MAE 545: Lecture 9 (3/1)

Shapes of growing sheets

Reminder: no lectures next week

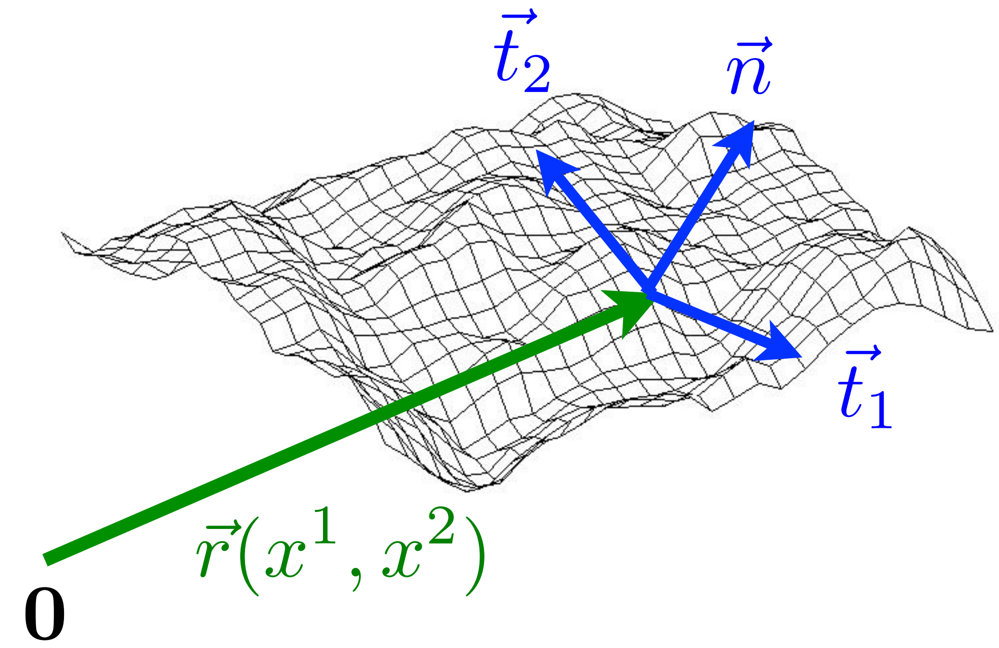
Metric tensor for measuring distances on surfaces

x^1, x^2 **parameters describing position along the surface**

$\vec{r}(x^1, x^2)$ **function describing shape of the surface**

$$\vec{t}_i = \frac{\partial \vec{r}}{\partial x^i}$$ **local tangent vectors to the surface**

$$\vec{n} = \frac{\vec{t}_1 \times \vec{t}_2}{|\vec{t}_1 \times \vec{t}_2|}$$ **unit normal vector of the surface**

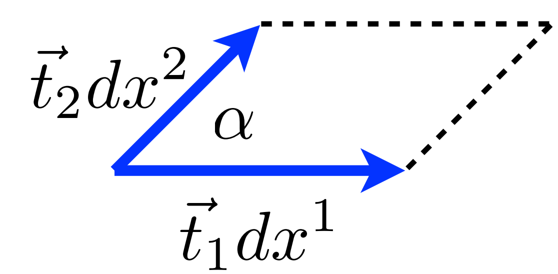


metric tensor for measuring lengths

$$d\ell^2 = d\vec{r}^{\,2} = \sum_{i,j} \vec{t}_i \cdot \vec{t}_j dx^i dx^j = \sum_{i,j} g_{ij} dx^i dx^j$$

$$\boxed{\begin{aligned} g_{ij} &= \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} \vec{t}_1 \cdot \vec{t}_1, & \vec{t}_1 \cdot \vec{t}_2 \\ \vec{t}_2 \cdot \vec{t}_1 & \vec{t}_2 \cdot \vec{t}_2 \end{pmatrix} \\ g &= \det(g_{ij}) = |\vec{t}_1 \times \vec{t}_2|^2 \end{aligned}}$$

area element

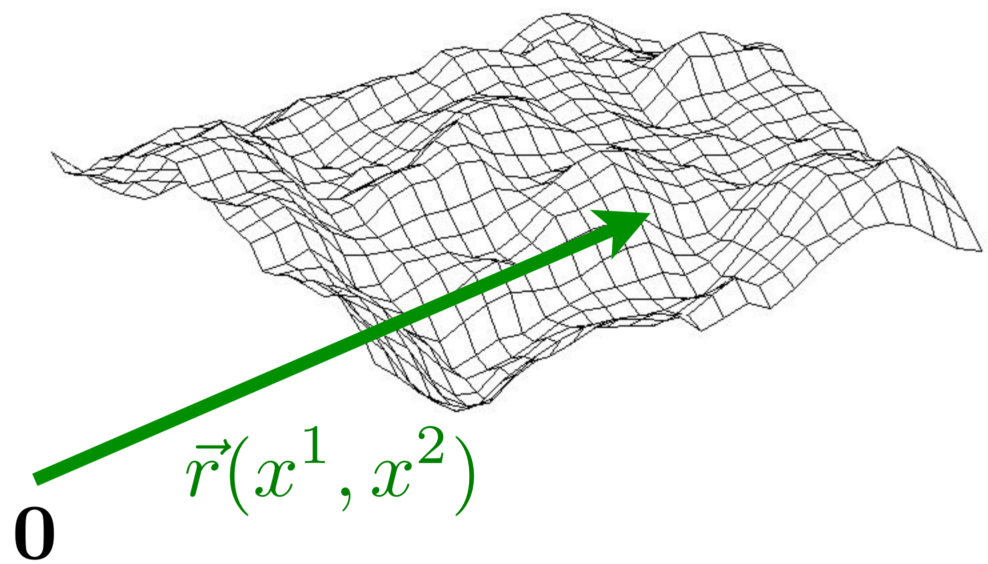


$$dA = |\vec{t}_1||\vec{t}_2| \sin\alpha dx^1 dx^2$$

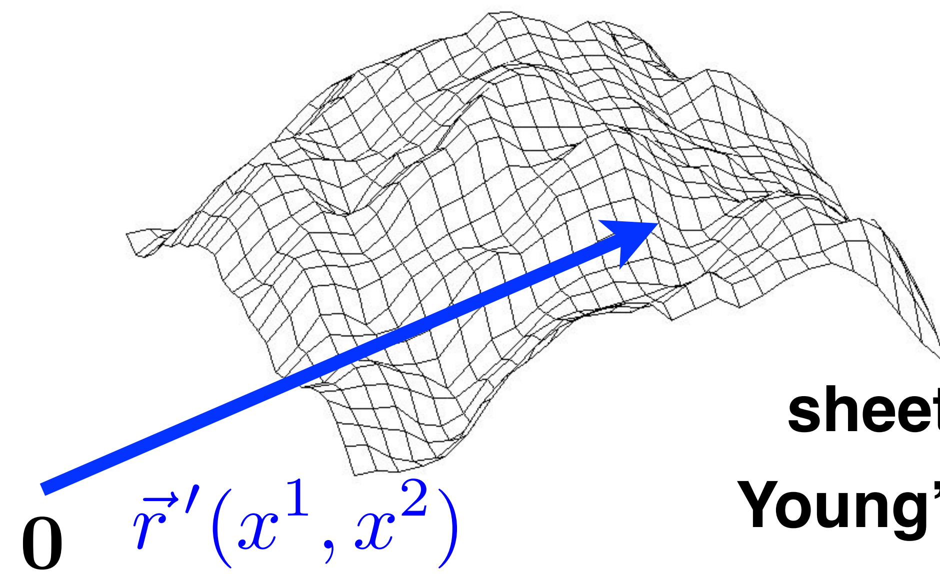
$$\boxed{dA = \sqrt{g}\, dx^1 dx^2}$$

Strain tensor and energy of shell deformations

undeformed shell



deformed shell



sheet thickness d

Young's modulus E

Poisson's ratio ν

$$g_{ij} = \frac{\partial \vec{r}}{\partial x^i} \cdot \frac{\partial \vec{r}}{\partial x^j}$$

$$d\ell^2 = \sum_{i,j} g_{ij} dx^i dx^j$$

$$g'_{ij} = \frac{\partial \vec{r}\,'}{\partial x^i} \cdot \frac{\partial \vec{r}\,'}{\partial x^j}$$

$$d\ell'^2 = \sum_{i,j} g'_{ij} dx^i dx^j$$

strain tensor

$$\boxed{u_{ij} = \frac{1}{2} \sum_k (g^{-1})_{ik} \left(g'_{kj} - g_{kj}\right)}$$

inverse metric tensor

$$\sum_k (g^{-1})_{ik} g_{kj} = \sum_k g_{ik} (g^{-1})_{kj} = \delta_{ij}$$

Energy cost for stretching, compressing and shearing

$$\boxed{U = \int \left(\sqrt{g}\, dx^1 dx^2\right) \frac{1}{2} \left[\lambda \left(\sum_i u_{ii}\right)^2 + 2\mu \sum_{i,j} u_{ij} u_{ji}\right]}$$

