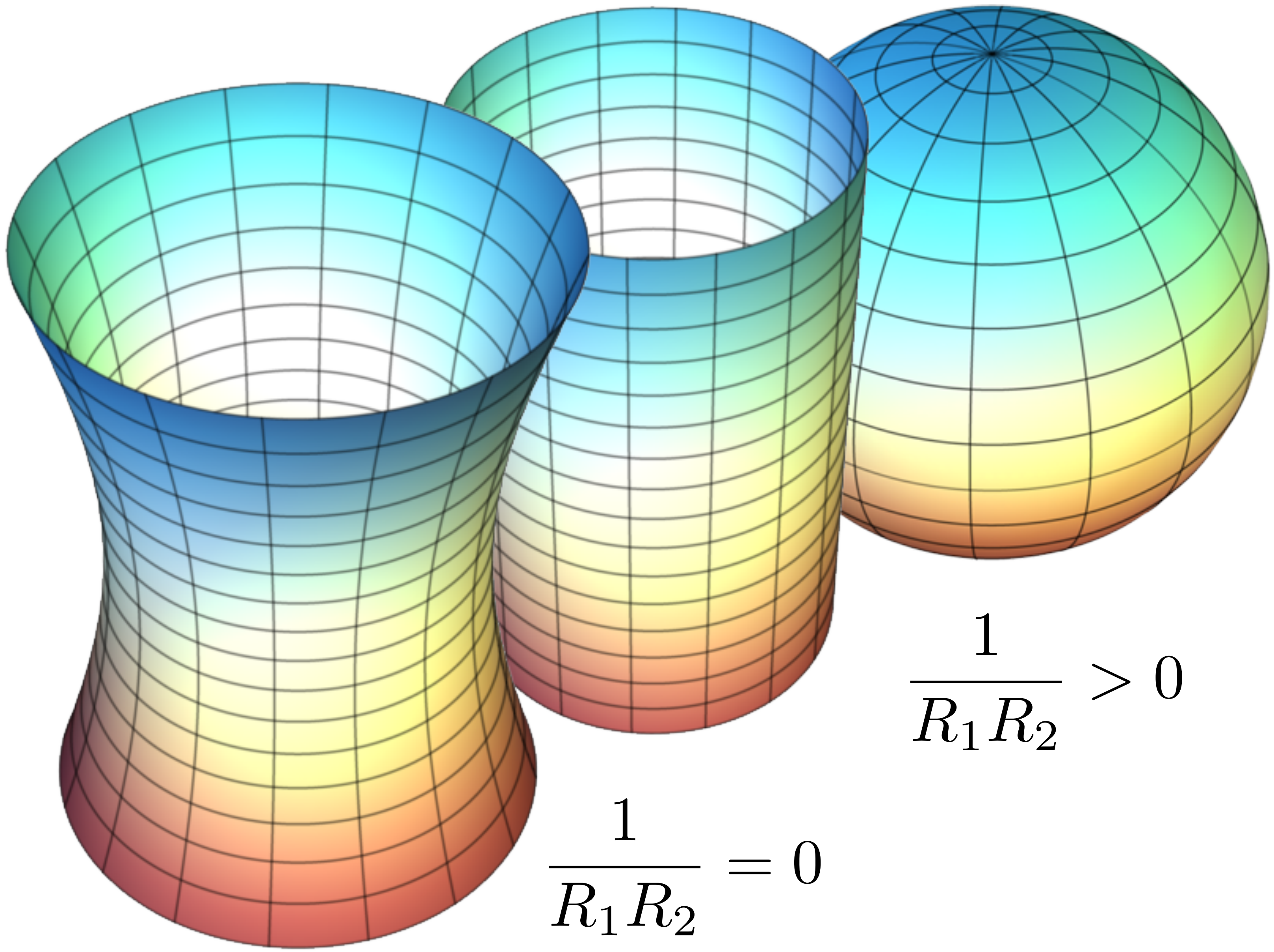
$$\frac{1}{R_1} = \frac{1}{r}$$

$$\frac{1}{R_2} = 0$$

$$\frac{1}{R_1} > 0$$

$$\frac{1}{R_2} < 0$$

Examples for Gaussian curvature



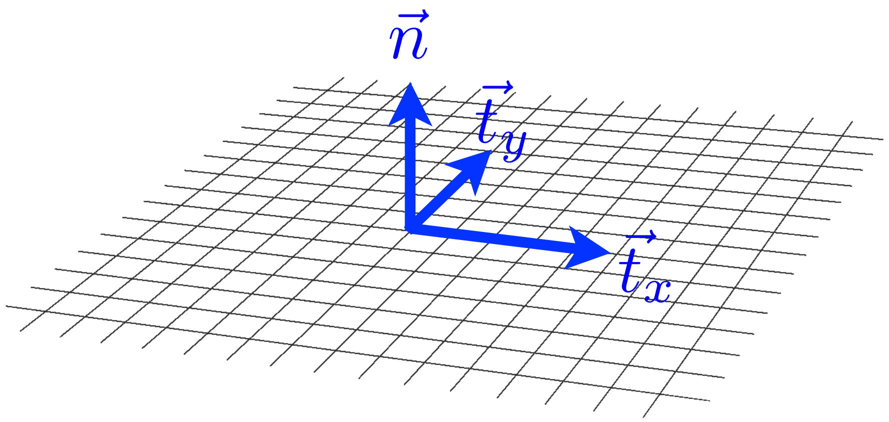
$$\frac{1}{R_1 R_2} > 0$$

$$\frac{1}{R_1 R_2} = 0$$

$$\frac{1}{R_1 R_2} < 0$$

Examples

$$K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)$$



$$\vec{r}(x,y) = (x,y,0)$$

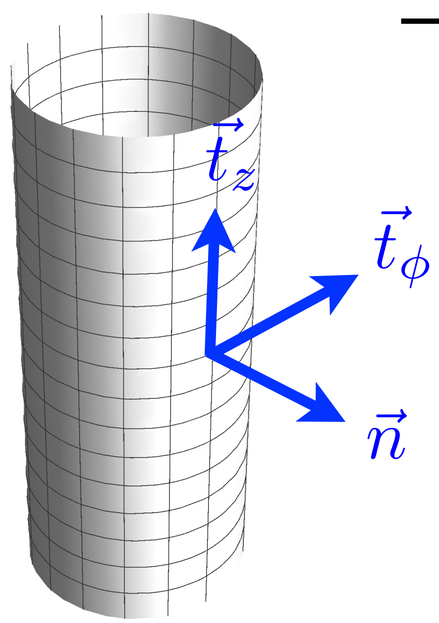
$$\vec{t}_x = \frac{\partial \vec{r}}{\partial x} = (1,0,0)$$

