Thermal Fluctuations of Active and Anisotropic Elastic Membranes

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Abstract

Atomically thin sheets, such as graphene, are widely used in nanotechnology. In the 80s it was shown that if one takes an isotropic elastic material and allows it to thermally fluctuate then beyond a thermal length scale, the effective Lamé constants scale as $\lambda_R(q), \mu_R(q) \sim q^{\eta_u}$ (where q is the Fourier scale and $\eta_u \approx .4$). On the other hand the effective bending rigidity diverged as $\kappa_R(q) \sim q^{-\eta}$ ($\eta \approx .8$). Given that this thermal length scale is generally around 2 nm for nano-materials at room temperature, it thus becomes of interest for us to study the effects of temperature on elastic membranes. However, the spectrum of 2-D materials is quite wide, including anisotropies (such as black phosphorus) and non-equilibrium properties. Motivated by this, we investigate elastic membranes in three different scenarios using field theory and simulations. Firstly, we examine the effect of a uni-axial stress on the scaling exponents η, η_u. We find that the scaling theory no longer remains the same and that the elastic moduli become explicitly anisotropic. We furthermore establish a non-linear stress-strain relation for intermediate stresses. Secondly, we investigate the effect of elastic anisotropies on the scaling exponents η, η_u; in particular we would like to know if an elastic modulus anisotropy persists (potentially diverging) or washes away. Our simulations indicate that the latter is the case whereas the theory is much less trivial and our work indicates that further calculations are necessary. Lastly, we investigate what impact non-equilibrium forces may have on the established equilibrium behavior. We do this by generalizing to odd elasticity, permitting the presence of moduli that break conservation of energy and angular momentum, A_{odd}, K_{odd}. A_{odd} couples torques to dilations whereas K_{odd} couples pure and simple shears. We find that fluctuation-dissipation is an unstable condition. If, however, it is satisfied then K_{odd} is an irrelevant parameter that converges to zero with an exponent $2\eta_u$ whereas A_{odd} acts as a marginal parameter that only converges to zero with exponent η_u.

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Chapter 1

Brief Summary Introducing Research on Fluctuating 2-Dimensional Elastic Membranes

The discovery of graphene (via mechanical exfoliation) in 2004 made by K.S. Novoselov, A.K. Geim and others marked the first known existence of 2-dimensional crystals [1, 2]. Graphene is a 2-dimensional sheet of carbon atoms arranged in a honeycomb (hexagonal) lattice in the absence of any defects. Each carbon atom is bonded to three other carbon atoms via covalent bonds, giving it a space group symmetry $p3$ which makes it elastically isotropic. For the exfoliation of graphene and other experimental findings on the 2-dimensional crystal (including Dirac points), Novoselov and Geim were awarded a Nobel Prize in 2010 [2, 3]. As the first 2-dimensional material, there is a whole host of interesting electronic properties to understand including the half-integer quantum hall effect and the recent "magic angle" superconductivity in bilayer graphene [2, 4, 5, 6, 7]. In addition, graphene is also quite notable because it is one of the strongest materials known with a Young's modulus of $342 J/m^2$ [8].

Despite this relatively recent discovery, geometric non-linearities are the mainstay of the study of the mechanics of slender structures [9]. Though this field is quite old, only over the last few decades have the effects of temperature on the mechanics of 2-D materials been studied, as is exemplified by polymerized membranes, graphene and a whole host of other 2D materials such as BN, WS_2 and MoS_2 which have been discovered over the last decade [10, 11, 12, 13, 2, 3, 14]. Free-standing layers of these 2D crystals offer an experimentally realizable system for exploring how mechanical behavior of thermalized elastic membranes. Further manipulation of these 2D crystals for the creation of metamaterials generates new opportunities for research on the interface of mechanical and electronic properties of 2D crystals. One such recent example shows the experimental realization of kirigami graphene where large effective strains did not affect its conductivity [15].

Although, 2-D elastic crystals may be viewed as a higher dimensional extension of the $D = 1$ elastic polymer, there are some major differences between these two physical systems. Analogous to the persistence length of thermalized polymers (the length scale over which a polymer is approximately straight) [16], 2-D materials have a temperature-dependent length scale, named the thermal length scale, ℓ_{th}, beyond which temperature plays a role in elastic responses to external stresses. However, due to the coupling between the Goldstone flexural phonons and the in-plane phonons, 2D elastic materials of arbitrarily large size avoid being subject to the Mermin-Wagner theorem when $D = 2$ [11, 17, 18] and thus remain flat at sufficiently low temperatures even beyond this thermal length scale. The result is a mean-field flat phase below a crumpling temperature that gives rise to elastic moduli that exhibit anomalous scale dependence. In [19] it was shown that beyond the thermal length scale, the effective Lamé constants scaled as $\lambda_R(q), \mu_R(q) \sim q^{\eta_u}$ (where q is the Fourier scale and $\eta_u \approx .4$). On the other hand the effective bending rigidity divered as $\kappa_R(q) \sim q^{-\eta}$ ($\eta \approx .8$). These results for the numerical exponents have been recently verified to 2 and 3-loop order [20, 21, 22, 23, 24, 25]. In addition, the arguments demonstrating that elasticity in D-dimensions exhibits scale but not conformal invariance [26] was extended to the case of D-dimensional elastic membranes embedded in $D + d_c$ dimensions (where d_c is the co-dimension) [27].

Experimental measurements of the scale-dependence of the elastic moduli of thermalized membranes and the resulting mechanical properties (such as the non-linear relation $\epsilon \sim \sigma^{\eta/(2-\eta)}$) in the absence of quenched disorder have not been realized yet. However, many theoretical and simulation efforts have been realized to extend the original results of [11] to a variety of interesting cases and to understand the mechanical response of 2-D materials. In particular, for isotropically stressed fluctuating membranes, when stresses are larger than the linear response but less than one that would flatten out all thermal wrinkles, a non-linear relation between stress and strain was obtained $\epsilon \sim \sigma^{\eta/(2-\eta)}$ [28, 29]. Thermal fluctuations also increase critical buckling load with respect to the Euler buckling load due to the divergent effective bending rigidity $\kappa_R(q) \sim q^{-\eta}$ [30, 31]. Extension of the theory to inversion-asymmetric tethered membranes (such as graphene coated with a material on one side) has been recently done in which a double spiral phase and long range orientational order was predicted [32]. Realistic considerations of single clamped boundary conditions have also been reported to introduce a spontaneous symmetry-breaking tilt [33]. A recent extension of the theory of mono-layer elastic membranes to bi-layers was also done and found that the effective scaling of the elastic moduli did not change in the infrared limit ($q \to 0$) [22]. In addition, a new universality class was obtained with different anomalous elastic exponents have been done in the presence of an external field that breaks the rotational symmetry of the embedding space [34]. Early theoretical studies focusing on estimating the Poisson ratio of stress-free membranes found a universal value of $-1/3$ to 1 loop order, which was confirmed by simulations measuring correlations [19, 35]. Other more recent studies [36, 37, 38] since then have attempted to refine this 1-loop estimation of the universal Poisson ratio. More investigations will be necessary to further comprehend the Poisson ratio as a function of exerted stress. Further simulations have been done in an effort to consider the effect of experimental realities such as the quenched rippling of graphene and defects [39]. These simulations showed that the Poisson ratio decreased with aspect ratio between the amplitude of the ripples and the system size, even making it negative.

1.1 Statistical Mechanics of Elastic Membranes

1.1.1 Reconciling The Existence of 2-Dimensional Crystals with the Mermin-Wagner Theorem

The discovery of graphene, a 2-dimensional crystal which exhibits long-range order in the laboratory, posed a very important problem due to the Hohenberg-Mermin-Wagner theorem (HMW theorem). The theorem states that for systems in 2 dimensions with continuous symmetry, no symmetry breaking phase transition exists. It is important to note that the theorem is meant to be applied to systems with sufficiently short range interactions and that HMW treated systems with only one order parameter [23, 40]. Indeed, there is a rigorous version proven by mathematicians for $O(N)$-symmetric models with a C^2 potential which establishes power-law decay of correlations [41, 42, 43]. (As an important note, although long-range translational order in 2-d cannot exist as we will see below, long-range order in bond orientation can [24, 25]. Further discussion of the limitations of the HMW theorem in various systems may be found in reference [25])

The free energy of a 2-dimensional crystal depends on whether one chooses the dimension of the elastic free energy to be $[d = 3, D = 2]$ (using this as notation for a D-dimensional crystal embedded in d physical dimensions) or $[d = 2, D = 2]$ (2-dimensional crystal occupying the whole space). The former is the realistic case for freely suspended membranes. But for now, consider the latter case. Then the relevant free energy takes the form [44]:

$$\mathcal{F} = \frac{1}{2}\int d^2\mathbf{r}[\lambda u_{ii}^2 + 2\mu u_{ij}^2] \tag{1.1}$$

where the strain tensor is defined as:

$$u_{ij} = \frac{1}{2}[\partial_i u_j + \partial_j u_i] \tag{1.2}$$

and u_i are the in-plane displacements along the i-th axis and λ and μ are the elastic moduli of an isotropic 2-dimensional material (known formally as the Lamé coefficients). Taking the Fourier transform one obtains:

$$\mathcal{F} = \frac{L^2}{2}\sum_{|\mathbf{q}|<\Lambda}[\mu q^2|\mathbf{u}(\mathbf{q})|^2 + (\mu+\lambda)(\mathbf{q}\cdot\mathbf{u}(\mathbf{q})^2] \tag{1.3}$$

We have taken the form of the Fourier transform to be: $G(\mathbf{r}) = G(0) + \sum_{\Lambda\geq|\mathbf{q}|\geq 2\pi/L} G(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}} = G(0) + \int_{2\pi/L}^{\Lambda}\frac{d^2\mathbf{q}}{A}G(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}$. In this definition of the Fourier transform, Λ is the UV cutoff introduced by the microscopic scale of the system where the continuum elastic theory breaks down i.e. $\Lambda = \pi/a$ where a is the lattice spacing of the material. The correspondence between real lengths and Fourier space inverse lengths is taken to be: $q = 2\pi/\ell$. One can now fix the corresponding Gibbs measure $e^{-\beta\mathcal{F}}$ to calculate correlations of u_i. Via a Gaussian integral the propagator is of the form:

$$\langle|\mathbf{u}(\mathbf{q})|^2\rangle = \frac{k_B T(\lambda+3\mu)}{L^2\mu(\lambda+2\mu)q^2} \tag{1.4}$$

From this, one can see that the square of the fluctuation amplitude, which is the integral of the propagator, will diverge logarithmically with system size L:

$$\Delta^2 = \frac{k_B T(\lambda + 3\mu)}{\mu(\lambda + 2\mu)} \int_{2\pi/L}^{2\pi/a} d^2 q \frac{1}{q^2} \tag{1.5}$$

As a consequence, in the long range, there should be no crystalline order and the HMW theorem holds (indeed, only quasi-long-range translational order exists).

Despite this certainty, one may consider the Lindemann criterion for melting [45], which states the condition that the fluctuation amplitude, Δ, in the positions of atoms should not exceed $c \cdot a \sim .14nm$ for graphene (we take graphene as the model 2-dimensional crystal). Here c serves as a constant satisfying $.15 < c < .5$. To estimate critical system size at which the fluctuation amplitude is $\sim c \cdot a$, one sums over the Green's function in Fourier space. Assuming room temperature and using the fact that the shear modulus of graphene is $\sim 40J/m^2$, one sees that the critical system size for which fluctuations in atom's positions are $\sim c \cdot a$ is on the order of $L_c \sim e^{\mu c^2 a^2 / k_B T} a$. If we take $c = .15$ then $L_c \sim 100a$ whereas if $c = .3$ then $L_c \sim 10^8 a \sim 1cm$. This is an enormous variation, however what is important to note is how quickly L_c increases as a function of c due to the logarithmic divergence of the propagator. Hence the argument for long-range translational order fails, but typically the scale at which translational order breaks down is much larger than the lattice spacing.