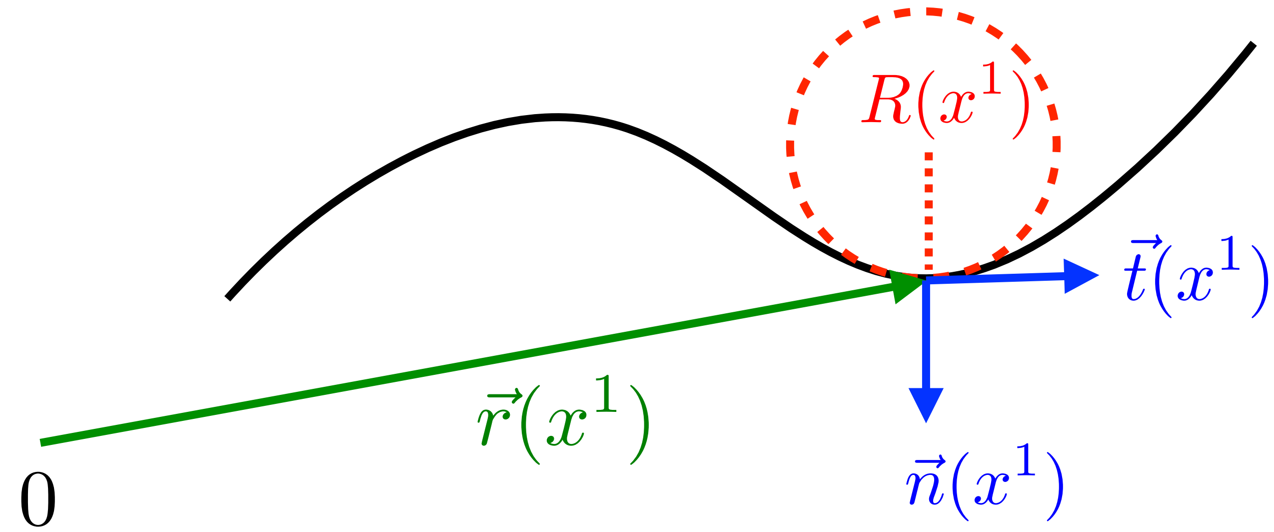
Lame constants

$$g = \det(g_{ij}) \qquad \lambda = \frac{E\nu d}{(1-\nu^2)} \quad \mu = \frac{Ed}{2(1+\nu)}$$

Curvature of curves

x^1 **parameter describing position along the curve**

$\vec{r}(x^1)$ **function describing shape of the curve**



$$\vec{t}(x^1) = \frac{d\vec{r}(x^1)}{dx^1}$$ **local tangent to the curve**

$\vec{n}(x^1)$ **local unit normal vector to the curve**

$g = \vec{t}^{\,2}$ **metric for measuring lengths**

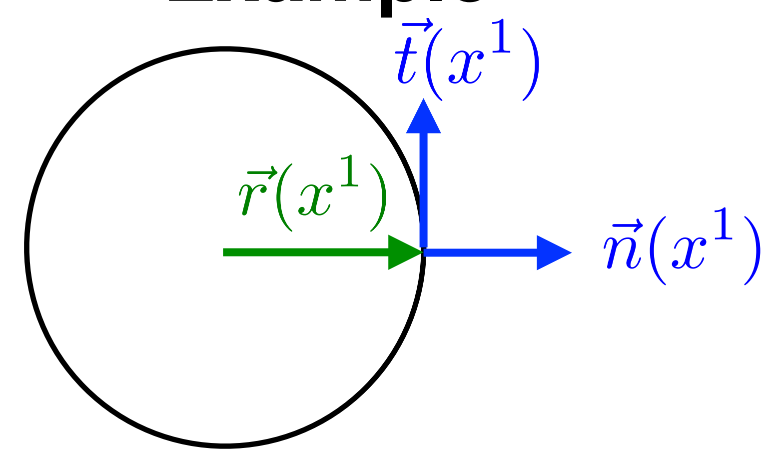
curvature of curve

$$\boxed{\frac{1}{R} = K = \frac{1}{g}\left(\vec{n}\cdot\frac{d^2\vec{r}}{d(x^1)^2}\right)}$$

in natural parametrization with *g*=1

$$\left|\frac{1}{R}\right| = \left|\frac{d^2\vec{r}}{d\left(x^1\right)^2}\right| = \left|\frac{d\vec{t}}{dx^1}\right|$$

Example



$$\vec{r}(x^1) = R\big(\cos(\omega x^1), \sin(\omega x^1)\big)$$

$$\vec{n}(x^1) = \big(\cos(\omega x^1), \sin(\omega x^1)\big)$$

$$g(x^1) = R^2\omega^2$$

$$K = -\frac{1}{R}$$

Curvature tensor for surfaces

x^1, x^2 **parameters describing position along the surface**

$\vec{r}(x^1, x^2)$ **function describing shape of the surface**

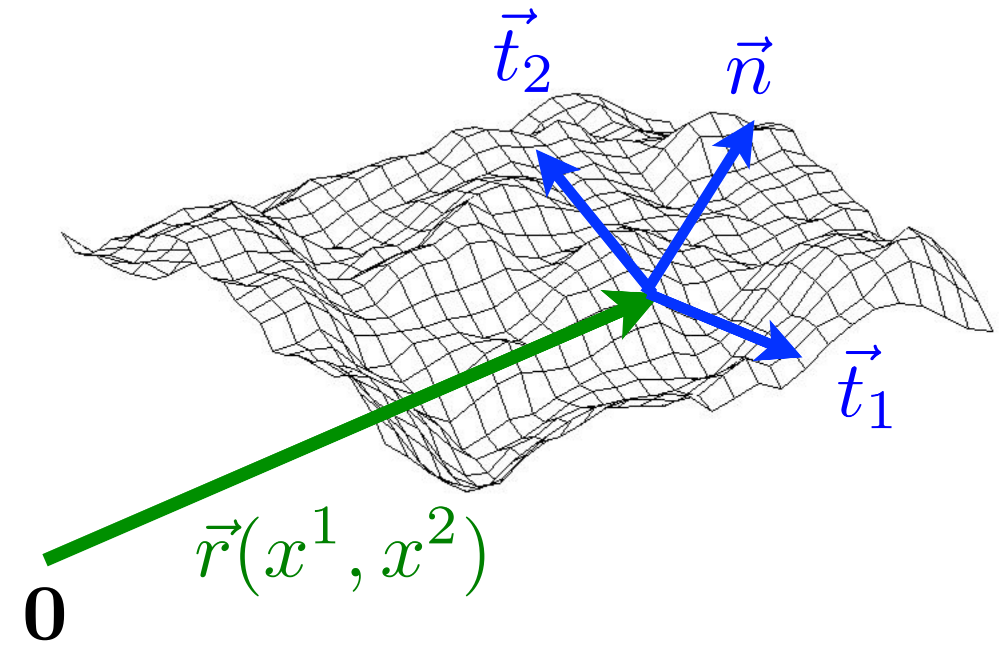
$\vec{t}_i = \frac{\partial \vec{r}}{\partial x^i}$ **local tangent vectors to the surface**

$\vec{n} = \frac{\vec{t}_1 \times \vec{t}_2}{|\vec{t}_1 \times \vec{t}_2|}$ **unit normal vector of the surface**