$$\vec{t}_y = \frac{\partial \vec{r}}{\partial y} = (0,1,0)$$

$$\vec{n} = \frac{\vec{t}_x \times \vec{t}_y}{|\vec{t}_x \times \vec{t}_y|} = (0,0,1)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

$$K_{ij} = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix}$$



$$\vec{r}(\phi,z) = (R\cos\phi, R\sin\phi, z)$$

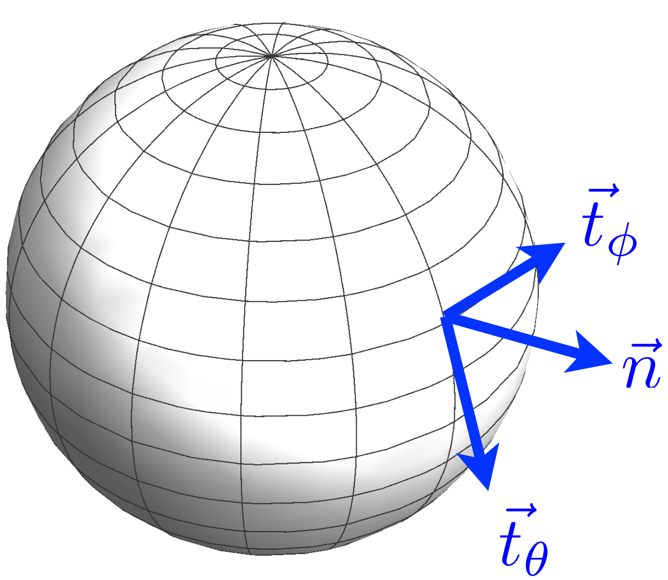
$$\vec{t}_\phi = \frac{\partial \vec{r}}{\partial \phi} = R(-\sin\phi, \cos\phi, 0)$$