One may also point out that most free 2-d membranes can fluctuate out-of-plane from a corresponding reference flat state (reference state meaning $\langle u_i \rangle = \langle f \rangle = 0$). Therefore one may choose to work with the former case in which $[d = 3, D = 2]$. The corresponding elastic free energy takes the following form:

$$\mathcal{F}[u_i, f] = \frac{1}{2} \int d^2\mathbf{r} [\lambda u_{ii}^2 + 2\mu u_{ij}^2 + \kappa(\Delta f)^2 - \kappa_G \det(\partial_i \partial_j f)] \tag{1.6}$$

where f is the out-of-plane deformation and κ is the zero-temperature bending rigidity whereas κ_G is the gaussian bending rigidity. In addition, the strain tensor is defined as:

$$u_{ij} = \frac{1}{2}[\partial_i u_j + \partial_j u_i + \partial_i f \partial_j f] \tag{1.7}$$

Despite that the crystalline membrane is allowed to fluctuate out-of-plane, the free energy still corresponds to a 2-dimensional theory. In addition, under the assumption of periodic boundary conditions, the last term of the free energy may be neglected via the Gauss-Bonnet theorem. The interactions so far seem local. However, note the two kinds of order parameters which are distinct: u_i and f. Therefore this free energy does not fall into the class treated by HMW. One may note the order of u is at most quadratic whereas there are an-harmonic interactions of the form f^4 and uf^2 (one can include an-harmonicities of the form u^4 but these turn out to be asymptotically irrelevant in the field-theoretic sense). Since the order of u_i is quadratic, a functional gaussian integral of the partition function over u_i leads to an effective free energy which is a function of only a single order parameter f:

$$\mathcal{Z} = \int \mathcal{D}[u_i, f] e^{-\beta \mathcal{H}[u_i, f]} = \int \mathcal{D}[f] e^{-\beta \mathcal{H}_{eff}[f]} \tag{1.8}$$

where $\mathcal{F}[u_i, f]$ is 1.6 and $\mathcal{F}_{eff}[f]$ is expressed below. This yields a free energy in Fourier space of the form:

$$\mathcal{F}_{eff}[f] = \frac{\kappa L^2}{2} \sum_{|\mathbf{q}|<\Lambda} q^4 f(\mathbf{q}) f(-\mathbf{q}) + \frac{Y L^2}{8} \sum_{\sum \mathbf{q}_i = 0} [q_{1i} P^T_{ij}(\mathbf{q}) q_{2j}][q_{3i} P^T_{ij}(\mathbf{q}) q_{4j}] \prod_{k=1}^{4} f(\mathbf{q}_k) \tag{1.9}$$

where

$$P^T_{ij}(\mathbf{q}) = \delta_{ij} - \frac{q_i q_j}{q^2} \tag{1.10}$$

with $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$. However it turns out that the f^4 interaction coefficient is no longer short-ranged as was noted by Nelson and Peliti [11, 25]. This can be checked by examining the four-vertex coefficient $P_{ij}^T(\mathbf{q})P_{\alpha\beta}^T(\mathbf{q})$ and taking its Fourier transform. Via the Fourier transform, power-law interactions in real space emerge which can be seen explicitly in reference [17]. Since the interaction is long-ranged, HMW should no longer apply. Indeed, contrasting with the case $[d = 2, D = 2]$ one sees that out-of-plane fluctuations serve to stabilize long-range order in 2-dimensional crystals such as graphene!

One may investigate further and consider a D-dimensional membranes embedded in d-dimensions ($d > D$). A renormalization group method known as the $1/d_c$ expansion (where d_c is the co-dimension) shows that the lower critical dimension D_{lc} (the dimension below which long-range order cannot exist) is greater than 1 and less than 2 [46, 12]. This once again provides affirmation of long range order for $D = 2$. This is quite remarkable since we saw that within the original multi-order-parameter free energy which seemed to have short range interactions Eq (1.6), long range order of Gaussian curvatures.

1.1.2 Crumpling Phase Transition

Due to the evasion of the HMW theorem by freely suspended 2-dimensional crystals, the "ordered" flat state should exist for low enough temperatures. Therefore it makes sense to ask (assuming $d = 3$ and $D = 2$) if there exists an order-disorder phase transition (ignoring steric interactions) at a critical temperature from an entropic "crumpled"-state (or disordered in shape) to a "flat"-state (ordered). Indeed such a phase transition exists since $D_{lc} < 2$ [10, 47, 46, 13, 48]. The un-crumpling phase transition can be characterized by the change in symmetry group from $O(d)$ to $O(d-D) \times O(D)$. Consider the above free energy (1.6) once again. Calculating the normal-normal correlation gives:

$$\langle \hat{n}(\mathbf{r}_a) \cdot \hat{n}(\mathbf{r}_b) \rangle \approx 1 - \sum_{|\mathbf{q}|<\Lambda} q^2 [1 - e^{i\mathbf{q}\cdot(\mathbf{r}_a - \mathbf{r}_b)}] \langle f(\mathbf{q}) f(-\mathbf{q}) \rangle \tag{1.11}$$

In this equation, one must use the correlation modified by the renormalization of elastic moduli, otherwise the sum diverges (renormalization shall be discussed in the next section). An explicit expression of Eq. (1.11) at large distances can be found in reference [28]:

$$\langle \hat{n}(\mathbf{r}_a) \cdot \hat{n}(\mathbf{r}_b) \rangle \approx 1 - \frac{k_B T}{2\pi\kappa_o}[\eta^{-1} + \ln(\ell_{th}\Lambda)] + \frac{k_B T}{5\kappa_o}\left(\frac{\ell_{th}}{|\mathbf{r}_b - \mathbf{r}_a|}\right)^{\eta} \tag{1.12}$$

where $\eta \approx .8$ (this turns out to be a critical exponent in the next section). Via calculation, one sees that at room temperature the normal-normal correlation decays to a constant very close to 1 implying long range order. Hence, the critical temperature can be quantified by the critical value of T at which the normal-normal correlation becomes zero at infinite separation distance. This turns out to be approximately $T_c \sim \kappa/k_B$. Therefore, given that $\kappa \approx 2 \cdot 10^{-19} J$ for graphene for example, $T_c \sim 50,000K$, which is a very high value. Indeed the melting temperature of graphene is estimated, via simulations, to be only a few thousand Kelvin and therefore the crumpling transition should not be relevant for 2-dimensional crystals like graphene [49]. From a theoretical standpoint, the melting of freely suspended 2-d crystals is not well understood, although some variation of the Kosterlitz-Thouless-Halperin-Nelson-Young theory is suspected [46]. The KTHNY phase transition is concerned with the melting of crystals (with quasi-long range translational order) strictly in 2 dimensions [50, 51, 52, 53, 54]. It asserts that as temperature is increased dislocations in a crystalline material unbind to form a hexatic phase (quasi-long-range six-fold orientational order). Eventually the dislocations dissociate into disclinations leading to a liquid phase. So for freely suspended membranes, the existence of a hexatic phase before a fluid phase may be possible (a convincing intuition is that dislocations have finite energy cost when crystalline membranes can buckle out of plane [55, 46]).

Returning to what is known, since the crumpling temperature is so high, one can confidently guess that the crumpling phase transition is most likely physically irrelevant for most freely suspended 2-dimensional crystals. So at room temperature, 2-dimensional crystalline membranes like graphene are in a reference flat state. Now it is possible to examine the elastic properties of these crystalline membranes as one does for polymers. This is the question addressed in the next section.

1.1.3 Momentum Shell Interpretation of Renormalization and Elastic Moduli of Graphene

Let us now examine some of the properties of free energy 1.6 at low temperatures where a flat un-melted state exists. Naively, one may expect the mechanical moduli of the free energy function to be the true effective moduli at all length scales. However, as noted before there are non-linear terms in the free energy of the form f^4 and uf^2. By a dimensional scaling analysis the upper critical dimension for free energy 1.6 satisfies $D_{uc} = 4$ (D_{uc} is such that for $d > D_{uc}$ non-linear terms may be ignored, whereas they must be considered for $d < D_{uc}$). Since $D_{uc} = 4$, one can expect that non-linear terms cannot be ignored for the physical case. However, intuitively, at small length scales and at low temperatures, harmonic theory (and therefore linear elastic theory) should be dominant. Quantitatively, the scale at which linear elastic theory breaks down may be identified with the scale at which the out-of-plane fluctuations $\sqrt{\langle f^2 \rangle}$ (where $\langle \cdot \rangle$ means harmonic average) become of the order of the "effective" thickness of the elastic material, $\sqrt{\kappa/Y}$. Indeed, for an elastic plate, non-linearities are important when deformations are of the order of the thickness of the plate [44]. Named the thermal length scale, it is approximately:

$$\langle f(\mathbf{q}) f(-\mathbf{q}) \rangle_0 = \frac{k_B T}{L^2 \kappa q^4} \sim \frac{\kappa}{Y} \rightarrow \ell_{th} \sim \sqrt{\frac{\kappa^2}{k_B T Y}} \tag{1.13}$$

where Y is the zero temperature Young's modulus (this length scale also could have been estimated via noting where perturbation theory fails). In general D dimensions $\ell_{\text{th}} \sim (\frac{\kappa^2(\lambda+2\mu)}{4\mu(\lambda+\mu)k_BT})^{\frac{1}{4-D}}$ [56] which can be derived by using the general effective thickness $t \sim \sqrt{\kappa(\lambda+2\mu)/(4\mu(\lambda+\mu))}$. This defines the thermal length scale, beyond which temperature affects the mechanical properties of the elastic membrane and anharmonic terms can no longer be ignored. For 2-dimensional crystals like graphene or h-BN at room temperature, $\ell_{th} \sim 1nm$.

Thus beyond this scale, we expect the effective theory to change due to renormalization of the elastic moduli due to the anharmonic f^4 and uf^2 terms becomes important in evaluating the Green's function. We can illustrate this more clearly by calculating the effective theory at a given scale $\ell^* = 2\pi/q^*$ via integrating out faster small-scale fluctuations. This can be done by splitting the phononic fields into pieces: $g_<(\mathbf{r}) = \sum_{|\mathbf{q}|<q^*} e^{i\mathbf{q}\cdot\mathbf{r}} g(\mathbf{q})$ and $g_>(\mathbf{r}) = \sum_{\Lambda>|\mathbf{q}|>q^*} e^{i\mathbf{q}\cdot\mathbf{r}} g(\mathbf{q})$ where $g \in \{u_i, f\}$ and integrating out the latter, $g_>$. By performing this integration we obtain [57]:

$$\mathcal{F}_{\ell^*}[u_{i<}, f_<] = \quad -k_BT \ln \int \mathcal{D}[u_{i>}, f_>] e^{-\mathcal{F}_a[u_{i<}, f_<, u_{i>}, f_>]/k_BT} \tag{1.14}$$

where $\mathcal{F}_a[u_{i<}, f_<, u_{i>}, f_>]$ is the full free energy function without any phononic modes having been integrated out. Since not all the terms in $\mathcal{F}_a[u_{i<}, f_<, u_{i>}, f_>]$ are harmonic in the high-frequency phononic fields, one may ask how this integration is exactly performed. To integrate them, one uses the quantum field theoretic method of Feynman diagrams which can be found in many texts [58, 59, 60, 61, 57]. Terms (even an-harmonic ones) purely made of high-frequency fields are not interesting to us as they just produce corrective numerical factors to the free energy. It is the terms that mix high and low frequency phononic fields that are of the most interest for us to integrate. This is because these terms can give rise to an-harmonic contribution in the low-frequency effective theory via integration. To summarize briefly, one assumes that the high frequency phononic fields are independent random variables. Thus one can then Taylor expand the an-harmonic high frequency phononic fields and use Wick's theorem (or Isserlis theorem as it is known in Probability theory) to integrate them out using the quadratic (Gaussian) high-frequency measure. This will generate terms that have low out-going frequencies and allow us to obtain the aforementioned an-harmonic contributions (formally known as renormalization) to the Gaussian low-frequency measure which is our effective theory.

Returning to the previously obtained results, to better understand the effect of an-harmonic terms, Aronovitz and Lubensky performed the ϵ-expansion [12]. The ϵ-expansion is one of the most rigorous forms of the renormalization group from the toolbox of a statistical physicist [62, 57]. The idea is to perform a momentum-shell renormalization group infinitesimally below the upper critical dimension $D_{uc} - \epsilon$ (where the fixed point bifurcation can occur and a non-trivial fixed point remain in the vicinity of the quadratic or Gaussian fixed point). Since the upper critical dimension is the dimension at which a bifurcation between Gaussian behavior (for $D > D_{uc}$) and non-Gaussian behavior (for $D < D_{uc}$) can occur, this form of the renormalization group is effective in giving a correct qualitative analysis.