$g_{ij} = \vec{t}_i \cdot \vec{t}_j$ **metric tensor for measuring lengths**

curvature tensor for surfaces

$$\boxed{K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)}$$



principal curvatures correspond to the eigenvalues of curvature tensor

$$\frac{1}{R_1}, \frac{1}{R_2}$$

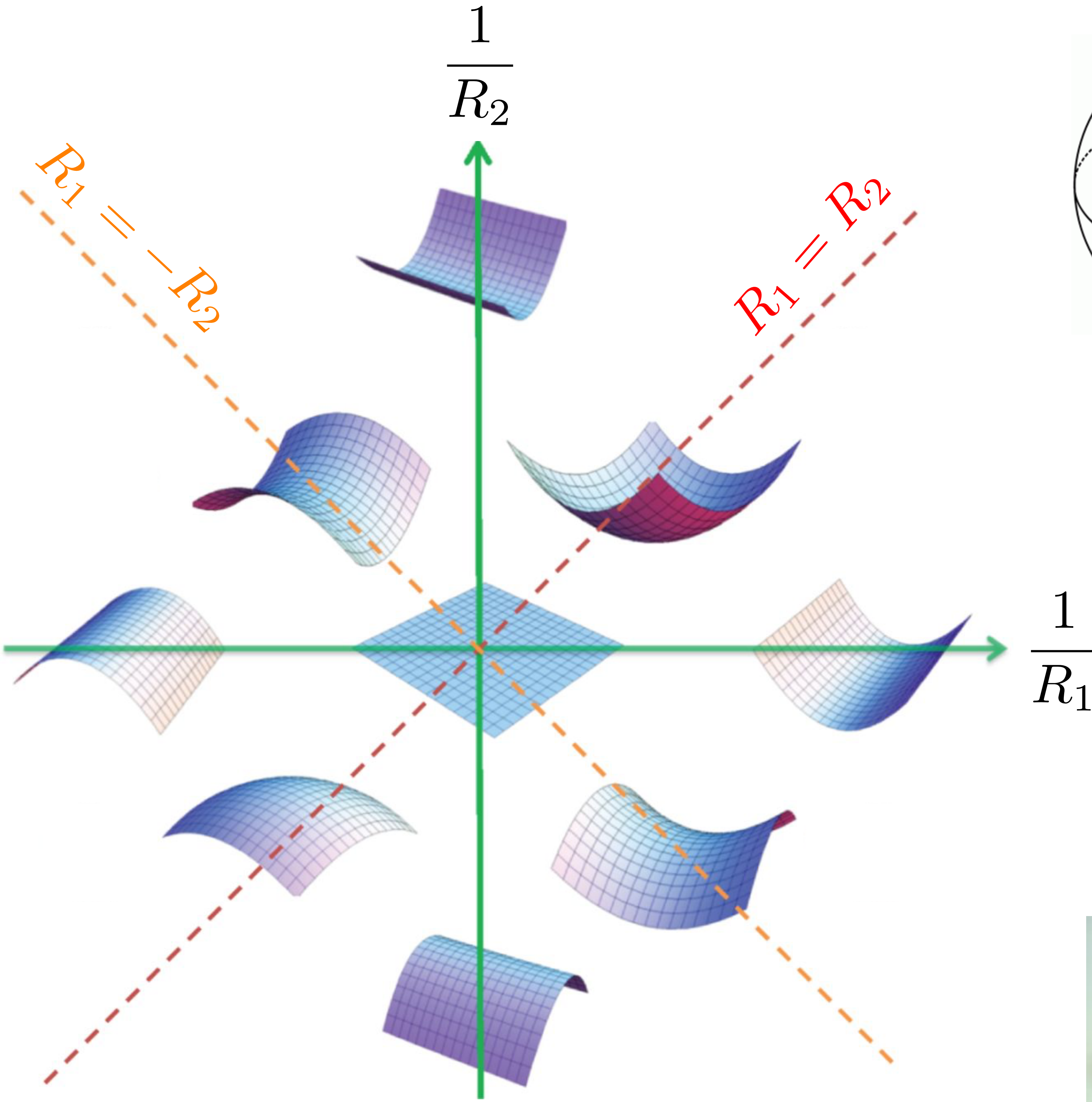
mean curvature

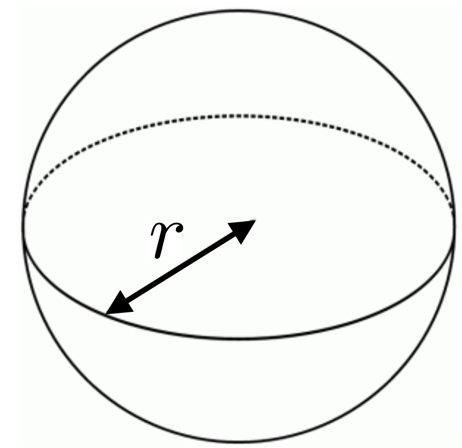
$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\sum_i K_{ii} = \frac{1}{2}\mathrm{tr}(K_{ij})$$

Gaussian curvature

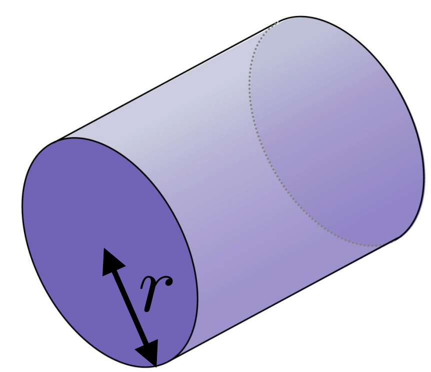
$$\frac{1}{R_1 R_2} = \det(K_{ij})$$

Surfaces of various principal curvatures





$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{r}$$



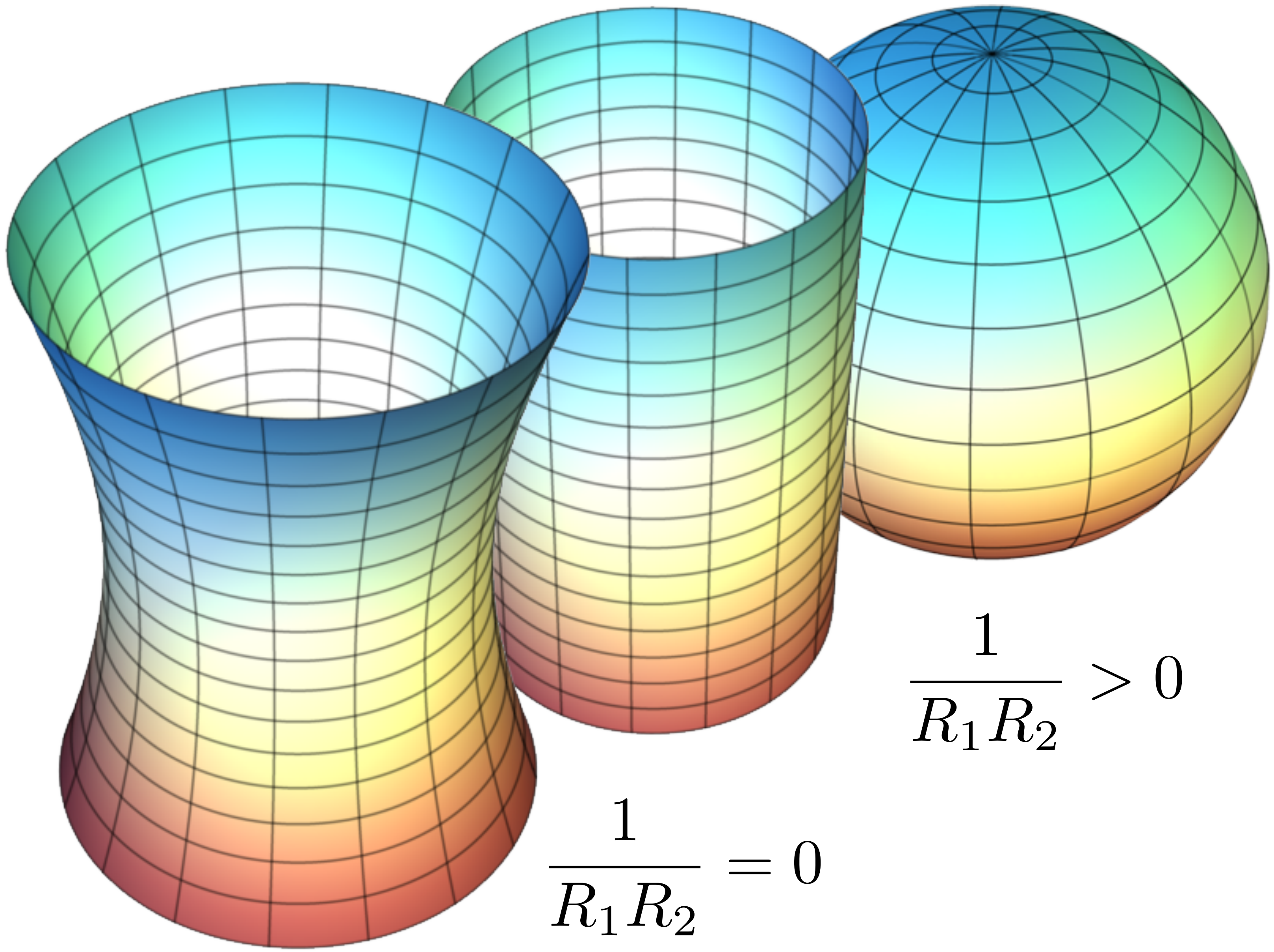
$$\frac{1}{R_1} = \frac{1}{r}$$

$$\frac{1}{R_2} = 0$$

$$\frac{1}{R_1} > 0$$

$$\frac{1}{R_2} < 0$$

Examples for Gaussian curvature



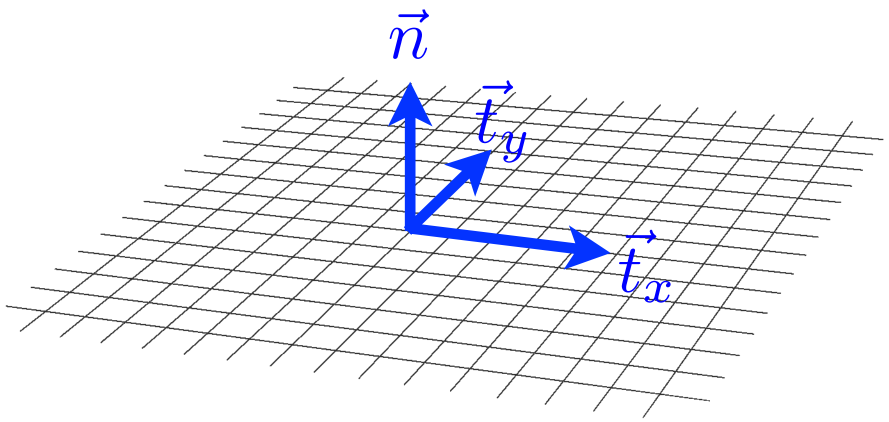
$$\frac{1}{R_1 R_2} > 0$$

$$\frac{1}{R_1 R_2} = 0$$

$$\frac{1}{R_1 R_2} < 0$$

Examples

$$\boxed{K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)}$$



$$\vec{r}(x,y) = (x,y,0)$$

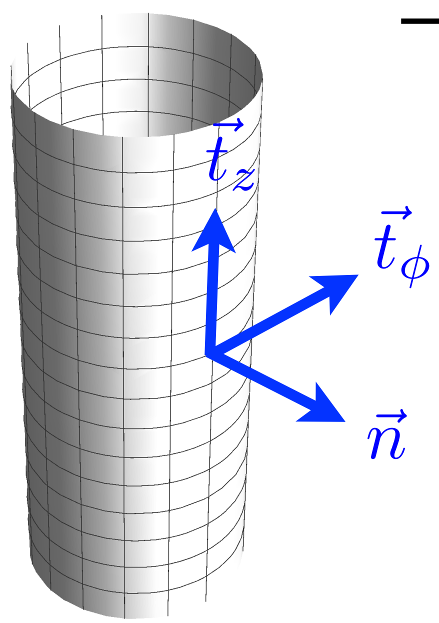
$$\vec{t}_x = \frac{\partial \vec{r}}{\partial x} = (1,0,0)$$