$$\vec{t}_z = \frac{\partial \vec{r}}{\partial z} = (0,0,1)$$

$$\vec{n} = \frac{\vec{t}_\phi \times \vec{t}_z}{|\vec{t}_\phi \times \vec{t}_z|} = (\cos\phi, \sin\phi, 0)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} R^2, & 0 \\ 0, & 1 \end{pmatrix}$$

$$K_{ij} = \begin{pmatrix} -\frac{1}{R}, & 0 \\ 0, & 0 \end{pmatrix}$$



$$\vec{r}(\theta,\phi) = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{t}_\theta = \frac{\partial \vec{r}}{\partial \theta} = R(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\vec{t}_\phi = \frac{\partial \vec{r}}{\partial \phi} = R\sin\theta(-\sin\phi, \cos\phi, 0)$$

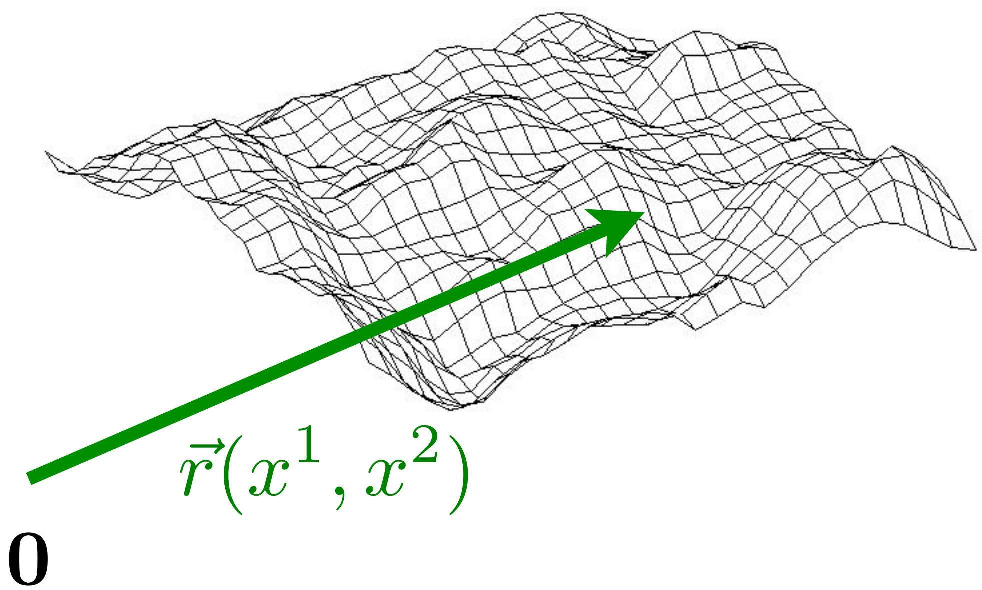
$$\vec{n} = \frac{\vec{t}_\theta \times \vec{t}_\phi}{|\vec{t}_\theta \times \vec{t}_\phi|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} R^2, & 0 \\ 0, & R^2\sin^2\theta \end{pmatrix}$$