Understanding the effect of the arithmetic transformation of the renormalization group, one can then obtain a set of ordinary differential equations. These are formally known as β-equations and describe the change in the moduli with changing scale (shown below). Via these ODEs, Aronovitz and Lubensky found 4 fixed points in non-dimensional parameters $\hat{\lambda} = \frac{k_B T \lambda}{\kappa \Lambda^2}$ and $\hat{\mu} = \frac{k_B T \mu}{\kappa \Lambda^2}$. The stability analysis and fixed points can be seen in Fig. 1.1.3, which are associated with the β-equations shown below.

$$\beta_{\hat{\mu}_R} = \frac{d\hat{\mu}_R}{ds} = 2\hat{\mu}_R - \frac{3\hat{Y}_R\hat{\mu}_R + \hat{\mu}_R^2}{8\pi} \tag{1.15}$$

$$\beta_{\hat{\lambda}_R} = \frac{d\hat{\lambda}_R}{ds} = 2\hat{\lambda}_R - \frac{3\hat{Y}_R\hat{\lambda}_R + \hat{\mu}_R^2 + 4\hat{\mu}_R\hat{\lambda}_R + 2\hat{\lambda}_R^2}{8\pi} \tag{1.16}$$

where s is the scale change (with a scale change the lattice spacing changes as $a \to ae^s$). In addition, the Young's modulus can be written in terms of the Lamé coefficients $\hat{Y} = 4\hat{\mu}(\hat{\mu} + \hat{\lambda})/(2\hat{\mu} + \hat{\lambda})$. Due to the presence of a stable non-Gaussian

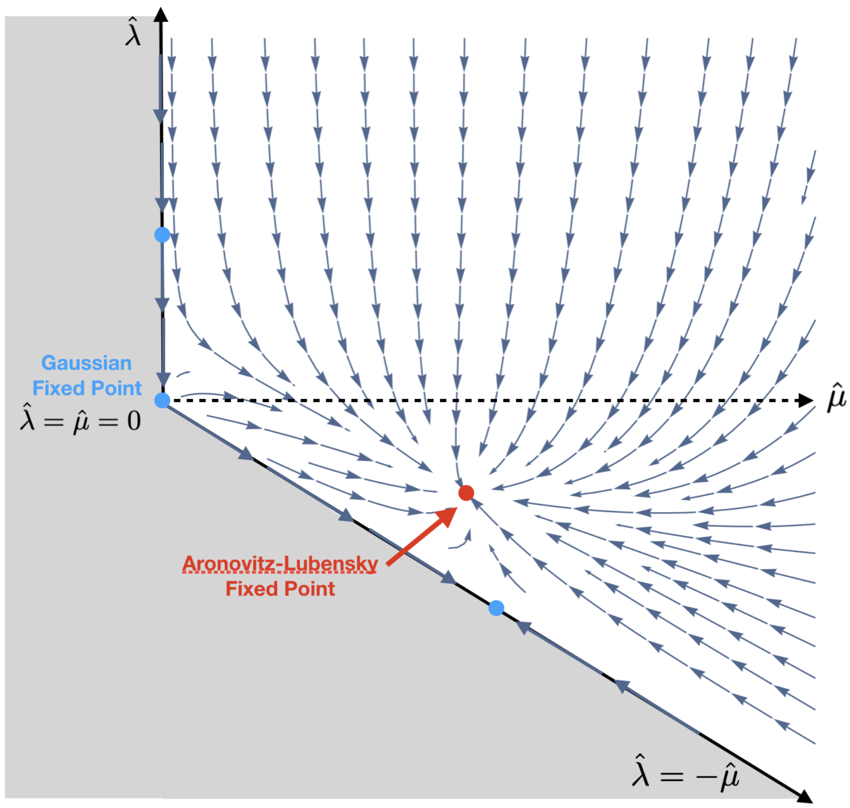

Figure 1.1: The ϵ-expansion analysis of free energy (3) reveals 4 different fixed points. The Gaussian is no longer stable slightly below the upper critical dimension $D_{uc} = 4$ and a new stable fixed point appears. The gray region is an un-physical region.

fixed point, non-trivial scaling of the elastic moduli is expected. Indeed, from Eq.1.9 one can see that non-trivial renormalization is caused by the transverse in-plane fluctuations (due to the transverse projection operator associated with shear). Nelson, Peliti, Leibler, Guitter and others [46, 13] carried out the less controlled but physically relevant renormalization group with $D = 2$ and $d = 3$. Via these calculations, it was found that beyond ℓ_{th}, the elastic moduli scale as $\kappa_R(\ell) \sim (\ell/\ell_{th})^\eta$ and

$\lambda_R(\ell) \sim \mu_R(\ell) \sim (\ell/\ell_{th})^{-\eta_u}$ with $\eta \sim .8 - .85$ and $\eta_u \sim .3 - .4$. These calculations can also be confirmed via SCSA which, like the ϵ-expansion, gives more accurate results than an uncontrolled RG [63]. Fig. 1.1.3 shows the change of elastic moduli with scale. In addition, due to the form of the strain tensor, the infinitesimal out-of-plane rotations allow one to derive the following Ward identity $2\eta + \eta_u = 2$ [13]. Although these results were confirmed by Bowick's simulations [64], they have not been confirmed experimentally due to the presence of defects and static ripples [65].

Most of the theory described thus far has been understood in the late 80s and early 90s. There are many other results concerning the critical scaling of elastic moduli for various other scenarios (such as including disorder or anisotropy).

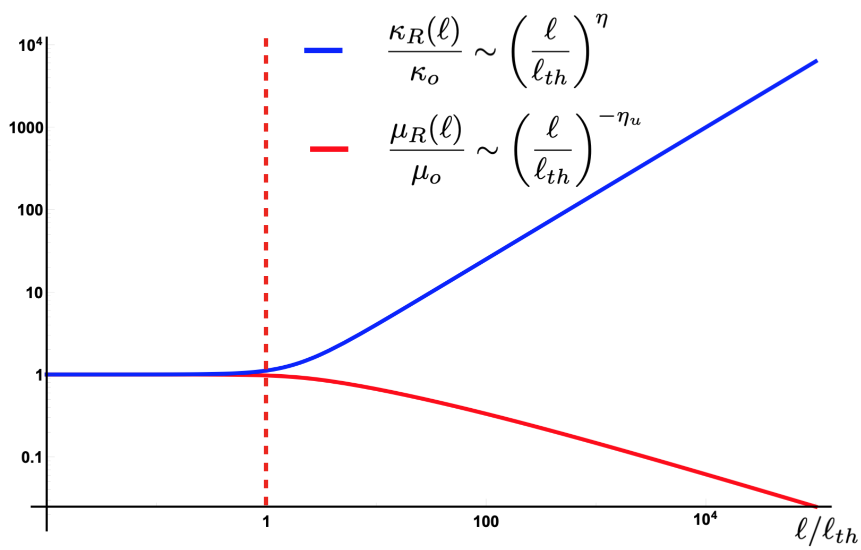

Figure 1.2: Using an uncontrolled renormalization group the renormalization of elastic moduli under the effects of temperature can be seen in a Log-Log plot. One can see that below the thermal length scale temperature does not play an important role whereas above, the elastic moduli take on critical exponents related by the exponent identity, $2\eta + \eta_u = 2$.

Chapter 2

Uni-Axially Stressed Thermalized Elastic Membranes

2.1 Introduction

Other investigations have been conducted for elastic membranes with an intrinsic anisotropy [66, 67]. For sufficiently high temperatures, due to the anisotropy, the mean field flat phase becomes un-stable and leads to a mean-field tubule phase, neglecting self-avoidance. This was further confirmed via non-perturbative approaches [68], which also better characterized the critical exponents associated with the phase transition. Anisotropies can be quite generic and thus the authors chose to focus on a tubule with effectively straight along one axis and crumpled along the other $D-1$ axes [66]. Within the tubule phase, assuming no self-avoidance, the effective elastic moduli become scale-dependent but with a behavior that is different from that found in the flat phase. Simulations done in [69] confirmed the existence of this flat-to-tubule phase transition by inserting an anisotropy in the bending rigidities. They further measured the gyration radius as a function of the length of the tubule and obtained the scaling $R_G \sim L^{\nu_F}$ where the Flory exponent is $\nu_F \approx .3$, and found it to be within close agreement with the theory, $\nu_F = 1/4$ [66].

In this section we focus on extending the theory of thermalized 2D elastic membranes to a scenario of physical interest in which a homogeneous uni-axial tension is exerted. A snapshot from a simulation can be seen in Fig. 2.1 and illustrates the physical scenario. Stress will introduce a new wave-vector, q_σ, which will render the theory dependent on the its relative magnitude with respect to $q_{\text{th}} = 2\pi/\ell_{\text{th}}$. We will explore the scaling of these elastic moduli at a variety of length scales and show an anomalous scaling at high stresses and temperatures that becomes identical to that of [66, 67] in the tubule phase. In the infrared limit ($q \to 0$) the modulus $C^R_{2222}(q) \sim q$ whereas $C^R_{1111} \sim$ constant, thus the system will exhibit strong anisotropy in the in-plane correlation functions. In the same limit, the moduli characterizing bending rigidities will exhibit anomalous behavior. Furthermore, as in the case of isotropic stress, we once again obtain a regime with a non-linear stress-strain relation, $\epsilon \sim \sigma^{\eta/(2-\eta)}$, when $2\pi/L < q_\sigma < q_{\text{th}}$ (where L is the system size).

We will first begin in Sec. 2.2 by introducing the theory in the absence of stress and explore the consequences of introducing uni-axial stresses for the symmetries of the free energy as well as the appearance of a new length scale beyond which stress becomes important. In Sec. 2.3, we study uni-axially stressed membranes via an engineering dimension analysis and Self-Consistent-Screening-Analysis (SCSA) equations to obtain the scaling of effective elastic moduli. Simulations performed in the NPT ensemble using the LAMMPS package confirm the scaling of this theory. In Sec. 2.4, we will study stress-strain relations using what we will have learned about the elastic moduli.

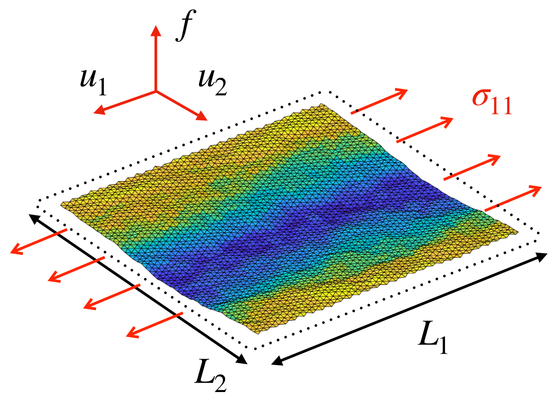

Figure 2.1: A snapshot of a simulation of a thermally fluctuating sheet placed under a uni-axial tension. u_i indicate the in-plane displacements whereas f is the flexural/out-of-plane field. The coloring of the membrane shows the scalar value of the height field (high frequency colors such as blue showing negative heights and low frequency colors such as yellow showing positive heights with respect to a zero-mean height). When a uni-axial stress is significant, transverse flexural fluctuations dominate as can be seen from the height coloration of the figure taken from a simulation. The dotted line marks the fact the $T = 0$ size which shows that elastic sheets shrink when temperature is present.

2.2 Statistical Mechanics Of Elastic Membranes Under Stress

We first repeat and discuss the free energy function of a general D-dimensional elastic membrane embedded in $(D+1)$-dimensions undergoing small deformations with respect to the reference flat state. Using Einstein notation in which repeated indices are implicitly summed over, such a function has the form [44]:

$$\mathcal{F} = \frac{1}{2}\int d^D\mathbf{r}[\lambda u_{ii}^2 + 2\mu u_{ij}^2 + \kappa K_{ii}^2 - 2\kappa_G \det(K_{ij})] \tag{2.1}$$

where λ and μ are the elastic Lamé constants and κ and κ_G are the bending and Gaussian bending rigidities. Here we use the strain and curvature tensor equations:

$$\begin{aligned} u_{ij} &= \frac{1}{2}[\partial_i u_j + \partial_j u_i + \partial_i f \partial_j f] \\ K_{ij} &= \partial_i \partial_j f \end{aligned} \tag{2.2}$$

where the indices i, j run through the D intrinsic dimensions of the elastic membrane. These describe the deformations from a reference flat metric and zero-curvature state, with u_i being the in-plane displacements along the i-th axis and f being the out-of-plane displacement. The strain tensor u_{ij} expresses stretching and shearing whereas K_{ij} expresses curvatures. Note that we have omitted $(\partial u)^2$ from the non-harmonic portion of the strain tensor due to the fact that in-plane stretching costs more energy than stretching due to the out-of-plane deformations represented by $(\partial f)^2$. By means of an engineering dimension analysis done in Sec. 2.3, one can show that $(\partial u)^2$ is irrelevant and can thus be ignored.