$$\vec{t}_y = \frac{\partial \vec{r}}{\partial y} = (0,1,0)$$

$$\vec{n} = \frac{\vec{t}_x \times \vec{t}_y}{|\vec{t}_x \times \vec{t}_y|} = (0,0,1)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

$$K_{ij} = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix}$$



$$\vec{r}(\phi, z) = (R\cos\phi, R\sin\phi, z)$$

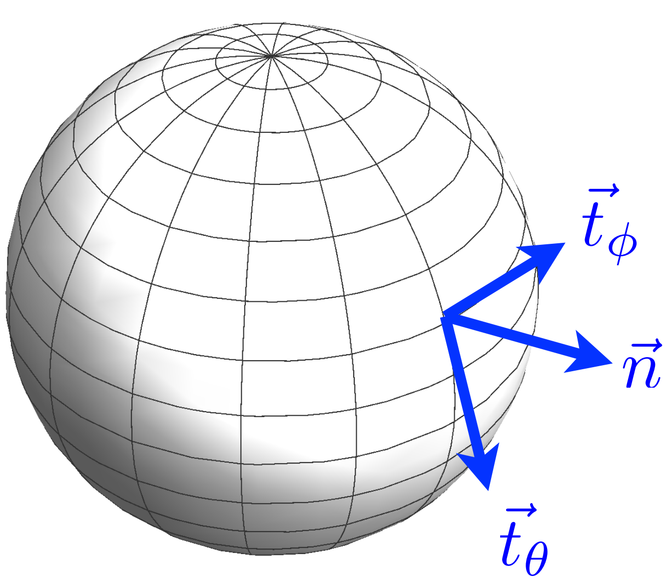
$$\vec{t}_\phi = \frac{\partial \vec{r}}{\partial \phi} = R(-\sin\phi, \cos\phi, 0)$$

$$\vec{t}_z = \frac{\partial \vec{r}}{\partial z} = (0,0,1)$$

$$\vec{n} = \frac{\vec{t}_\phi \times \vec{t}_z}{|\vec{t}_\phi \times \vec{t}_z|} = (\cos\phi, \sin\phi, 0)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} R^2, & 0 \\ 0, & 1 \end{pmatrix}$$

$$K_{ij} = \begin{pmatrix} -\frac{1}{R}, & 0 \\ 0, & 0 \end{pmatrix}$$



$$\vec{r}(\theta, \phi) = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{t}_\theta = \frac{\partial \vec{r}}{\partial \theta} = R(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\vec{t}_\phi = \frac{\partial \vec{r}}{\partial \phi} = R\sin\theta(-\sin\phi, \cos\phi, 0)$$

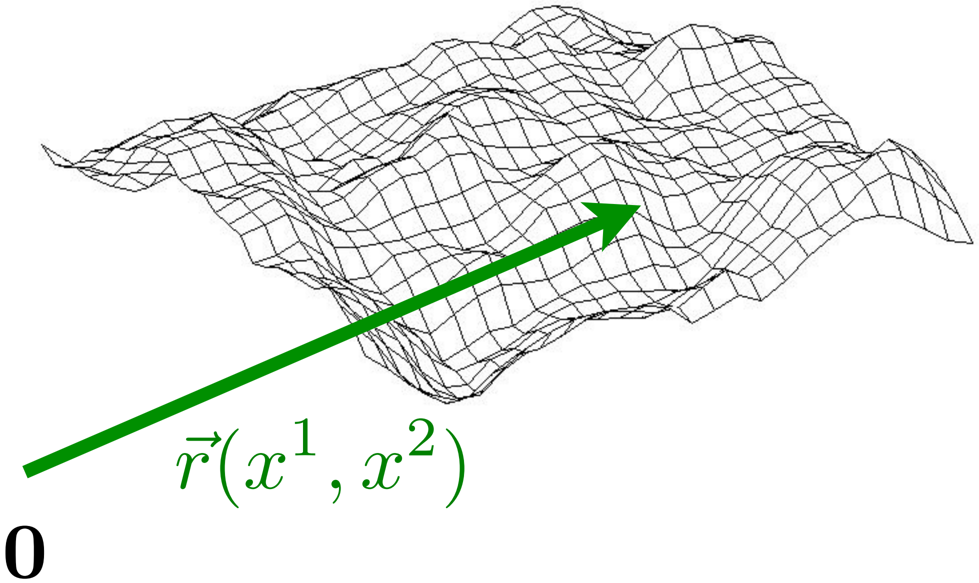
$$\vec{n} = \frac{\vec{t}_\theta \times \vec{t}_\phi}{|\vec{t}_\theta \times \vec{t}_\phi|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$g_{ij} = \vec{t}_i \cdot \vec{t}_j = \begin{pmatrix} R^2, & 0 \\ 0, & R^2\sin^2\theta \end{pmatrix}$$

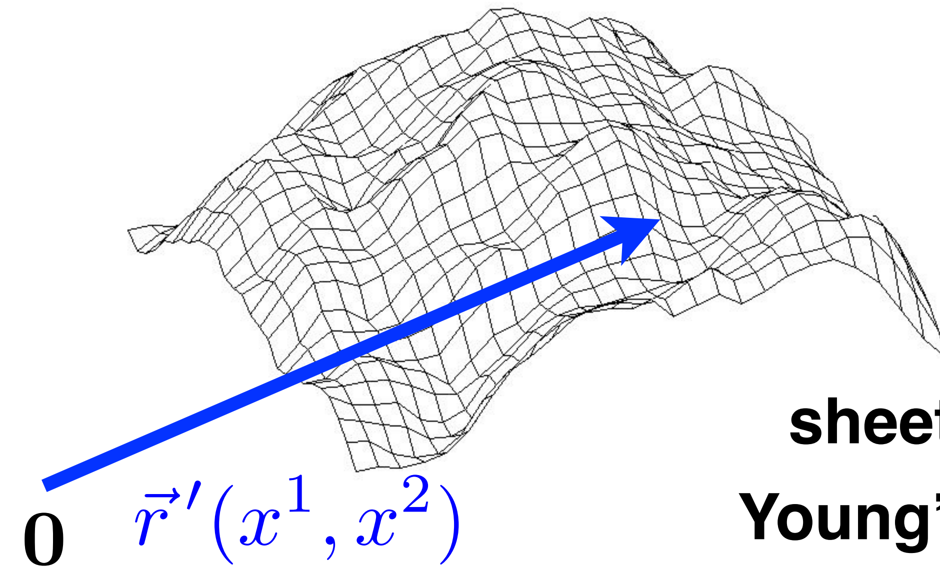
$$K_{ij} = \begin{pmatrix} -\frac{1}{R}, & 0 \\ 0, & -\frac{1}{R} \end{pmatrix}$$

Bending energy for deformation of shells

undeformed shell



deformed shell



sheet thickness d

Young's modulus E

Poisson's ratio ν

$$K_{ij} = \sum_k \left(g^{-1}\right)_{ik} \left(\vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial x^k \partial x^j}\right)$$

$$K'_{ij} = \sum_k \left(g'^{-1}\right)_{ik} \left(\vec{n}' \cdot \frac{\partial^2 \vec{r}\,'}{\partial x^k \partial x^j}\right)$$

bending strain tensor

$$\boxed{b_{ij} = K'_{ij} - K_{ij}}$$

(local measure of deviation from preferred curvature)

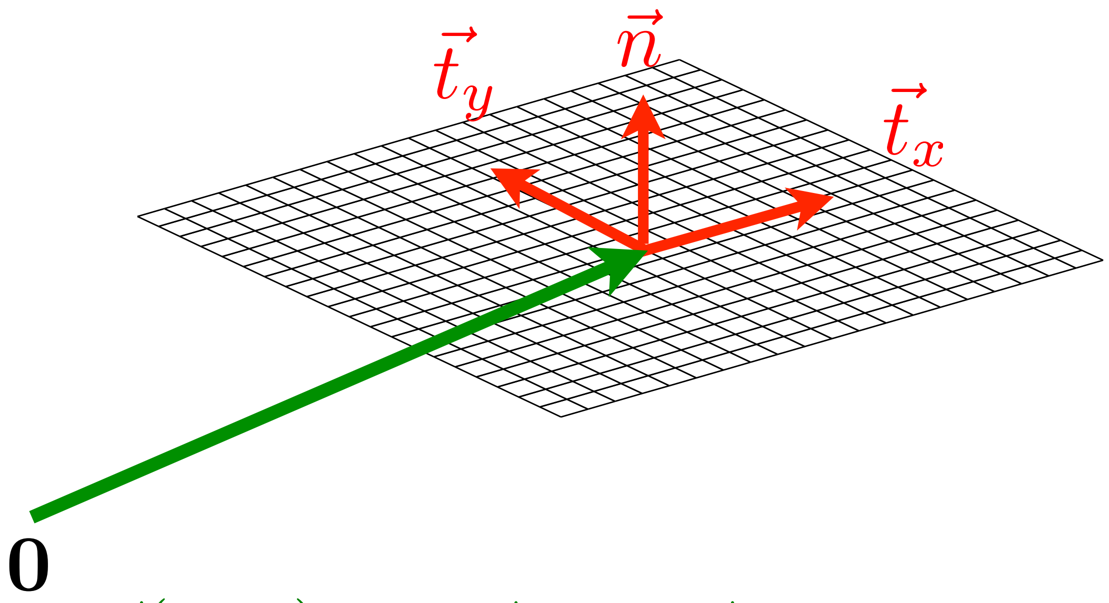
Energy cost of bending

$$\boxed{U = \int \left(\sqrt{g}\,dx^1 dx^2\right) \left[\frac{1}{2}\kappa \left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij})\right]}$$