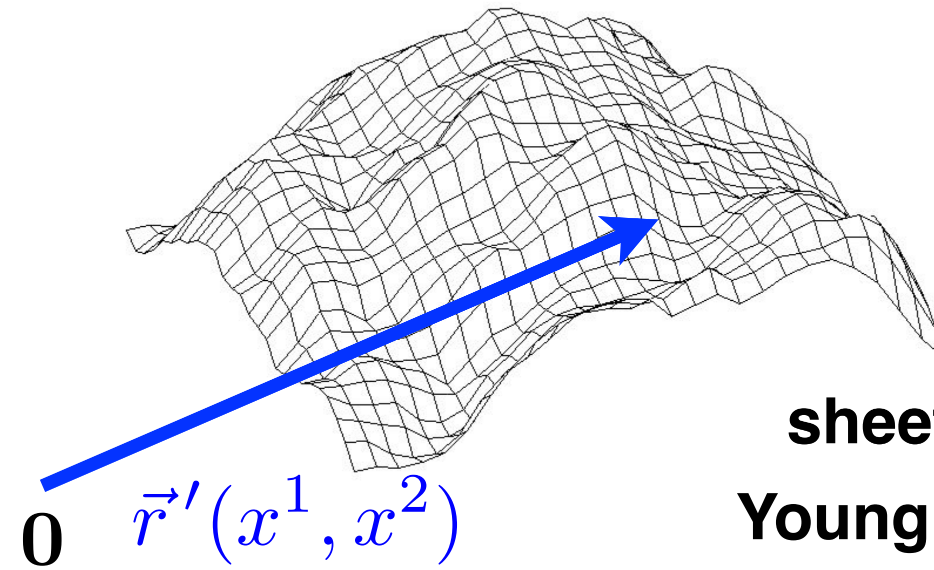
$$K_{ij} = \begin{pmatrix} -\frac{1}{R}, & 0 \\ 0, & -\frac{1}{R} \end{pmatrix}$$

Bending energy for deformation of shells

undeformed shell



deformed shell



sheet thickness d

Young's modulus E

Poisson's ratio ν

$$K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)$$

$$K'_{ij} = \sum_k \left(g'^{-1}\right)_{ik} \left(\vec{n}' \cdot \frac{\partial^2 \vec{r}\,'}{\partial x^k \partial x^j}\right)$$

bending strain tensor

$$b_{ij} = K'_{ij} - K_{ij}$$

(local measure of deviation from preferred curvature)

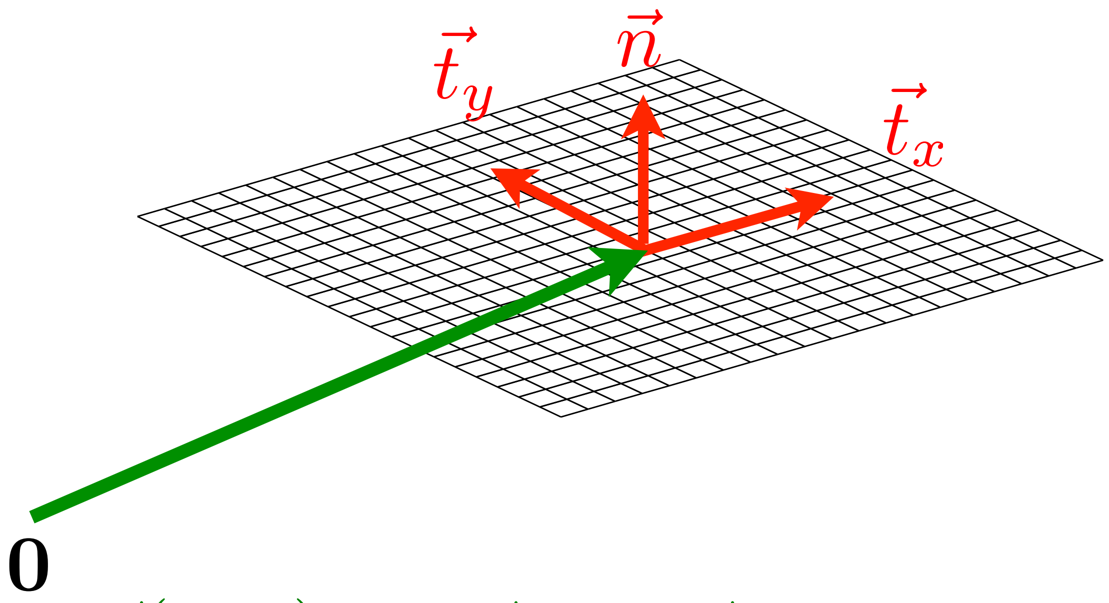
Energy cost of bending

$$U = \int \left(\sqrt{g}\, dx^1 dx^2\right) \left[\frac{1}{2}\kappa \left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij})\right]$$