The effect of thermal fluctuations in a system with free energy function $\mathcal{F}$ can be extracted from the correlation functions, obtained via functional integrals over all membrane configurations [57]:

$$\begin{aligned} \mathcal{G}^R_{u_i u_j}(\mathbf{r}_2 - \mathbf{r}_1) &= \frac{1}{\mathcal{Z}} \int \mathcal{D}[u_i, f] u_i(\mathbf{r}_2) u_j(\mathbf{r}_1) e^{-\mathcal{F}/k_B T} \\ \mathcal{G}^R_{ff}(\mathbf{r}_2 - \mathbf{r}_1) &= \frac{1}{\mathcal{Z}} \int \mathcal{D}[u_i, f] f(\mathbf{r}_2) f(\mathbf{r}_1) e^{-\mathcal{F}/k_B T} \end{aligned} \tag{2.3}$$

where $e^{-\mathcal{F}/k_B T}$ is the temperature dependent Boltzman weight and $\mathcal{Z}$ is the normalizing partition function, $\mathcal{Z} = \int \mathcal{D}[u_i, f] e^{-\mathcal{F}/k_B T}$. Due to the form of the in-plane strain tensor, the free energy function is not harmonic in the displacement parameters u_i and f. In the absence of such an-harmonic terms and stress and under periodic boundary conditions (so we may integrate out the Gaussian bending term via the Gauss-Bonnet theorem), the correlation function of the flexural phonons and in-plane phonons of a system of Fourier scale q take the form [11, 46]:

$$\mathcal{G}_{u_i u_j}(\mathbf{q}) = \frac{k_B T P^T_{ij}(\mathbf{q})}{A\mu q^2} + \frac{k_B T(\delta_{ij} - P^T_{ij}(\mathbf{q}))}{A(2\mu+\lambda)q^2} \tag{2.4}$$

$$\mathcal{G}_{ff}(\mathbf{q}) = \frac{k_B T}{A\kappa q^4} \tag{2.5}$$

where A is the membrane area and $P^T_{ij}(q) = \delta_{ij} - q_i q_j/q^2$ is the transverse projection operator. We have taken the form of the Fourier transform to be: $G(\mathbf{r}) = G(0) + \sum_{\Lambda \geq |\mathbf{q}| \geq 2\pi/L} G(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}} = G(0) + \int_{2\pi/L}^{\Lambda} \frac{d^2\mathbf{q}}{A} G(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}$. In this definition of the Fourier transform, Λ is the UV cutoff introduced by the microscopic scale of the system where the continuum elastic theory breaks down i.e. $\Lambda = \pi/a$ where a is the lattice spacing of the material. The correspondence between real lengths and Fourier space inverse lengths is taken to be: $q = 2\pi/\ell$. Via scaling analysis, a length scale subtly emerges due to the presence of temperature. It is well known from plate theory that an-harmonic terms play a role once the magnitude of deflection becomes comparable to the thickness of the plate [44]. Considering specifically $D = 2$ materials such as graphene, though it is atomically thin, we can assign an effective thickness, derivable via the elastic formula, $t \sim \sqrt{\kappa/Y}$ where Y is the 2D Young's modulus ($Y = 4\mu(\lambda + \mu)/(\lambda + 2\mu)$) [44]. Assuming we take a plate thickness t under-going an out-of-plane displacement deformation of amplitude f over a length L, a scaling analysis can be performed to compare the bending energy and stretching energy cost from Eq. (2.1) [44]:

$$\begin{aligned} \mathcal{F}_{\text{stretching}} &\sim Y\left(\frac{f^2}{L^2}\right)^2 \\ \mathcal{F}_{\text{bending}} &\sim Yt^2\left(\frac{f}{L^2}\right)^2 \end{aligned} \tag{2.6}$$

When the two energy costs are of comparable order, an-harmonic terms can no longer be ignored. Indeed, one can notice from the form of Eq. (2.6) that this occurs when

$f \approx t \sim \sqrt{\kappa/Y}$. Inserting this equivalence of length scales and the Fourier form $q \sim 1/L$ into Eq. (2.5) and solving for L, we obtain a length scale $\ell_{\text{th}} \sim \sqrt{\frac{\kappa^2}{Yk_BT}}$ when $D = 2$. In general D dimensions $\ell_{\text{th}} \sim (\frac{\kappa^2(\lambda+2\mu)}{4\mu(\lambda+\mu)k_BT})^{\frac{1}{4-D}}$ [56] which can be derived by using the general effective thickness $t \sim \sqrt{\kappa(\lambda+2\mu)/(4\mu(\lambda+\mu))}$. This defines the thermal length scale, beyond which temperature affects the mechanical properties of the elastic membrane and an-harmonic terms can no longer be ignored.

The non-linear form of the in-plane strain tensor produces long-ranged coupling of Gaussian curvatures and induces a non-trivial scaling of the correlation functions beyond the thermal length scale, ℓ_{th} [17, 11, 46]. In the absence of stress, the scaling of the correlation functions is known in the long-wavelength limit and temperature renormalizes the moduli and renders them scale-dependent [12, 56, 11]:

$$\begin{aligned}
\mathcal{G}^R_{u_iu_j}(\mathbf{q}) &= \frac{k_BTP^T_{ij}(\mathbf{q})}{A\mu_R(q)q^2} + \frac{k_BT(\delta_{ij} - P^T_{ij}(\mathbf{q}))}{A(2\mu_R(q)+\lambda_R(q))q^2} \sim \left(\frac{q}{q_{\text{th}}}\right)^{-2-\eta_u} \\
\mathcal{G}^R_{ff}(\mathbf{q}) &= \frac{k_BT}{A\kappa_R(q)q^4} \sim \left(\frac{q}{q_{\text{th}}}\right)^{-4+\eta}
\end{aligned} \tag{2.7}$$

where $q_{\text{th}} = 2\pi/\ell_{\text{th}}$. The anomalous exponents take the approximate values $\eta \approx .8$ and $\eta_u \approx .4$ for $D = 2$ [28]. These exponents are not distinct but are related due to the form of the in-plane strain tensor u_{ij}, where ∂u and $(\partial f)^2$ must scale together [13]. This leads to the exponent identity $2\eta + \eta_u = 4 - D$ [28, 13]. It is important to note that $D_{\text{uc}} = 4$ is the upper critical dimension of the theory and thus no anomalous exponents will be present when $D > D_{\text{uc}}$. For $D < D_{\text{uc}}$, these exponents imply that the renormalized bending rigidity diverges as $\kappa_R(q) \sim (q/q_{\text{th}})^{-\eta}$ whereas the renormalized in-plane moduli converge to zero as $\mu_R(q), \lambda_R(q), Y_R(q) \sim (q/q_{\text{th}})^{\eta_u}$ [11].

Naively, in the presence of an arbitrary edge stress, σ_{ij}, applied to an edge with unit normal $\hat{\mathbf{n}}$, one would write the following free energy function [28]:

$$
\begin{aligned}
\mathcal{F} =& \frac{1}{2}\int d^D\mathbf{r}[\lambda u_{ii}^2 + 2\mu u_{ij}^2 + \kappa K_{ii}^2 - 2\kappa_G \det(K_{ij})] \\
&- \oint d^{D-1}\mathbf{S}\,\hat{n}_i\sigma_{ij}u_j
\end{aligned}
\tag{2.8}
$$

where the boundary term expresses the work done by an external stress. However, the effective theory at a given scale $\ell^* = 2\pi/q^*$ can be extracted by integrating out faster small-scale fluctuations. This can be done by splitting the phononic fields into pieces: $g_<(\mathbf{r}) = \sum_{|\mathbf{q}|<q^*} e^{i\mathbf{q}\cdot\mathbf{r}}g(\mathbf{q})$ and $g_>(\mathbf{r}) = \sum_{\Lambda>|\mathbf{q}|>q^*} e^{i\mathbf{q}\cdot\mathbf{r}}g(\mathbf{q})$ where $g \in \{u_i, f\}$ and integrating out the latter, $g_>$. By performing this integration we obtain [57]:

$$
\begin{aligned}
&\mathcal{F}_{\ell^*}[u_{i<}, f_<] = \\
&\quad - k_BT\ln\int \mathcal{D}[u_{i>}, f_>]e^{-\mathcal{F}_a[u_{i<},f_<,u_{i>},f_>]/k_BT}
\end{aligned}
\tag{2.9}
$$

where $\mathcal{F}_a[u_{i<}, f_<, u_{i>}, f_>]$ is the full free energy function without any phononic modes having been integrated out. Trivially, homogeneous isotropic stress will not cause renormalized anisotropies to develop since the stress will not break any rotational or mirror symmetries in the free energy. However, in the presence of a homogeneous anisotropic stress the isotropy of the free energy will be broken.

Thus it must be considered that these moduli can develop effective anisotropies for a non-isotropic stress and not only a scale dependence due to temperature. Despite the need for a generalization of the free energy, some symmetries will remain assuming the form of the stress to be "uni-axial". We clarify here that in general D-dimensions we define "uni-axial" stress as the case in which all axes experience an equal tension $\sigma_{\alpha\alpha} = \sigma$ with $\alpha \in \{1, ..., D-1\}$ and $\sigma_{DD} = 0$ (we call this the case of "uni-axial" tension since in $D = 2$, which is the case of interest, it is indeed the correct physical scenario). As a brief aside, note here that we will use Greek letters to range over indices between

$\{1, ..., D-1\}$ and Roman letters as indices that range over $\{1, ..., D\}$. We take this unusual definition of "uni-axial" to mimic the exact same theoretical formulation of tubules in Radzihovsky and Toner's theory [66, 67]. Examining Eq. (2.8), one can see that for "uniaxial" stresses the free energy will have at least orthorhombic symmetry; those being D mirror symmetries across each of the D axes. These orthorhombic symmetries will remain in renormalized free energies.

Accommodating orthorhombic anisotropy into the free energy, the generalization takes the form [44]:

$$\begin{aligned}\mathcal{F} = \frac{1}{2}\int d^D\mathbf{r}[C_{ijkl}u_{ij}u_{kl} + B_{ijkl}K_{ij}K_{kl}] \\ - \oint d^{D-1}\mathbf{S}\hat{n}_i\sigma_{ij}u_j\end{aligned} \tag{2.10}$$

where, the bare elastic moduli tensors have the fundamental major and minor symmetries: $C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$ and $B_{ijkl} = B_{klij} = B_{jikl} = B_{ijlk}$ [70]. In addition to these, the orthorhombic symmetries will enforce that $C_{iiij} = C_{iijk} = C_{ijkl} = 0$ where each distinct index is taken to be a distinct number between 1 and D. The same will hold true for the B_{ijkl} tensor. In this notation, an isotropic elastic material will have the following decomposition: $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$ and $B_{ijkl} = (\kappa - \kappa_G)\delta_{ij}\delta_{kl} + \kappa_G/2[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$.

Under stress the effective flexural phonon correlation function may be defined:

$$\mathcal{G}^R_{ff}(\mathbf{q}) = \frac{k_BT}{A[B^R_{ijkl}(\mathbf{q})q_iq_jq_kq_l + \sigma_{ij}q_iq_j]} \tag{2.11}$$

whereas the correlation function for in-plane phonons takes the form:

$$\mathcal{G}^R_{u_iu_j} = \frac{k_BT}{A}[C^R_{ikjl}(\mathbf{q})q_kq_l]^{-1} \tag{2.12}$$

These renormalized elastic constants have been defined based on the analogous correlation functions in the harmonic approximation similarly as one would define $\lambda_R(q), \mu_R(q), \kappa_R(q)$ in Eq. (2.7) based on Eqs. (2.5) and (2.4). For isotropic systems that aren't under any stress, the correlation function of the in-plane phonons reduces to same form as Eq. (2.7). However, as previously explained, in the presence of anisotropic stress the renormalized moduli may also become anisotropic.