$$\kappa = \frac{Ed^3}{12(1-\nu^2)} \qquad \kappa_G = -\frac{Ed^3}{12(1+\nu)}$$

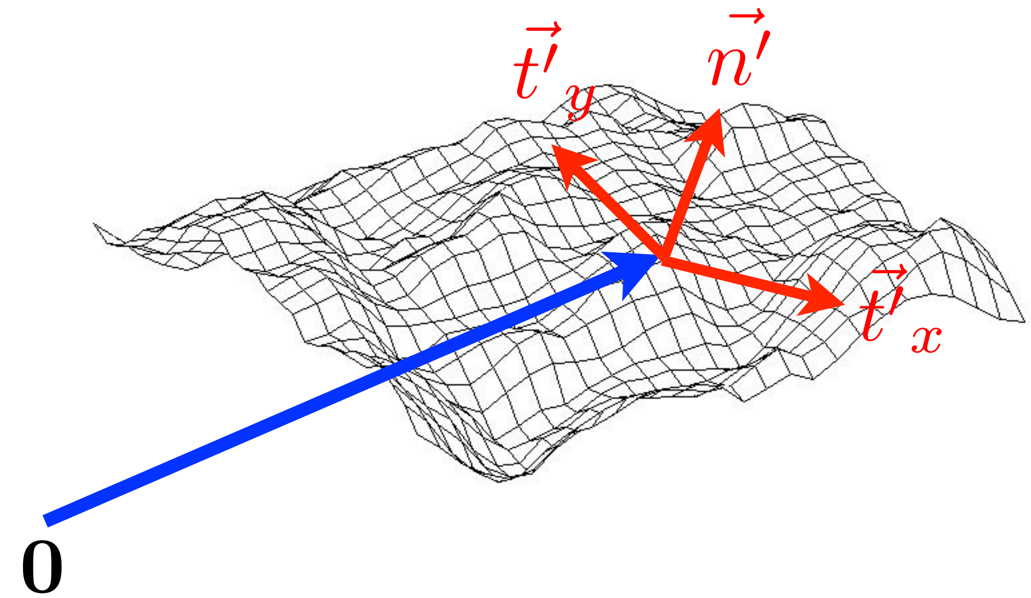
Bending strain for deformation of flat plates

undeformed plate



$$\vec{r}(x,y) = x\vec{e}_x + y\vec{e}_y$$

deformed plate



$$\vec{r'}(x,y) = \vec{r}(x,y) + u_x(x,y)\vec{e}_x + u_y(x,y)\vec{e}_y + h(x,y)\vec{e}_z$$

local normal

$$\vec{n} = \frac{\vec{t}_x \times \vec{t}_y}{|\vec{t}_x \times \vec{t}_y|} = \vec{e}_z$$

local normal (neglecting in-plane deformations)

$$\vec{n'} \approx \frac{\vec{e}_z - (\partial_x h)\,\vec{e}_x - (\partial_y h)\,\vec{e}_y}{\sqrt{1 + (\partial_x h)^2 + (\partial_y h)^2}}$$

reference curvature tensor

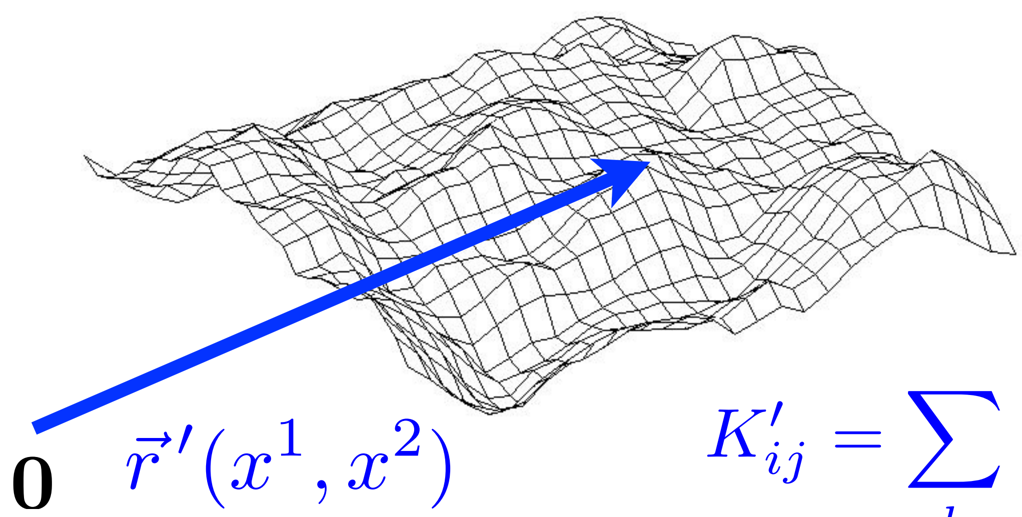
$$K_{ij} = \vec{n} \cdot \partial_i \partial_j \vec{r} = 0$$

bending strain tensor

$$\boxed{b_{ij} = K'_{ij} \approx \partial_i \partial_j h + \cdots}$$

Mechanics of growing sheets

Growth defines preferred metric tensor g_{ij}, and preferred curvature tensor K_{ij}.



$$g'_{ij} = \frac{\partial \vec{r}\,'}{\partial x^i} \cdot \frac{\partial \vec{r}\,'}{\partial x^j}$$

$$K'_{ij} = \sum_k \left(g'^{-1}\right)_{ik} \left(\vec{n}' \cdot \frac{\partial^2 \vec{r}\,'}{\partial x^k \partial x^j}\right)$$

strain tensors

$$u_{ij} = \frac{1}{2} \sum_k \left(g^{-1}\right)_{ik} \left(g'_{kj} - g_{kj}\right)$$

$$b_{ij} = K'_{ij} - K_{ij}$$

The equilibrium membrane shape $\vec{r}\,'(x^1, x^2)$ corresponds to the minimum of elastic energy:

$$U = \int \left(\sqrt{g} dx^1 dx^2\right) \left[\frac{1}{2}\lambda \left(\sum_i u_{ii}\right)^2 + \mu \sum_{i,j} u_{ij} u_{ji} + \frac{1}{2}\kappa \left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij})\right]$$

Growth can independently tune the metric tensor g_{ij} and the curvature tensor K_{ij}, which may not be compatible with any surface shape that would produce zero energy cost!

Zero energy shape exists only when preferred metric tensor g_{ij} and preferred curvature tensor K_{ij} satisfy Gauss-Codazzi-Mainardi relations!

Mechanics of growing sheets

One of the Gauss-Codazzi-Mainardi equations (Gauss's Theorema Egregium) relates the Gauss curvature to metric tensor

$$\boxed{\det(K'_{ij}) = \mathcal{F}(g'_{ij})}$$

The equilibrium membrane shape $\vec{r}\,'(x^1, x^2)$ corresponds to the minimum of elastic energy:

$$U = \int (\sqrt{g}dx^1 dx^2) \left[\frac{1}{2}\lambda \left(\sum_i u_{ii} \right)^2 + \mu \sum_{i,j} u_{ij}u_{ji} + \frac{1}{2}\kappa \left(\mathrm{tr}(b_{ij})\right)^2 + \kappa_G \det(b_{ij}) \right]$$

scaling with membrane thickness d

$$\lambda, \mu \sim Ed$$

$$\kappa, \kappa_G \sim Ed^3$$

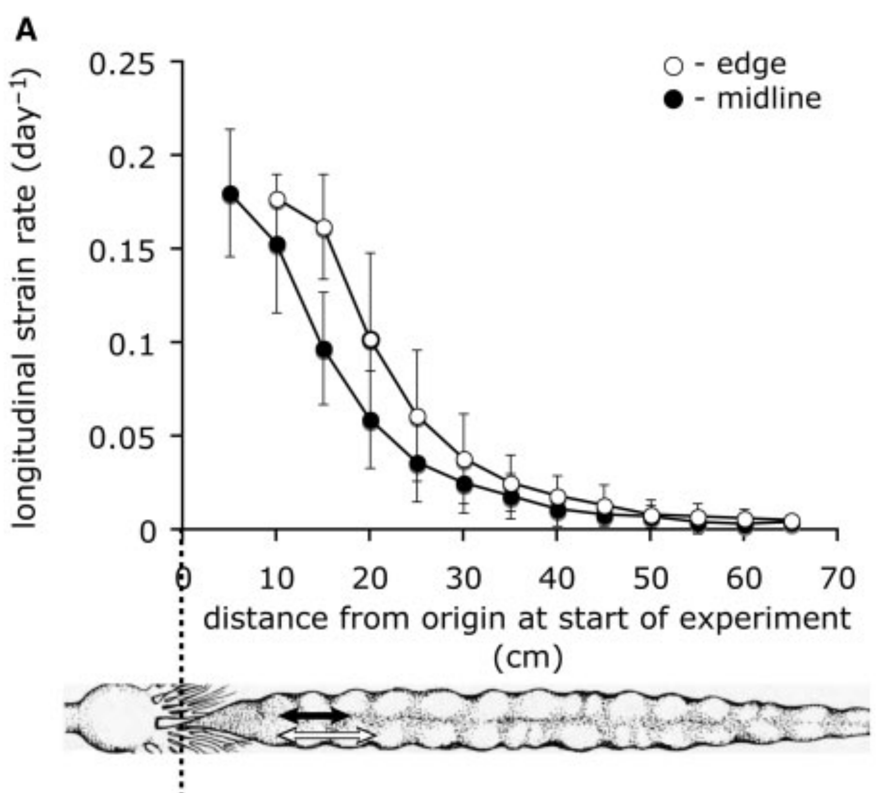
For very thin membranes the equilibrium shape matches the preferred metric tensor to avoid stretching, compressing and shearing. This also specifies the Gauss curvature!