$$\kappa = \frac{Ed^3}{12(1-\nu^2)} \qquad \kappa_G = -\frac{Ed^3}{12(1+\nu)}$$

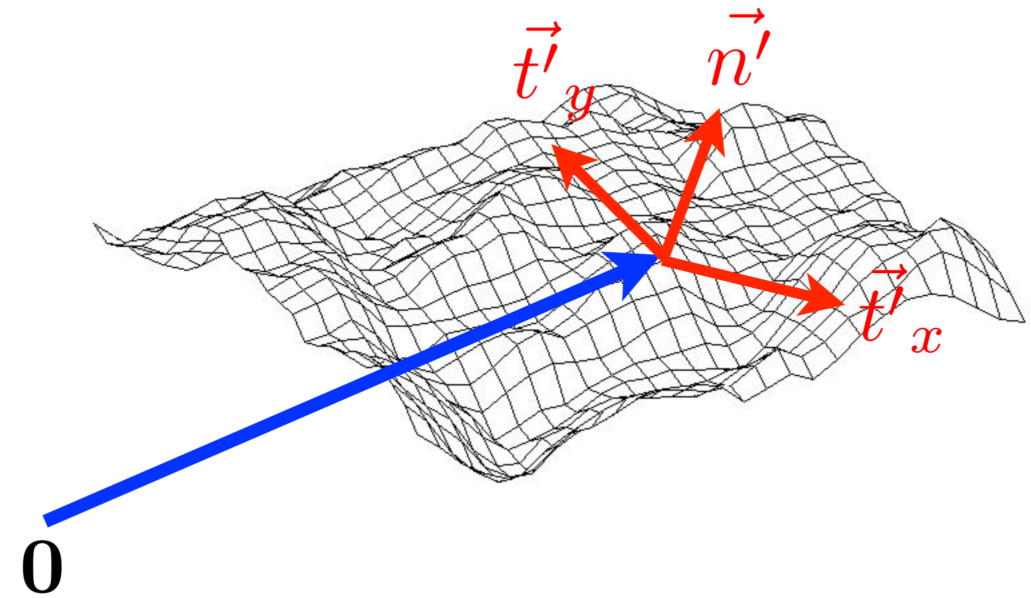
Bending strain for deformation of flat plates

undeformed plate



$$\vec{r}(x,y) = x\vec{e}_x + y\vec{e}_y$$

deformed plate



$$\vec{r'}(x,y) = \vec{r}(x,y) + u_x(x,y)\vec{e}_x + u_y(x,y)\vec{e}_y + h(x,y)\vec{e}_z$$

local normal

$$\vec{n} = \frac{\vec{t}_x \times \vec{t}_y}{|\vec{t}_x \times \vec{t}_y|} = \vec{e}_z$$

local normal (neglecting in-plane deformations)

$$\vec{n'} \approx \frac{\vec{e}_z - (\partial_x h)\,\vec{e}_x - (\partial_y h)\,\vec{e}_y}{\sqrt{1 + (\partial_x h)^2 + (\partial_y h)^2}}$$