We can now ask ourselves at what scale such a stress becomes important and thus induces anisotropy. Observing Eq. (2.11), it can be seen that under tensile stresses, one may note the introduction of a new length scale when $\sigma_{\alpha\alpha} q_\alpha^2 \sim B^R_{ijkl}(q) q_i q_j q_k q_l$ [28]. This length scale can be identified with the scale beyond which stress plays a dominant role along the axes in which it is present. For sufficiently small stresses such that $q_\sigma \ll q_{\text{th}}$ and assuming that the material is isotropic at $T = 0$ and thus we can assume that $B^R_{ijkl}(q) \sim \kappa_R(q) \sim (q/q_{\text{th}})^{-\eta}$. Thus the wave vector takes the form [28]:

$$q_\sigma = \left(\frac{\sigma_{\alpha\alpha}}{\kappa}\right)^{1/(2-\eta)} q_{\text{th}}^{-\eta/(2-\eta)} \tag{2.13}$$

The value of stress for which these two length scales are equal can be solved for $\sigma_{q_{th}} = \kappa q_{\text{th}}^2 = \kappa(\frac{4\mu(\lambda+\mu)k_B T}{\kappa^2(\lambda+2\mu)})^{\frac{1}{4-D}}$. For very large stress, $\sigma \gg \sigma_{q_{th}}$, where $q_\sigma \gg q_{\text{th}}$, then anomalous behaviors will not enter into the comparison between the stress and bending portions of the flexural correlation function and thus the bare parameters can be used, $\sigma q^2 \sim \kappa q^4$, resulting in:

$$q_\sigma = \left(\frac{\sigma}{\kappa}\right)^{1/2} \tag{2.14}$$

With this pre-amble we may now begin to investigate the scaling theory of "uni-axial" stresses imposed upon thermalized elastic membranes.

2.3 Scaling Behavior Of Elastic Moduli

In the following text, we aim to derive the scaling of the correlation functions in different regimes which will depend on the ordering of $q, q_\sigma, q_{\rm th}$ (note that order of these Fourier scales can change by tuning the parameters σ, κ, T, Y as well as trivially changing the inverse length scale q). In order to develop a convention for naming these different regimes, we define un-ambiguously that a system under "low" stress to be such that $q > q_\sigma$ and "high" stress such that $q < q_\sigma$. We also define systems under "low" temperature conditions to be such that $q > q_{\rm th}$ and high temperature such that $q < q_{\rm th}$.

To obtain the scaling of correlation function we must commence with the calculation of the engineering dimensions when stress is "low" and when stress is "high". Engineering dimensions will tell us whether terms are relevant or irrelevant to the theory. Specifically, terms with negative engineering dimension are called irrelevant and can be ignored from the scaling theory of the free energy. On the other hand, terms with positive engineering dimensions are called relevant and cannot be ignored from the scaling of the theory. An-harmonic terms with positive engineering dimension can induce anomalous scaling of the elastic moduli of the theory that is different from the linear theory (such as those for the un-stressed isotropic elastic membranes in Eq. (2.7)). Thus it will be necessary to derive the Self-Consistent Screening Analysis (SCSA) equations, which allow us to describe the effect of an-harmonic terms in the free energy. With these two tools, we will then derive the scaling of the correlation functions in each regime.

2.3.1 Engineering Dimensions

Low Stress $q > q_\sigma$

Before engineering dimensions are calculated, it is important to establish what the dominant term of the harmonic portion of the free energy is in order to proceed further into our scale-dependent analysis. In the presence of vanishingly small stresses, one can consider the non-anomalous correlation functions (also known as harmonic propagators) to scale as $\mathcal{G}_{ff}(q) \sim q^{-4}$, $\mathcal{G}_{uu}(q) \sim q^{-2}$ (as can be seen from Eqs. (2.5) and (2.4)), and see that the flexural modes fluctuate with a larger amplitude for small enough q and thus produce the dominant modes of the harmonic portion of the free energy, otherwise known as the harmonic/Gaussian theory. In other words, the term $B_{ijkl}\partial_i\partial_j f \partial_k \partial_l f$ is the dominant term in the Gaussian theory.

Thus, in the presence of the scale-invariant dominant term, $B_{ijkl}\partial_i\partial_j f \partial_k \partial_l f$, we may calculate how fields f should be re-scaled $f(\mathbf{q}) \equiv b^{-\Delta_f} f'(\mathbf{q}')$ as a result of the scale transformation $\mathbf{q} = b\mathbf{q}' \equiv b^{\Delta_q}\mathbf{q}'$ where $b > 1$ is a rescaling parameter.

Engineering dimensions give the exponent with which parameters of a theory rescale (in this case $C_{ijkl}, \sigma, B_{ijkl}$ though B_{ijkl} will be scale-invariant since we set it to our dominant term), $O \equiv b^{-\Delta_O} O'$, under the scale transformation $q = bq'$. If an engineering dimension, Δ_O, of a parameter is positive then it cannot be ignored as $q \to 0$ since it grows with scale. If on the other hand it is negative, then the parameter rescales to zero as $q \to 0$ and can thus be ignored (unless it is dangerously irrelevant) [58].

Proceeding to the counting of engineering dimensions, at low stresses, the dominant term of our theory is $B_{ijkl} q_i q_j q_k q_l f(\mathbf{q}) f(-\mathbf{q})$. This will automatically imply that

$q > q_\sigma$, $\sigma_{\alpha\alpha} = \sigma, \sigma_{DD} = 0$	
Term	Eng. Dim.
Δ_q	1
Δ_f	$(4-D)/2$
Δ_u	$3-D$
$\Delta_{C_{ijkl}}$	$4-D$
$\Delta_{B_{ijkl}}$	0
$\Delta_{\sigma_{\alpha\alpha}}$	2

Table 2.1: In the presence of small "uniaxial" stress and high temperatures, engineering dimensions of the order parameters and the elastic moduli of the theory.

the engineering dimensions $\Delta_{B_{ijkl}} = 0$. This implies then:

$$\begin{aligned} &A \sum_{|\mathbf{q}|<\Lambda} B_{ijkl}\ q_i q_j q_k q_l f(\mathbf{q}) f(-\mathbf{q}) \\ &= b^{-D} A' \sum_{|\mathbf{q}'|<\Lambda/b} B_{ijkl}\ b^4 q_i' q_j' q_k' q_l' f(b\mathbf{q}') f(-b\mathbf{q}') \end{aligned} \tag{2.15}$$

where $b^{-D}A' = A$ due to the fact that the area is a D-dimensional area in real space. In order for this term to be scale invariant then we must enforce that $b^{(4-D)/2} f(b\mathbf{q}') = f'(\mathbf{q}')$. Thus, we have obtained $\Delta_f = (4-D)/2$, and we can use this to obtain how the order parameter u and all other coefficients of terms in the free energy re-scale.

Due to the rotational symmetry of the free energy, the strain tensor will be preserved despite renormalization [13, 61]. Thus, the scale-invariance of the theory will enforce that the u field rescales $u(\mathbf{q}) \equiv b^{-\Delta_u} u'(\mathbf{q}')$ in such a way that the individual terms of the strain tensor $q_i u_j$ and $q_i f q_j f$ also re-scale the same way. Thus, equating $q_i u_j \sim q_i f q_j f$ leads to

$$\begin{aligned} &q_i u_j(\mathbf{q}) = b q_i' u_j(b\mathbf{q}') \equiv b^{-\Delta_u+1} q_i' u_j'(\mathbf{q}') \sim \\ &q_i q_j f(\mathbf{q}) f(-\mathbf{q}) = b^2 q_i' q_j' f(b\mathbf{q}') f(-b\mathbf{q}') \\ &\equiv b^{-2\Delta_f+2} q_i' q_j' f'(\mathbf{q}') f'(-\mathbf{q}') \end{aligned} \tag{2.16}$$

In other words, $\Delta_u - 1 = 2\Delta_f - 2$ which gives $\Delta_u = 3 - D$. With this, we can finally calculate how the coefficients of an-harmonic terms, C_{ijkl}, of the free energy in Eq. (2.10) should re-scale. For example, one can obtain:

$$\begin{aligned} &A \sum_{|\mathbf{q}|<\Lambda} C_{ijkl}\ q_i q_k u_j(\mathbf{q}) u_l(-\mathbf{q}) \\ &= b^{-D} A' \sum_{|\mathbf{q}'|<\Lambda/b} C_{ijkl}\ b^2 q'_i q'_k b^{2D-6} u_j(b\mathbf{q}') u_l(-b\mathbf{q}') \end{aligned} \tag{2.17}$$

which implies then that $C'_{ijkl} \equiv C_{ijkl} b^{D-4}$ and gives us the engineering dimension $\Delta_C = 4 - D$. Given these results then we know that when $D < 4$, that an-harmonic terms with coefficients C_{ijkl} will be relevant to physical behavior in the limit $q \to 0$. If stresses are not vanishingly small, a similar analysis can be done to show that $\Delta_\sigma = 2$ which indicates that it will be strongly relevant and that once $q < q_\sigma$, it can no longer be ignored. Thus the dominant term of the theory would have to be reconsidered in the "high" stress case which we will deal with in the very next section. All engineering dimensions for the "low" stress case are summarized in Table 2.1.

High Stress $q < q_\sigma$

As previously mentioned, when stress is significant, the dominant portion of the Gaussian theory has to be reconsidered. A "uni-axial" stress term, σq_α^2, will dominate over the bending rigidities in the flexural correlation function along every axis except for the D^{th} axis. Thus the new dominant term of the Gaussian theory becomes:

$$[B_{DDDD}\ q_D^4 + \sigma_{\alpha\alpha}\ q_\alpha^2] f(\mathbf{q}) f(-\mathbf{q}) \tag{2.18}$$

Rendering these free energy terms scale invariant requires axes co-linear with the "uni-axial" stress must be re-scaled such that $q_1 \sim ... \sim q_{D-1} \sim q_D^2$ [71, 72, 73]. Thus if the wave vectors rescale as $(\mathbf{q}_{D-1}, q_D) \equiv (b^2 \mathbf{q}'_{D-1}, b q'_D)$ where $\mathbf{q}_{D-1} = (q_1, ..., q_{D-1})$,

$q < q_\sigma, \sigma_{\alpha\alpha} > 0, \sigma_{DD} = 0$	
Term	Eng. Dim.
Δ_{q_α}	2
Δ_{q_D}	1
Δ_f	$(5-2D)/2$
Δ_{u_α}	$3-2D$
Δ_{u_D}	$4-2D$
$\Delta_{C_{\alpha\alpha\beta\beta}}$	$1-2D$
$\Delta_{C_{\alpha\beta\alpha\beta}}$	$1-2D$
$\Delta_{C_{\alpha D\alpha D}}$	$3-2D$
$\Delta_{C_{\alpha\alpha DD}}$	$3-2D$
$\Delta_{C_{DDDD}}$	$5-2D$
$\Delta_{B_{\alpha\alpha\beta\beta}}$	-4
$\Delta_{B_{\alpha\alpha DD}}$	-2
$\Delta_{B_{DDDD}}$	0
$\Delta_{\sigma_{\alpha\alpha}}$	0

Table 2.2: In the presence of a large "uni-axial" tension, engineering dimensions are shown for spatial dimensions, order parameters and the elastic moduli of the theory.