$$\boxed{\begin{array}{c} g'_{ij} = g_{ij} \\ \det(K'_{ij}) = \mathcal{F}(g_{ij}) \end{array}}$$

Wrinkled and straight blades in macroalgae

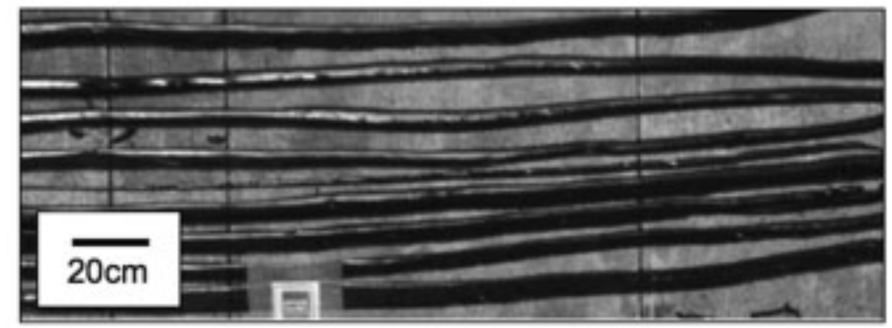
Slow water flow environment (v~0.5 m/s)

edges of blades grow faster than the midline

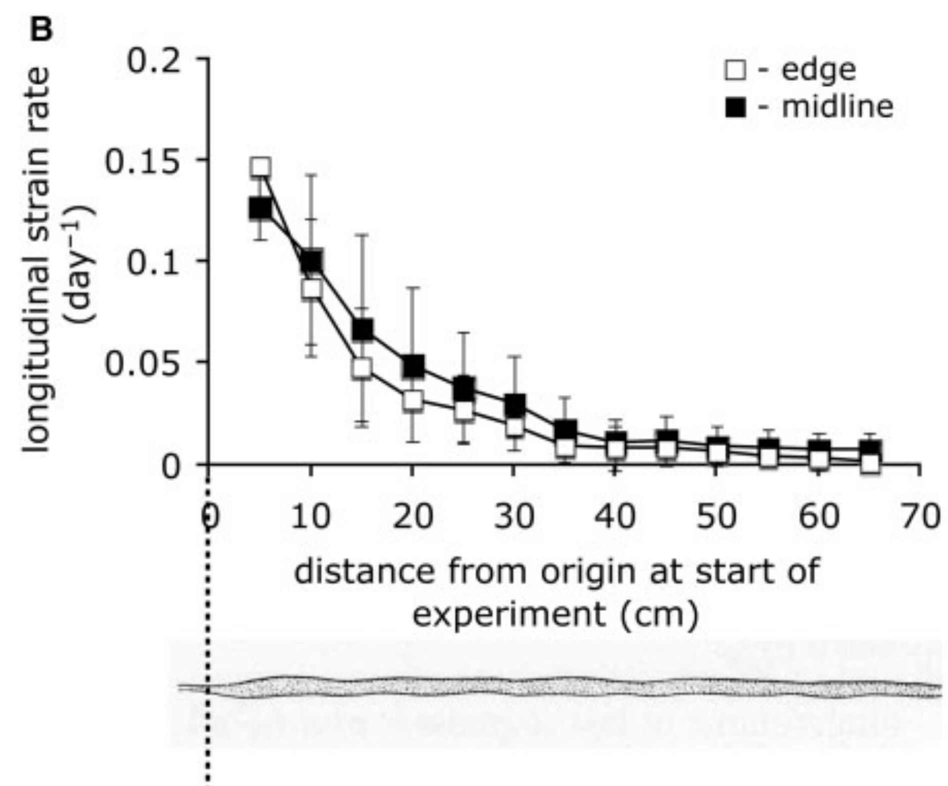


What is the effect of differential growth rate between the edge and the midline of the blade?

Fast water flow environment (v~1.5 m/s)



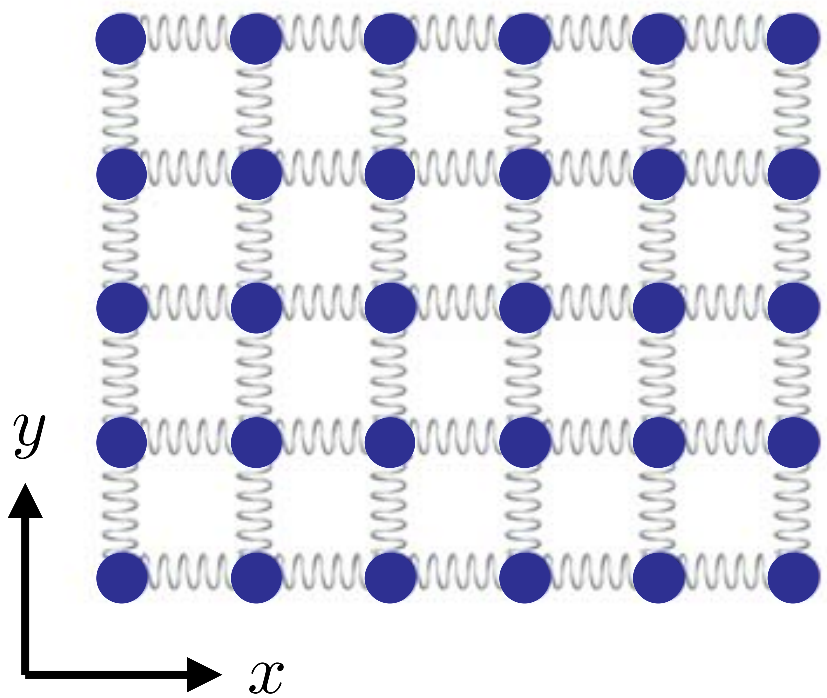
edges of blades grow at the same speed as the midline



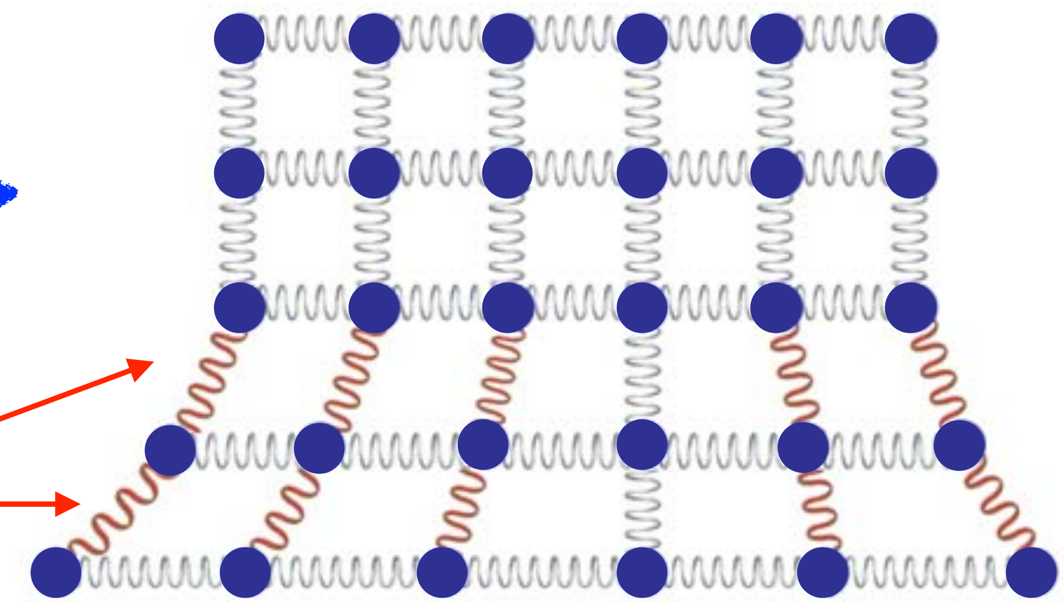
M. Koehl et al., Integ. Comp. Biol. **48**, 834 (2008)

Differential growth produces internal stress

before growth



faster growth of the bottom edge in x direction



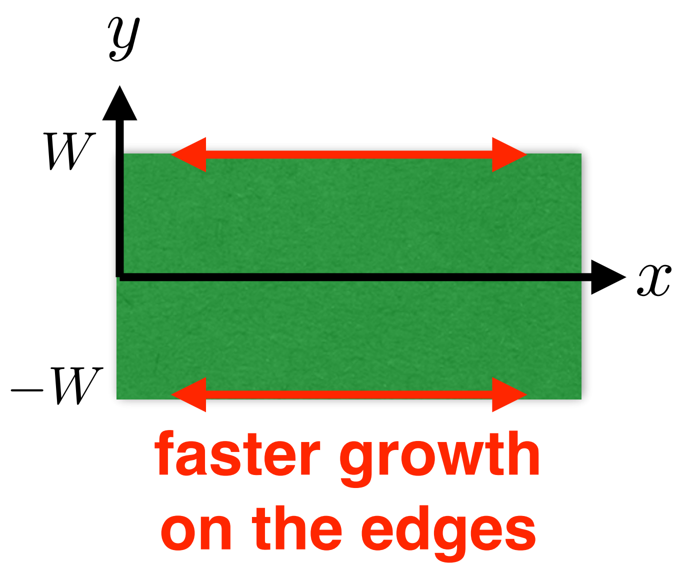
$$d\ell^2 = \sum_{i,j} g_{ij} dx^i dx^j$$

Growth modifies the metric tensor of sheet!

$$g_{ij} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} \longrightarrow g_{ij} = \begin{pmatrix} f(y), & 0 \\ 0, & 1 \end{pmatrix}$$

Note: If growth is different between the top and bottom of the sheet, then the curvature tensor K_{ij} is modified as well!