reference curvature tensor

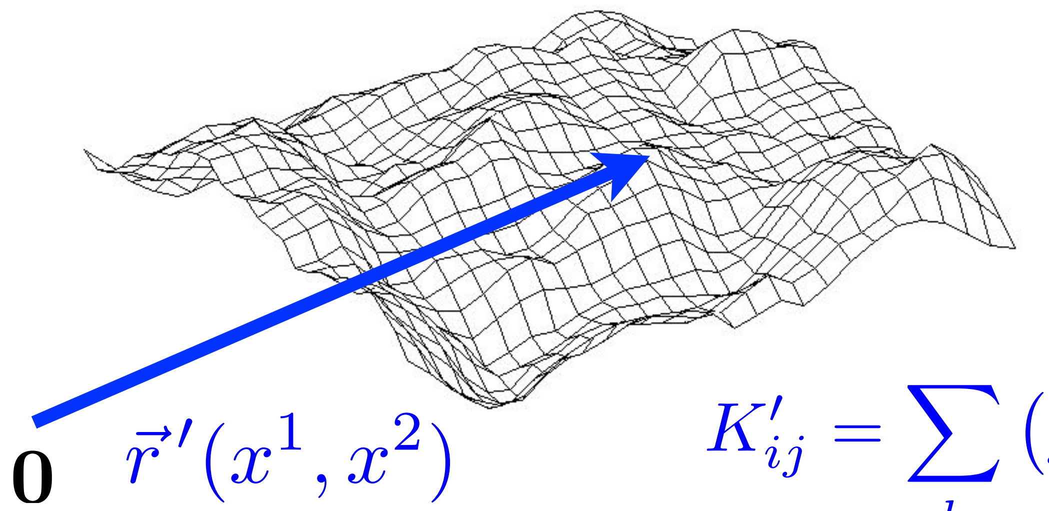
$$K_{ij} = \vec{n} \cdot \partial_i \partial_j \vec{r} = 0$$

bending strain tensor

$$b_{ij} = K'_{ij} \approx \partial_i \partial_j h + \cdots$$

Mechanics of growing sheets

Growth defines preferred metric tensor g_{ij}, and preferred curvature tensor K_{ij}.



$$g'_{ij} = \frac{\partial \vec{r}\,'}{\partial x^i} \cdot \frac{\partial \vec{r}\,'}{\partial x^j}$$

$$K'_{ij} = \sum_k \left(g'^{-1}\right)_{ik} \left(\vec{n}' \cdot \frac{\partial^2 \vec{r}\,'}{\partial x^k \partial x^j}\right)$$

strain tensors

$$u_{ij} = \frac{1}{2} \sum_k \left(g^{-1}\right)_{ik} \left(g'_{kj} - g_{kj}\right)$$

$$b_{ij} = K'_{ij} - K_{ij}$$

The equilibrium membrane shape $\vec{r}\,'(x^1, x^2)$ corresponds to the minimum of elastic energy:

$$U = \int \left(\sqrt{g}\, dx^1 dx^2\right) \left[\frac{1}{2}\lambda \left(\sum_i u_{ii}\right)^2 + \mu \sum_{i,j} u_{ij} u_{ji} + \frac{1}{2}\kappa \left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij})\right]$$

Growth can independently tune the metric tensor g_{ij} and the curvature tensor K_{ij}, which may not be compatible with any surface shape that would produce zero energy cost!

Zero energy shape exists only when preferred metric tensor g_{ij} and preferred curvature tensor K_{ij} satisfy Gauss-Codazzi-Mainardi relations!

Mechanics of growing sheets

One of the Gauss-Codazzi-Mainardi equations (Gauss's Theorema Egregium) relates the Gauss curvature to metric tensor

$$\det(K'_{ij}) = \mathcal{F}(g'_{ij})$$

The equilibrium membrane shape $\vec{r}\,'(x^1, x^2)$ corresponds to the minimum of elastic energy:

$$U = \int (\sqrt{g}dx^1dx^2) \left[\frac{1}{2}\lambda \left(\sum_i u_{ii}\right)^2 + \mu \sum_{i,j} u_{ij}u_{ji} + \frac{1}{2}\kappa\left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij})\right]$$

scaling with membrane thickness d

$$\lambda, \mu \sim Ed$$

$$\kappa, \kappa_G \sim Ed^3$$

For very thin membranes the equilibrium shape matches the preferred metric tensor to avoid stretching, compressing and shearing. This also specifies the Gauss curvature!

$$g'_{ij} = g_{ij}$$

$$\det(K'_{ij}) = \mathcal{F}(g_{ij})$$