then keeping the term $A\sum_{\mathbf{q}}[B_{DDDD}q_D^4 + \sigma_{\alpha\alpha}q_\alpha^2]f^2$ scale invariant leads to

$$\begin{aligned}
&A\sum_{|\mathbf{q}|<\Lambda}[B_{DDDD}\ q_D^4 + \sigma_{\alpha\alpha}\ q_\alpha^2]f(\mathbf{q}_{D-1}, q_D)f(-\mathbf{q}_{D-1}, -q_D) \\
&= b^{1-2D}A' \sum_{|\mathbf{q}'|<\Lambda/b}[B_{DDDD}\ b^4 {q'_D}^4 + \sigma_{\alpha\alpha} b^4 {q'_\alpha}^2] \times \\
&\qquad f(b^2\mathbf{q}'_{D-1}, bq'_D)f(-b^2\mathbf{q}'_{D-1}, -bq'_D)
\end{aligned} \tag{2.19}$$

where $A = b^{1-2D}A'$ since $A = L_D\prod_{\alpha=1}^{D-1}L_\alpha = b^{-1}L'_D\prod_{\alpha=1}^{D-1}b^{-2}L'_\alpha$ (where L_i are the system dimensions along axis i and re-scale inverse to how q_i re-scale). This equation thus implies the engineering dimension $\Delta_f = (5-2D)/2$. Observing the difference between the two terms $B_{DDDD}q_D^4$ and $B_{\alpha\alpha DD}q_\alpha^2 q_D^2 f(\mathbf{q})^2$ and since $\Delta_{B_{DDDD}} = 0$ and $q_\alpha \sim q_D^2$, we can conclude that $\Delta_{B_{\alpha\alpha DD}} = -2$. Likewise, $\Delta_{B_{\alpha\alpha\beta\beta}} = -4$. As was done in the previous section we can compare the terms within the strain tensor $q_i u_j \sim q_i f q_j f$

and we get that

$$
\begin{aligned}
& q_i u_j(\mathbf{q}) = b^{\Delta_{q_i}} q_i' u_j(b\mathbf{q}') \equiv b^{-\Delta_u + \Delta_{q_i}} q_i' u_j'(\mathbf{q}') \sim \\
& q_i q_j f(\mathbf{q}) f(-\mathbf{q}) = b^{\Delta_{q_i} + \Delta_{q_j}} q_i' q_j' f(b\mathbf{q}') f(-b\mathbf{q}') \\
& \equiv b^{-2\Delta_f + \Delta_{q_i} + \Delta_{q_j}} q_i' q_j' f'(\mathbf{q}') f'(-\mathbf{q}')
\end{aligned} \tag{2.20}
$$

and thus $\Delta_{u_i} = 2\Delta_f - \Delta_{q_i}$. Because of the difference in how q_α and q_D re-scale, we obtain two different engineering dimensions for the in-plane displacement fields: $\Delta_{u_\alpha} = 3 - 2D$ and $\Delta_{u_D} = 4 - 2D$. Due to the anisotropic re-scaling of spatial dimensions, this causes C_{ijkl} with differing indices to be re-scaled differently as well. As an example, consider $C_{\alpha\alpha\beta\beta}$:

$$
\begin{aligned}
& A \sum_{|\mathbf{q}|<\Lambda} C_{\alpha\alpha\beta\beta}\ q_\alpha q_\beta u_\alpha(\mathbf{q}_{D-1}, q_D) u_\beta(-\mathbf{q}_{D-1}, -q_D) \\
& = b^{1-2D} A' \sum_{|\mathbf{q}'|<\Lambda/b} C_{\alpha\alpha\beta\beta}\ b^4 q_\alpha' q_\beta' \times \\
& \qquad b^{4D-6} u_\alpha(b^2 \mathbf{q}_{D-1}', b q_D') u_\beta(-b^2 \mathbf{q}_{D-1}', -b q_D')
\end{aligned} \tag{2.21}
$$

thus implying $\Delta_{C_{\alpha\alpha\beta\beta}} = 1 - 2D$. Similar analysis yields as well that $\Delta_{C_{\alpha\beta\alpha\beta}} = 1 - 2D$. On the other hand, for $C_{\alpha D\alpha D}$:

$$
\begin{aligned}
& A \sum_{|\mathbf{q}|<\Lambda} C_{\alpha D\alpha D}\ q_\alpha q_\alpha\ u_D(\mathbf{q}_{D-1}, q_D) u_D(-\mathbf{q}_{D-1}, -q_D) \\
& = b^{1-2D} A' \sum_{|\mathbf{q}'|<\Lambda/b} C_{\alpha D\alpha D}\ b^4 q_\alpha' q_\alpha' \times \\
& \qquad b^{4D-8} u_D(b^2 \mathbf{q}_{D-1}', b q_D') u_D(-b^2 \mathbf{q}_{D-1}', -b q_D')
\end{aligned} \tag{2.22}
$$

thus implying $\Delta_{C_{\alpha D\alpha D}} = 3 - 2D$. Similar conclusions can be drawn for $\Delta_{C_{\alpha\alpha DD}} = 3 - 2D$ whereas $\Delta_{C_{DDDD}} = 5 - 2D$ as summarized in Table 2.2.

Thus, we see that when $D > 5/2$, all C^R_{ijkl} become irrelevant and thus an-harmonic terms do not contribute to the theory and can be ignored in the limit $q \to 0$. On the other hand, in the case of interest $D = 2$, we can remove all irrelevant terms $(C^R_{1111}, C^R_{1122}, C^R_{1212}, B^R_{1111}, B^R_{1122})$ in the expression of the free energy as they only add technical complications and do not contribute to the qualitative scaling behavior. One can then integrate out the in-plane phonons, $u_i(\mathbf{r})$, with only the constant C^R_{2222} present in the free energy since all other C^R_{ijkl} are irrelevant. This can be done by means of the functional integral:

$$e^{-\beta \mathcal{F}_{eff}} = \int \prod_i D[u_i(\mathbf{r})] e^{-\beta \mathcal{F}} \tag{2.23}$$

Such an integration will cause all f^4 vertices to disappear and thus the effective free energy will take a simplified form:

$$\frac{\mathcal{F}_{eff}}{A} = \frac{1}{2} \sum_{|\mathbf{q}|<q_\sigma} [B^R_{2222}(q_\sigma) q_2^4 + \sigma_{11} q_1^2] f(\mathbf{q}) f(-\mathbf{q}) \tag{2.24}$$

Observing this equation, one may note the absence of an-harmonic terms. This implies that B_{2222} will no longer renormalize once the "uni-axial" stress is dominant. Thus the bending rigidity satisfies $B^R_{2222}(\mathbf{q}) = B^R_{2222}(q_\sigma)$ (and more generally B_{DDDD}) and actually remains a constant when $q < q_\sigma$.

We will now calculate the SCSA equations corresponding to the theory so that we may later on calculate the potential anomalous scaling of elastic moduli due to the presence of relevant an-harmonic terms in the free energy.

2.3.2 SCSA And β Equations

In order to derive anomalous exponents of the elastic moduli and more precise values for the cross-over scales q_σ , q_{th} where scaling of the correlation functions change, it is important to take a moment to calculate the SCSA equations that take into account the effect of an-harmonic terms into the calculation of effective elastic moduli.

Before proceeding to the derivation of the SCSA equations, we will take a brief aside to mention that one can integrate out all in-plane phonons to obtain an effective free energy for the flexural field. This will be necessary to obtain the SCSA equation for the flexural correlation function $\mathcal{G}^R_{ff}(\mathbf{q})$. For the purpose of obtaining useful expressions, we assume the general orthorhombic symmetry of the elastic tensors. Integrating out the in-plane phonons for general D gives rise to a complicated coefficient of the f^4 vertex. However, the effective energy for flexural phonons under periodic boundary conditions in the presence of a general stress takes the following form in $D = 2$ [46]:

$$\begin{aligned}\frac{\mathcal{F}_{eff}}{A} &= \frac{1}{2}\sum_{|\mathbf{q}|<\Lambda}[B_{ijkl}\ q_i q_j q_k q_l + \sigma_{ij} q_i q_j] f(\mathbf{q}) f(-\mathbf{q}) \\ &+ \frac{1}{8}\sum_{\substack{\mathbf{q}_1+\mathbf{q}_2= \\ -\mathbf{q}_3-\mathbf{q}_4=\mathbf{q}\neq 0, |\mathbf{q}_i|_{i=1,\ldots,4}<\Lambda}} q^4 [q_{1i} P^T_{ij}(\mathbf{q}) q_{2j}][q_{3i} P^T_{ij}(\mathbf{q}) q_{4j}] \frac{N}{E(\mathbf{q})} f(\mathbf{q}_1) f(\mathbf{q}_2) f(\mathbf{q}_3) f(\mathbf{q}_4)\end{aligned} \tag{2.25}$$

where N and $E(\mathbf{q})$ are:

$$\begin{aligned}N =& C_{1212}[C_{1111}C_{2222} - C_{1122}^2] \\ E(\mathbf{q}) =& \mathrm{Det}[C_{ijkl} q_i q_k] \\ =& C_{1111}C_{1212} q_1^4 + C_{2222}C_{1212} q_2^4 \\ &+ (C_{1111}C_{2222} - 2C_{1122}C_{1212} - C_{1122}^2) q_1^2 q_2^2\end{aligned} \tag{2.26}$$

Returning now to the derivation of the SCSA equation, one can compute one-loop SCSA equations for the in-plane moduli. In principle one can do calculations to higher order loops to obtain more accurate results, however one can often gain a qualitative understanding from a one-loop analysis. From an SCSA shown in Fig. 2.2 we can derive the following self-consistent equations for C^R_{ijkl}.

$$
\begin{aligned}
&C^R_{ijkl}(\mathbf{q}) = C_{ijkl} \\
&- \frac{k_BT}{4(2\pi)^D}\int_{|\mathbf{p}|<\Lambda} d^D\mathbf{p}[C^R_{ijmn}(\mathbf{q})(q_m-p_m)p_n][C_{abkl}(q_a-p_a)p_b]\frac{A\mathcal{G}^R_{ff}(\mathbf{q}-\mathbf{p})}{k_BT}\frac{A\mathcal{G}^R_{ff}(\mathbf{p})}{k_BT} \\
&- \frac{k_BT}{4(2\pi)^D}\int_{|\mathbf{p}|<\Lambda} d^D\mathbf{p}[C_{ijmn}(q_m-p_m)p_n][C^R_{abkl}(\mathbf{q})(q_a-p_a)p_b]\frac{A\mathcal{G}^R_{ff}(\mathbf{q}-\mathbf{p})}{k_BT}\frac{A\mathcal{G}^R_{ff}(\mathbf{p})}{k_BT}
\end{aligned} \tag{2.27}
$$

The symmetrization of the diagrammatic contributions seen in Fig. 2.2 is due to the major symmetry of the tensor $C_{ijkl} = C_{klij}$ which enforces conservation of energy. This symmetry should remain even through renormalization. One can also obtain identical SCSA equations renormalizing the $C_{ijkl}\partial_i u_j \partial_k f \partial_l f$ vertex since the form of Hamiltonian will be preserved via renormalization [13]. Similarly a self-consistent equation can be written down for the flexural correlation function:

$$
\mathcal{G}^R_{ff}(\mathbf{q}) = \mathcal{G}_{ff}(\mathbf{q}) - \frac{A}{k_BT}\sum_{|\mathbf{p}|<\Lambda} q^4[p_i P^T_{ij}(\mathbf{q}-\mathbf{p})q_j]^2\frac{N}{E(\mathbf{q}-\mathbf{p})}\mathcal{G}^R_{ff}(\mathbf{p})\mathcal{G}^R_{ff}(\mathbf{q})\mathcal{G}_{ff}(\mathbf{q}) \tag{2.28}
$$

One can also obtain the corresponding momentum shell Renormalization-Group equations, or β equations, which will be of use to derive the values of q_{th} and q_σ more precisely [59]. One can apply the operator, $-q\partial_q \equiv \partial_s$ to the SCSA equation (2.27), and convert the Fourier sum to a momentum shell integral [56, 28, 62] in which we have integrate a momentum shell of Fourier space $\Lambda/b < p < \Lambda$ of the $\mathbf{p}$ integral:

$$
\begin{aligned}
\partial_s C^R_{ijkl}(s) &= 2(2\Delta_f - 1)C^R_{ijkl}(s) \\
&- \frac{k_BT\Lambda^{D-4}}{2(2\pi)^D}\int d^{D-1}\hat{\mathbf{p}}\frac{[C^R_{ijmn}(s)\hat{p}_m\hat{p}_n][C^R_{abkl}(s)\hat{p}_a\hat{p}_b]}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l + \frac{\sigma_{ij}}{\Lambda^2}\hat{p}_i\hat{p}_j]^2}
\end{aligned} \tag{2.29}
$$

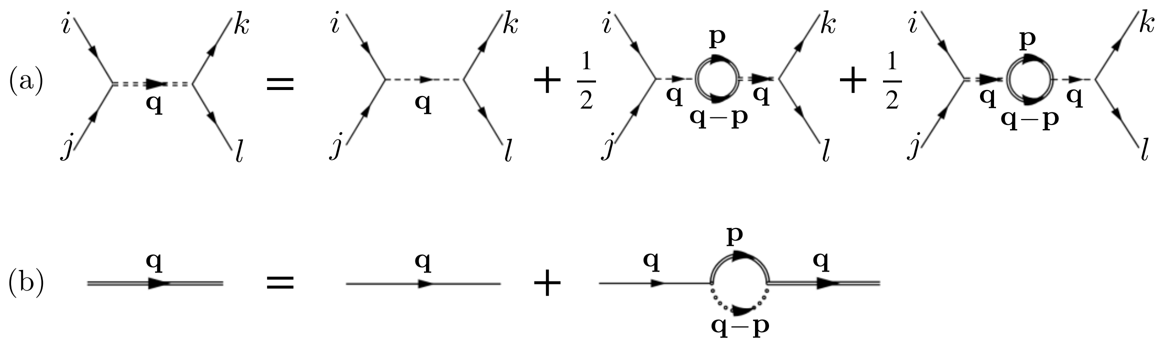

Figure 2.2: (a) The SCSA equation is shown graphically using the $C_{ijkl}\partial_i f \partial_j f \partial_k f \partial_l f$ vertex. This equation is used to obtain a scaling of the C_{ijkl} via a self-consistent analysis. The symmetrization is due to the major symmetry of the Hamiltonian. The the dashed line indicates C_{ijkl} and the doubled dashed line C^R_{ijkl}. The solid lines indicate $\mathcal{G}_{ff}$ whereas the doubled solid lines indicate $\mathcal{G}^R_{ff}$. (b) The SCSA equation corresponding to the flexural correlation function is shown using the effective f^4 vertex in Eq. (2.25). The renormalized structure of the vertex is marked by the doubled dotted line.

where s is the rescaling parameter, with $b = e^s; 0 < s \ll 1$. We have extracted the powers of Λ and this leaves us with an angular integral over the unit vectors $\hat{p}$. Furthermore Δ_f is the engineering dimension of the order parameter f, though in renormalization, it is generally treated as a degree of freedom [28, 57, 56]. Once the momenta in the window $\Lambda/b < p < \Lambda$ have been integrated over, the UV cutoff Λ/b is re-scaled to Λ.