Example



Assume that differential growth in x direction produces metric tensor of the form

$$g_{ij} = \begin{pmatrix} f(y), & 0 \\ 0, & 1 \end{pmatrix} \qquad f(y) = 1 + ce^{(|y|-W)/\lambda}$$

For thin membranes the metric tensor wants to be matched

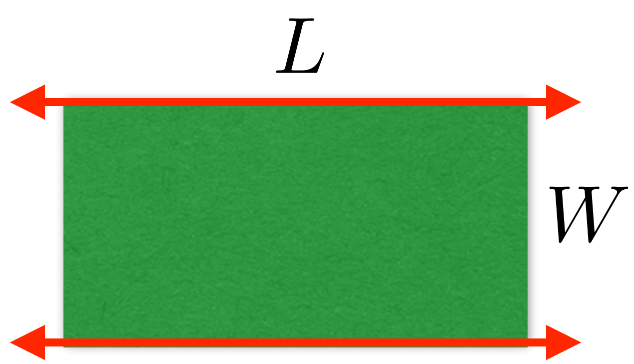
$$g'_{ij} = g_{ij}$$

Gauss's Theorema Egregium provides Gauss curvature

$$\det(K'_{ij}(y)) = \mathcal{F}(g_{ij}) = -\frac{1}{f}\frac{d^2 f(y)}{dy^2} = -\frac{1}{\lambda^2} \times \frac{ce^{(|y|-W)/\lambda}}{\left(1 + ce^{(|y|-W)/\lambda}\right)} < 0$$

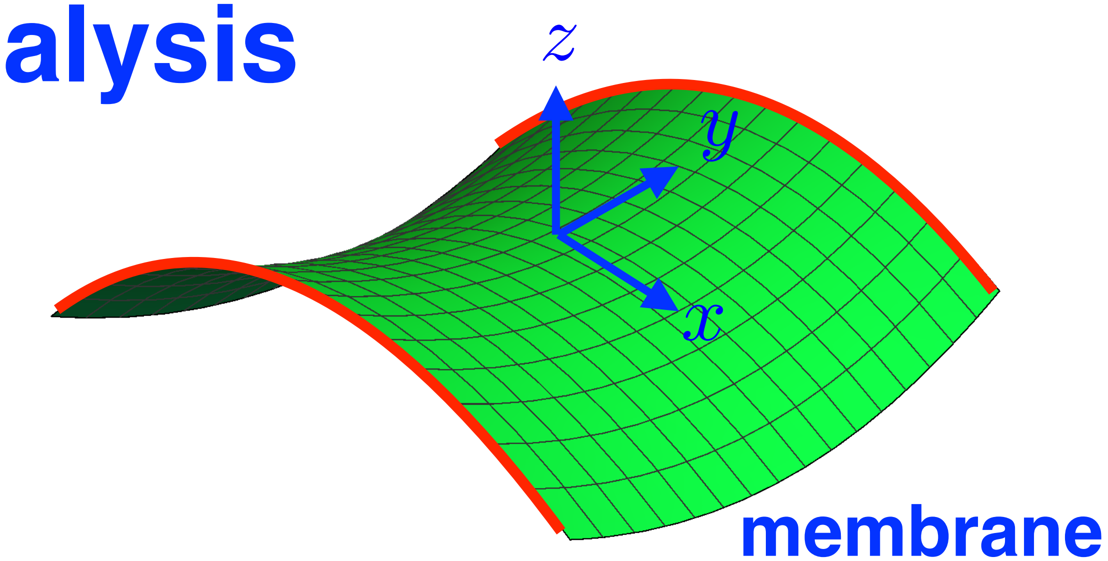
For thin membranes faster growth on edges produces shapes that locally look like saddles!

Scaling analysis



faster growth increases the edge length by factor ϵ

stress released by bending

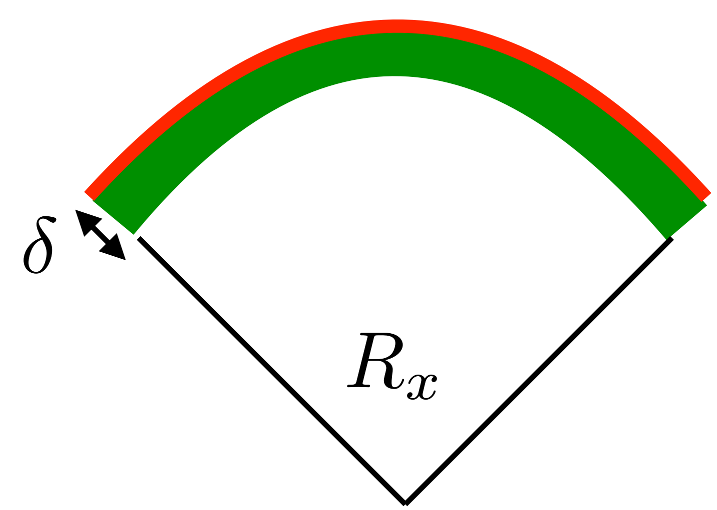


membrane thickness d

y-z cross-section

$$\frac{1}{R_y} \sim \frac{\delta}{W^2}$$

projection to x-z plane



$$\frac{L(1+\epsilon)}{L} \sim \frac{R_x+\delta}{R_x}$$

$$\frac{1}{R_x} \sim \frac{\epsilon}{\delta}$$

Membrane bending energy

$$U_b \sim A \times \kappa \times \left(\frac{1}{R_x^2} + \frac{1}{R_y^2} + \frac{1}{R_x R_y}\right) \sim A \times E_m d^3 \times \left(\frac{\epsilon^2}{\delta^2} + \frac{\delta^2}{W^4} + \frac{\epsilon}{W^2}\right)$$

Minimize U_b with respect to δ :

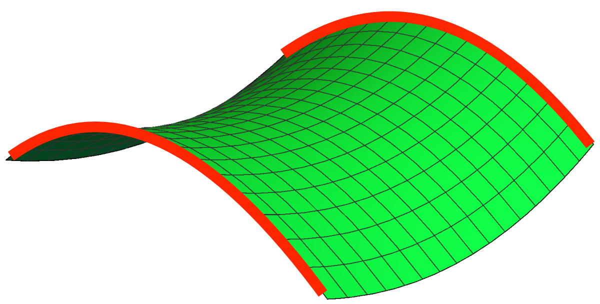
$$\boxed{\delta \sim W\sqrt{\epsilon}} \rightarrow \boxed{U_b \sim \frac{A E_m d^3 \epsilon}{W^2}}$$

Scaling analysis

membrane compression

$$U_c \sim A E_m d \epsilon^2$$

membrane bending



$$U_b \sim \frac{A E_m d^3 \epsilon}{W^2}$$

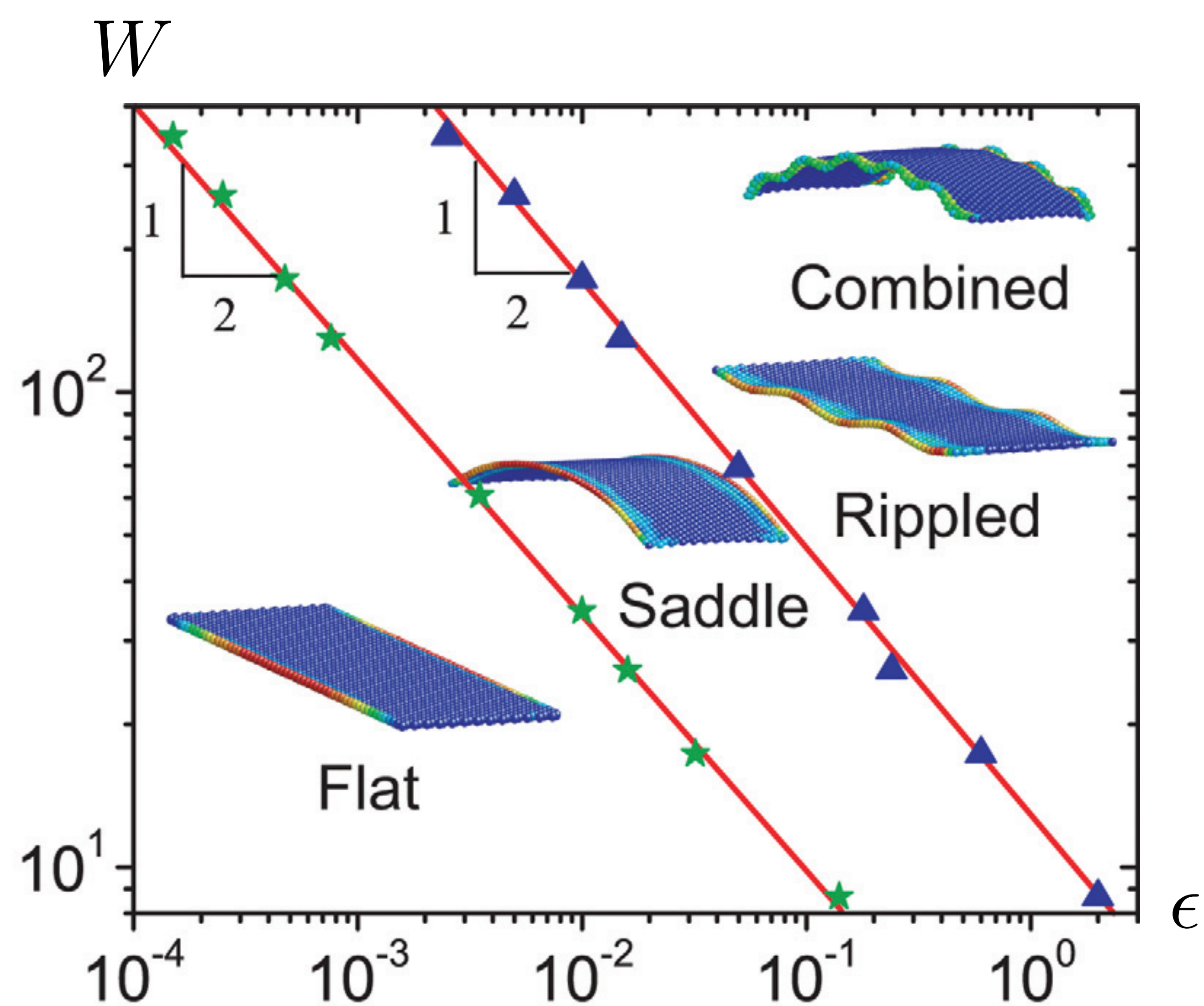
membrane bends above the critical strain