In the limit of vanishing stress one can additionally write down a general D-dimensional β function for the isotropic bending rigidity κ [56]:

$$\begin{aligned} \partial_s \kappa_R(s) &= (2\Delta_f - D - 4)\kappa_R(s) \\ &\quad + \frac{4[D^2-1]\mu_R(s)[\lambda_R(s)+\mu_R(s)]S_D k_B T}{[D^2+2D][\lambda_R(s)+2\mu_R(s)](2\pi)^D \kappa_R(s)\Lambda^{4-D}} \end{aligned} \tag{2.30}$$

where S_D is the area of a D-dimensional unit sphere.

With the SCSA equations and β-functions calculated, we can now proceed to obtaining the scaling of correlation functions in each regime and obtain the scale limits of each regime. We order the regimes by investigating regimes with different dominant harmonic terms independently: in other words low stress, $q > q_\sigma$ and high stress $q < q_\sigma$. Within each of these regimes, we investigate each sub-regime depending on whether temperature is significant or not: in other words low temperature, $q > q_{\text{th}}$ and high temperature $q < q_{\text{th}}$.

2.3.3 Scaling At Low Stress $q > q_\sigma$

Scaling At Low Temperature $q > \max\{q_\sigma, q_{\mathbf{th}}\}$

The positive engineering dimension of C_{ijkl} when $D < 4$ signifies the importance of the parameter when $q \to 0$, however this does not mean that an-harmonic effects need to be considered at smaller finite length scales.

Indeed, when $q > \max\{q_\sigma, q_{\text{th}}\}$, the scaling of the correlation functions is trivial since both temperature and stress do not contribute significantly, thus the harmonic low stress correlation functions are sufficient to understand the theory and no anomalous effects should be observed in the theory. Reminding the reader once again that we are assuming the Roman letter indices such as i, j, k, l range over $\{1, ..., D\}$ and Greek letter indices range over $\{1, ..., D-1\}$ we have:

$$\begin{aligned} \mathcal{G}_{ff}(\mathbf{q}) =& \frac{k_B T}{A[B_{ijkl} q_i q_j q_k q_l + \sigma_{\alpha\alpha} q_\alpha^2]} \\ \approx& \frac{k_B T}{A[B_{ijkl} q_i q_j q_k q_l]} \end{aligned} \tag{2.31}$$

where and all B_{ijkl} are the bare parameters of the theory since no thermal anomalous effects are relevant. Assuming the bare material is isotropic, we then have:

$$\mathcal{G}_{ff}(\mathbf{q}) \approx \frac{k_B T}{A \kappa q^4} \tag{2.32}$$

where the stress term is insignificant to the scaling of the flexural correlation function. And for the in-plane phonons we have:

$$\mathcal{G}_{u_i u_j} = \frac{k_B T}{A}[C_{ikjl} q_k q_l]^{-1} \tag{2.33}$$

where the C_{ijkl} are also the bare parameters of the theory.

Assuming a vanishing stress such that $q_\sigma \ll q_{\text{th}} < q$, one can ask at what q_{th} these harmonic approximations are no longer appropriate to use and thus an-harmonic terms play an important role. To do this, it is necessary to resort to the β equations presented in Sec. 2.3.2 and use the bare values κ, λ, μ. Specifically we can examine Eq. (2.30) and look at what re-scaled UV cutoff Λ_{th} the an-harmonic contribution is of the order of κ. Assuming isotropy and ignoring infinitesimal stresses, we have [56]:

$$\kappa \approx \frac{4(D^2-1)k_B T \mu(\lambda+\mu) S_D \Lambda_{\text{th}}^{D-4}}{(\lambda+2\mu)(D^2+2D)\kappa(2\pi)^D} \tag{2.34}$$

where S_D is the area of a unit D-dimensional sphere. Therefore we obtain:

$$q_{\text{th}} \equiv \Lambda_{\text{th}} = \left(4\frac{(D^2-1)k_B T \mu(\lambda+\mu) S_D}{(\lambda+2\mu)(D^2+2D)(2\pi)^D \kappa^2}\right)^{\frac{1}{4-D}} \tag{2.35}$$

Likewise, assuming very small temperatures such that $q_{\text{th}} \ll q_\sigma < q$, one can similarly ask at what q_σ the Gaussian theory breaks down due to stress. The answer to this is already in our derivation of Eq. (2.14) for q_σ. When $q_{\text{th}} < q < q_\sigma$, the dominant terms in the harmonic theory will have to be re-examined, which will be done in the following Sec. 2.3.4.

Scaling At High Temperature $q_{\text{th}} > q > q_\sigma$

Having obtained the engineering dimensions we can make certain inferences of the behavior of these elastic sheets beyond q_{th}. Indeed when $q > q_\sigma$, the engineering dimensions seen in Table 2.1 indicate that all the in-plane elastic moduli have positive engineering dimensions and must thus be considered in the theory. Indeed these are the same engineering dimensions as were found in [13]. One can then perform an SCSA or renormalization group analysis [19, 13, 11] to obtain the anomalous exponents assuming negligible stress. Thus, the non-trivial scaling analysis of stress-free thermally fluctuating membranes will hold [28]. Thus, assuming a large separation of the three Fourier scales and $q_\sigma \ll q \ll q_{\text{th}}$ and $D < 4$, we expect to observe the anomalous thermalized exponents η, η_u since the bending modes are dominant and the stress term is not significant. Thus we have:

$$
\begin{aligned}
\mathcal{G}_{ff}(\mathbf{q}) =& \frac{k_B T}{A[B^R_{ijkl}(\mathbf{q}) q_i q_j q_k q_l + \sigma_{\alpha\alpha} q_\alpha^2]} \\
\approx& \frac{k_B T}{A[B^R_{ijkl}(\mathbf{q}) q_i q_j q_k q_l]}
\end{aligned}
\tag{2.36}
$$

where $B^R_{ijkl}(\mathbf{q})$ can be taken to be the isotropic $\kappa_R(\mathbf{q})$ assuming the material is isotropic at $T = 0$ and thus $\kappa_R(\mathbf{q}) \sim (q/q_{\text{th}})^{-\eta}$. Stress is still not significant in its contributions to the flexural correlation functions and for this reason it can be Taylor expanded. Analogously,

$$
\mathcal{G}_{u_i u_j} = \frac{k_B T}{A} [C^R_{ikjl}(\mathbf{q}) q_k q_l]^{-1} \tag{2.37}
$$

and $C^R_{ikjl}(\mathbf{q}) \sim (q/q_{\text{th}})^{\eta_u}$. The scaling of the elastic moduli can be observed in Table 2.3.

Assuming, $q_\sigma < q < q_{\text{th}}$ we can ask ourselves once more, up to what q_σ will this non-trivial scaling hold. To do this we merely repeat the steps indicated to derive Eq. (2.13). Thus we are aware of what is the range of this scaling regime in the presence of small stresses and must always be sure to use the anomalous exponents η, η_u only when $q_\sigma < q < q_{th}$.

2.3.4 Scaling At High Stress $q < q_\sigma$

Scaling At Low Temperature $q_{\text{th}} < q < q_\sigma$

The stress length scale of this regime, establishing one of the bounds, is once again given by Eq. (2.14). Since we are considering the case of low temperature, the renormalizing effect of an-harmonic terms can be ignored. Since no anomalous behaviors are expected, the flexural correlation function when stress is "uni-axial" is easily written down as:

$$\mathcal{G}_{ff}(\mathbf{q}) \approx \frac{k_B T}{A[B_{DDDD} q_D^4 + \sigma_{\alpha\alpha} q_\alpha^2]} \tag{2.38}$$

where B_{DDDD} is the $T = 0$ value (which is once again κ for an isotropic material). Similarly, the in-plane phonon correlation functions should not show any anomalous behavior and we should observe:

$$\mathcal{G}_{u_i u_j}(\mathbf{q}) = \frac{k_B T}{A}[C_{ikjl} q_k q_l]^{-1} \tag{2.39}$$

with the $T = 0$ parameters of the theory being used. These scaling laws can be observed in Table 2.4.

In the high stress case, we can once again ask where is the breakdown of the harmonic theory, in other words the new value of q_{th}. It need not be the same as the formula for the low stress case in Eq. (2.35) due to stress now being significant. However, for $D > 2$, there is no breakdown of the harmonic theory since all an-harmonic terms in the free energy are irrelevant. A length scale where the harmonic theory breaks down can be evaluated for $D = 2$ for which $C_{DDDD} = C_{2222}$ is a relevant parameter. Thus we will calculate the value of q_{th} explicitly in the case of $D = 2$.

From the engineering dimensions, we understood that even beyond q_{th}, the flexural phonons should not show any anomalous behavior since $B^R_{DDDD} = B^R_{2222}$ will remain a constant. Indeed the only relevant parameter that can show anomalous behavior is $C^R_{DDDD} = C^R_{2222}$. Thus we repeat our derivation of q_{th} in a similar manner as the derivation of Eq. (2.35), except we use the β equation of C^R_{2222} given in Eq. (3.5):

$$\partial_s C^R_{2222} = 2C^R_{2222} - \frac{k_B T}{2(2\pi\Lambda_{th})^2} \int_0^{2\pi} d\theta \frac{[C^R_{2222}\sin^2\theta + C^R_{1122}\cos^2\theta]^2}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l + \frac{\sigma_{11}}{\Lambda^2_{th}}\cos^2\theta]^2} \tag{2.40}$$

where $\Delta_f = 1$. Anomalous effects cannot be ignored once Λ takes on a value such that the two terms on the right hand side are of the same order. Here we have once again named this inverse length scale as Λ_{th}. In other words we are interested to know at what scale the anomalous contribution becomes significant with respect to the linear term in the β equation:

$$2C^R_{2222} \approx \frac{k_B T}{2(2\pi\Lambda_{th})^2} \int_0^{2\pi} d\theta \frac{[C^R_{2222}\sin^2\theta + C^R_{1122}\cos^2\theta]^2}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l + \frac{\sigma_{11}}{\Lambda^2_{th}}\cos^2\theta]^2} \tag{2.41}$$

We can approximately rewrite the denominator with the bare parameters as:

$$B^R_{ijkl}\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l + \frac{\sigma_{11}}{\Lambda^2_{\text{th}}}\cos^2\theta \longrightarrow \kappa + \frac{\sigma_{11}}{\Lambda^2_{\text{th}}}\cos^2\theta \tag{2.42}$$

where we set $\kappa = B_{1111} = B_{2222} = B_{1122}$ as the bare isotropic bending rigidity. This is justified since an-harmonic contributions are insignificant and thus the bare bending rigidities can be used. Replacing C^R_{1122} and C^R_{2222} by their bare isotropic values λ, $\lambda+2\mu$ respectively, one can then perform the integral identity holds when q is less than:

$$\Lambda_{\text{th}} \equiv q_{\text{th}} = \frac{k_B T(\lambda + 2\mu)}{16\pi\sqrt{\kappa^3\sigma}} \tag{2.43}$$

Thus in the large stress limit we have a different form for q_{th} which still matches with the low stress limit formula, Eq. 2.35, for q_{th} when $q_{\text{th}} \approx q_\sigma$ and $D = 2$.

Scaling At High Temperature $q < \min\{q_\sigma, q_{\text{th}}\}$ for $D = 2$

As was shown in Sec. 2.3.1, we know that below q_σ, B^R_{DDDD} should not be anomalous and should be a constant at some finite value, we can use the SCSA equations to obtain the scaling behavior of the moduli in the long wavelength limit when $q < \min\{q_\sigma, q_{\text{th}}\}$. Such an analysis is done in [66] for tubules as well for general D.