$$\boxed{\epsilon > \epsilon_c \sim \frac{d^2}{W^2}}$$

amplitude of bending at the critical strain

$$\boxed{\delta^* \sim W\sqrt{\epsilon_c} \sim d}$$

numerical simulations



H. Liang and L. Mahadevan, PNAS **106**, 22049 (2009)

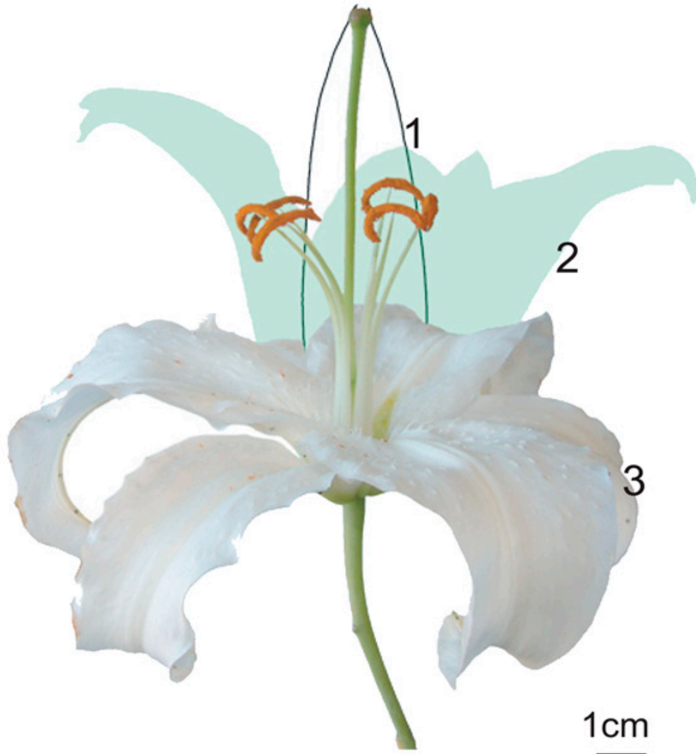
Shapes of flowers and leaves

Faster growth of the edge is consistent with observed saddles and edge wrinkles, which indeed correspond to the negative Gauss curvature!

saddles

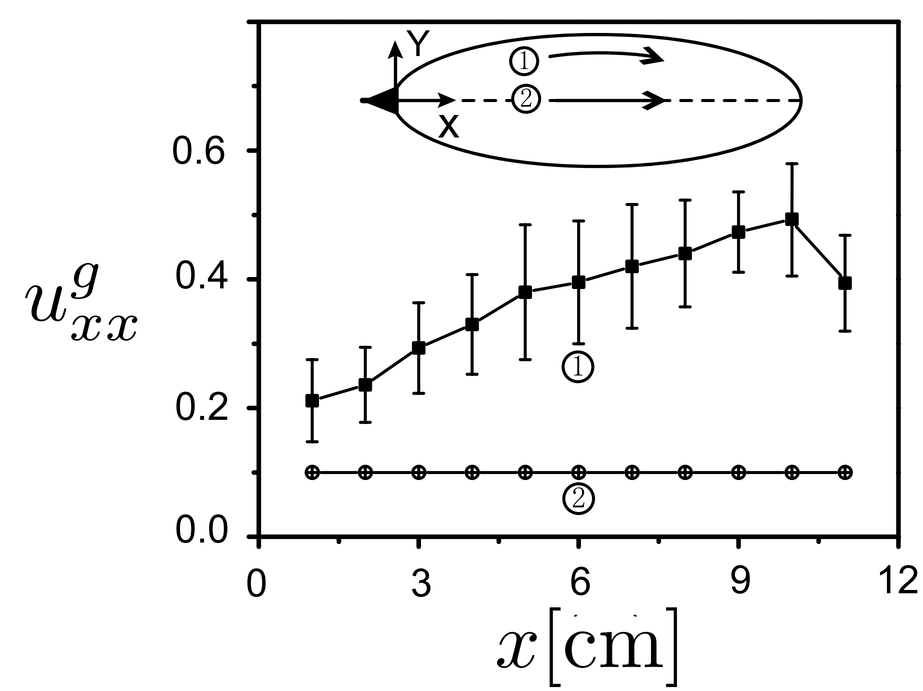
wrinkled edges (+saddles)

Growth of a blooming lily



in lab blooming takes 4.5 days under constant fluorescent light (1 frame/min)

faster growth of the edge



H. Liang and L. Mahadevan, PNAS **108**, 5516 (2011)

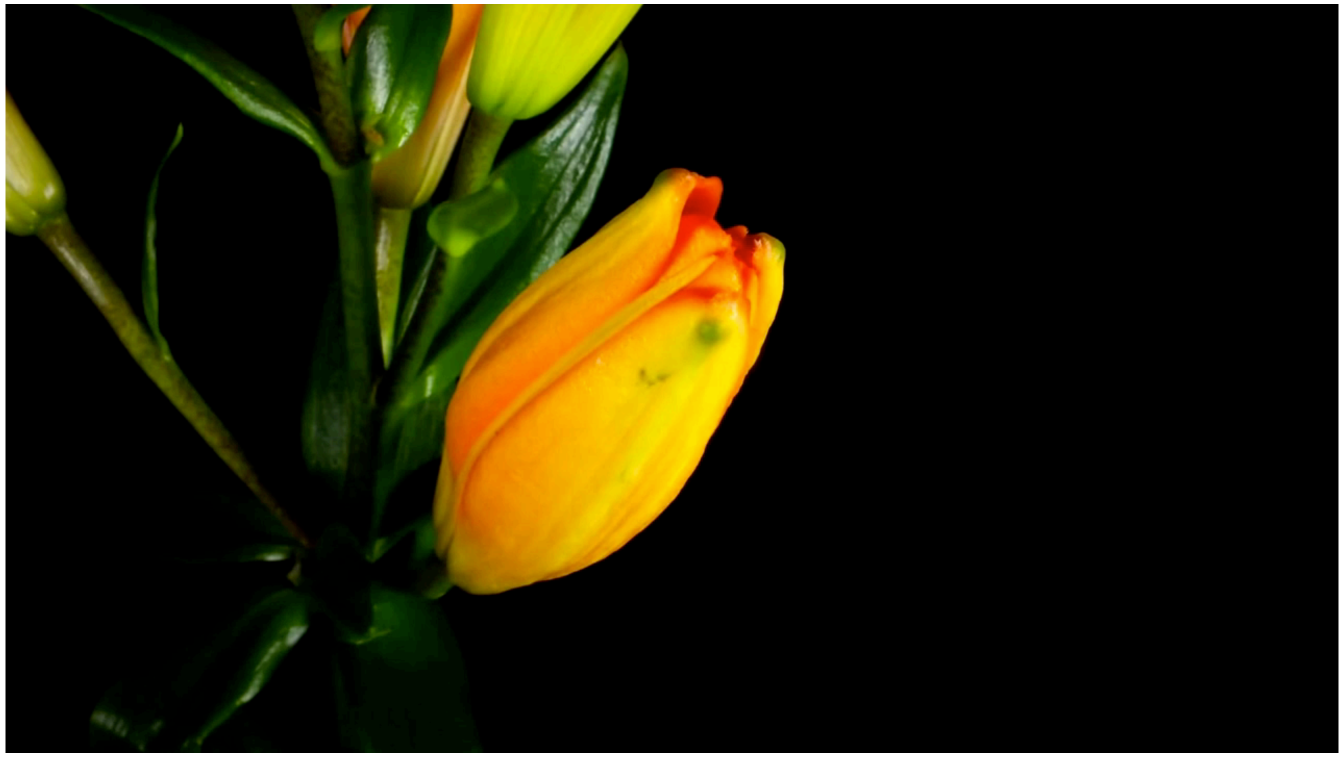
How flowers open in the morning and close in the evening?

https://vimeo.com/98276732

How flowers open in the morning and close in the evening?

morning

evening



When temperature increases in the morning, flowers regulate their growth pattern to grow more new cells on the inside of flower leaves. This results in curling of leaves and opening of flowers.

When temperature drops in the evening, flowers regulate their growth pattern to grow more new cells on the outside of flower leaves. This results in straightening of leaves and closing of flowers.