However, we again restrict our focus to the scaling analysis of the elastic moduli in strictly $D = 2$. This is not only the physically interesting case but also the least trivial due to the relevance of C_{2222} for $D = 2$ (whereas for larger dimensions all C_{ijkl}, including C_{DDDD}, become irrelevant as $q \to 0$). Furthermore, the analysis of the SCSA equations will be technically clearer in $D = 2$.

We can now proceed to the SCSA using the same idea of removing irrelevant an-harmonic coefficients. We shall further assume that we have integrated all high-frequency modes with $q > \min\{q_\sigma, q_{\text{th}}\}$ and thus all un-integrated wave-vectors in the following analysis satisfy $q < \min\{q_\sigma, q_{\text{th}}\}$.

Before beginning the analysis we note once again from Table 2.2 that when $q < \min\{q_\sigma, q_{\text{th}}\}$ all bending rigidities except for B_{2222} are irrelevant. In addition B^R_{2222} becomes a constant so if we define $q_{\min} \equiv \min\{q_\sigma, q_{\text{th}}\}$ then we can approximate the correlation function for $q < q_{\min}$ as:

$$
\begin{aligned}
\mathcal{G}^R_{ff}(\mathbf{q}) &= \frac{k_B T}{A[B^R_{ijkl}(\mathbf{q}) q_i q_j q_k q_l + \sigma_{11} q_1^2]} \\
&\approx \frac{k_B T}{A[B^R_{2222}(\mathbf{q}) q_2^4 + \sigma_{11} q_1^2]} \\
&\approx \frac{k_B T}{A[B^R_{2222}(q_{\min}) q_2^4 + \sigma_{11} q_1^2]}
\end{aligned}
\tag{2.44}
$$

As a final preliminary step we use a one loop SCSA analysis to obtain an-harmonic corrections to the elastic moduli resulting in Eq. (2.27). We will divide this equation by $C_{ijkl} C^R_{ijkl}(\mathbf{q})$ and use the Eq. 2.44 as the correlation functions leading to:

$$\begin{aligned}\frac{1}{C_{ijkl}} =&\frac{1}{C^R_{ijkl}(\mathbf{q})} \\ &-\frac{k_BT}{4(2\pi)^2}\int_{|\mathbf{p}|<q_{\min}} dp_1dp_2 \\ &\frac{[C^R_{ijmn}(\mathbf{q})(q_m-p_m)p_n][C_{abkl}(q_a-p_a)p_b]}{C_{ijkl}C^R_{ijkl}(\mathbf{q})[B^R_{2222}(q_{\min})(p_2-q_2)^4+\sigma_{11}(p_1-q_1)^2][B^R_{2222}(q_{\min})p_2^4+\sigma_{11}p_1^2]} \\ &-\frac{k_BT}{4(2\pi)^2}\int_{|\mathbf{p}|<q_{\min}} dp_1dp_2 \\ &\frac{[C_{ijmn}(q_m-p_m)p_n][C^R_{abkl}(\mathbf{q})(q_a-p_a)p_b]}{C_{ijkl}C^R_{ijkl}(\mathbf{q})[B^R_{2222}(q_{\min})(p_2-q_2)^4+\sigma_{11}(p_1-q_1)^2][B^R_{2222}(q_{\min})p_2^4+\sigma_{11}p_1^2]},\end{aligned} \tag{2.45}$$

where $\mathbf{p} = (p_1, p_2)$. In the analysis we will drop the bounds of the integral and take it as a given that $|\mathbf{p}| < q_{\min}$. We can now examine the scaling behavior of the elastic moduli as $q \to 0$. However, before beginning, it should be stated that though these results are derived from a one-loop SCSA, that the scalings for the elastic moduli in the rest of this section will be correct to all loops [66].

Scaling Behavior of C_{2222}. For C_{2222}, the corresponding self-consistent perturbative equation is given by Eq. (2.45) as:

$$\begin{aligned}\frac{1}{C_{2222}} =&\frac{1}{C^R_{2222}(\mathbf{q})} - \frac{k_BT}{2(2\pi)^2}\int dp_1dp_2 \\ &\frac{[(p_2-q_2)p_2+\frac{C^R_{1122}(\mathbf{q})}{C^R_{2222}(\mathbf{q})}(p_1-q_1)p_1][(p_2-q_2)p_2+\frac{C_{1122}}{C_{2222}}(p_1-q_1)p_1]}{[B^R_{2222}(q_{\min})p_2^4+\sigma_{11}p_1^2][B^R_{2222}(q_{\min})(p_2-q_2)^4+\sigma_{11}(p_1-q_1)^2]}\end{aligned} \tag{2.46}$$

where C_{2222} and C_{1122} are the bare un-renormalized moduli. We examine Eq. (2.46) in the long wavelength limit when $q \to 0$, where $q < \min\{q_\sigma, q_{\text{th}}\}$. From Table 2.2 we know that $q_1 \sim q_2^2$ and $C_{1111}, C_{1122}, C_{1212}, B_{1111}, B_{1122}$ are all irrelevant.

We can then extract powers of q_2:

$$\frac{1}{C_{2222}} = \frac{1}{C^R_{2222}(\mathbf{q})} - \frac{k_B T}{2(2\pi)^2} \int d\tilde{p}_1 d\tilde{p}_2\ q_2^3 \frac{[\frac{1}{q_2^2}(\tilde{p}_2 - 1)\tilde{p}_2 + \frac{C^R_{1122}(\mathbf{q})}{C^R_{2222}(\mathbf{q})}(\tilde{p}_1 - \tilde{q}_1)\tilde{p}_1][\frac{1}{q_2^2}(\tilde{p}_2 - 1)\tilde{p}_2 + \frac{C_{1122}}{C_{2222}}(\tilde{p}_1 - \tilde{q}_1)\tilde{p}_1]}{[B^R_{2222}(q_{\min})\tilde{p}_2^4 + \sigma_{11}\tilde{p}_1^2][B^R_{2222}(q_{\min})(\tilde{p}_2 - 1)^4 + \sigma_{11}(\tilde{p}_1 - \tilde{q}_1)^2]} \tag{2.47}$$

where $\tilde{p}_2 = p_2/q_2$, $\tilde{p}_1 = p_1/q_2^2$ and $\tilde{q}_1 = q_1/q_2^2$. We must collect the most divergent terms as $q_2 \to 0$ since the left hand side of the equation is a constant. Since C_{1122} is irrelevant we may also remove this term from the numerator and thus obtain the following equation:

$$\frac{1}{C_{2222}} = \frac{1}{C^R_{2222}(\mathbf{q})} - \frac{k_B T}{2(2\pi)^2} \frac{1}{q_2} \int d\tilde{p}_1 d\tilde{p}_2 \frac{(\tilde{p}_2 - 1)^2 \tilde{p}_2^2}{[B^R_{2222}(q_{\min})\tilde{p}_2^4 + \sigma_{11}\tilde{p}_1^2][B^R_{2222}(q_{\min})(\tilde{p}_2 - 1)^4 + \sigma_{11}(\tilde{p}_1 - \tilde{q}_1)^2]} \tag{2.48}$$

In the above equation, one can easily see that the integral is a homogeneous function of $\tilde{q}_1$, thus the self-consistent equation is solved by the ansatz:

$$C^R_{2222}(\mathbf{q}) \approx C^R_{2222}(q_{\min}) \frac{q_2}{q_{\min}} \Omega_C^2(1/\sqrt{\tilde{q}_1}) \tag{2.49}$$

where $\Omega_C^2(1/\sqrt{\sqrt{q}_1})$ is a universal scaling function that is a constant when $\tilde{q}_1 \to 0$. The pre-factor $C^R_{2222}(q_{\min})q_2/q_{\min}$ is meant to ensure that the correlation functions for $q < q_{\min}$ and $q > q_{\min}$ transition smoothly at $q_{\min}$. The full form of Ω_C^2 can be determined from the fact that $C^R_{2222}(\mathbf{q})$ should be independent of q_1 when $q_1 \to 0$ and it should be independent of q_2 when $q_2 \to 0$. Thus:

$$\Omega_C^2(s) \sim \begin{cases} \text{constant} & s \to \infty \\ s^{-1} & s \to 0 \end{cases} \tag{2.50}$$

Assembling the pieces together we obtain:

$$C_{2222}^R(\mathbf{q}) \sim \begin{cases} q_2 & q_2 \gg \sqrt{q_1} \\ \sqrt{q_1} & q_2 \ll \sqrt{q_1} \end{cases} \tag{2.51}$$

Therefore, despite the fact that the effective theory in Eq. 2.24 does not possess anomalous scaling, the full theory with in-plane phonons does have anomalous exponents due to thermal fluctuations. The significance of this intuitively is that sinusoidal waves can form transverse to the axis of stress and are not flattened out by it.

In addition, this result will be correct to all loops since one can check that at higher orders, the leading contribution to the SCSA equation of C_{2222}^R at every order is $q_2\Omega_C^2(1/\sqrt{\tilde{q}_1})$ [66].

Scaling Behavior of C_{1122} and C_{1111}. Although the remaining moduli are irrelevant, we can repeat this same analysis to obtain how they scale. We can check for example how C_{1122}^R and C_{1111}^R should scale.

We can use Eq. (2.45) and keep in mind that if we look at Table 2.2, we see that C_{1111}^R is more irrelevant with respect to C_{1122}^R and thus we can omit the C_{1111} contributions to the SCSA equation of C_{1122}^R. This will result in:

$$\begin{aligned} \frac{1}{C_{1122}} = \frac{1}{C_{1122}^R(\mathbf{q})} - \frac{k_BT}{4(2\pi)^2}\int dp_1 dp_2 \\ \Bigg[\frac{[(p_2-q_2)p_2 + \frac{C_{1111}}{C_{1122}}(p_1-q_1)p_1][(p_1-q_1)p_1 + \frac{C_{2222}^R(\mathbf{q})}{C_{1122}^R(\mathbf{q})}(p_2-q_2)p_2]}{[B_{2222}^R(q_{\min})p_2^4 + \sigma_{11}p_1^2][B_{2222}^R(q_{\min})(p_2-q_2)^4 + \sigma_{11}(p_1-q_1)^2]} \\ + \frac{[(p_2-q_2)p_2 + \frac{C_{1111}^R(\mathbf{q})}{C_{1122}^R(\mathbf{q})}(p_1-q_1)p_1][(p_1-q_1)p_1 + \frac{C_{2222}}{C_{1122}}(p_2-q_2)p_2]}{[B_{2222}^R(q_{\min})p_2^4 + \sigma_{11}p_1^2][B_{2222}^R(p_2-q_2)^4 + \sigma_{11}(p_1-q_1)^2]}\Bigg] \end{aligned} \tag{2.52}$$

and

$$\frac{1}{C_{1111}} = \frac{1}{C^R_{1111}(\mathbf{q})} - \frac{k_B T}{2(2\pi)^2} \int dp_1 dp_2 \frac{[\frac{C^R_{1122}(\mathbf{q})}{C^R_{1111}(\mathbf{q})}(p_2 - q_2)p_2 + (p_1 - q_1)p_1][\frac{C_{1122}}{C_{1111}}(p_2 - q_2)p_2 + (p_1 - q_1)p_1]}{[B^R_{2222}(q_{\min})(p_2 - q_2)^4 + \sigma_{11}(p_1 - q_1)^2][B^R_{2222}(q_{\min})p_2^4 + \sigma_{11}p_1^2]} \tag{2.53}$$

Similarly as before we can extract powers of q_2, keeping in mind also that $C^R_{2222}(\mathbf{q}) \sim q_2$. Thus giving:

$$\begin{aligned} \frac{1}{C_{1122}} = \frac{1}{C^R_{1122}(\mathbf{q})} - \frac{k_B T}{4(2\pi)^2} \int \frac{d\tilde{p}_1 d\tilde{p}_2}{q_2^5} \\ \Bigg[\frac{[(\tilde{p}_2 - \tilde{q}_2)\tilde{p}_2 q_2^2 + \frac{C_{1111}}{C_{1122}}(\tilde{p}_1 - \tilde{q}_1)\tilde{p}_1 q_2^4][\tilde{p}_1(\tilde{p}_1 - \tilde{q}_1)q_2^4 + \frac{C^R_{2222}(q_{\min})\Omega^2_C(\frac{1}{\sqrt{\tilde{q}_1}})}{q_{\min} C^R_{1122}(\mathbf{q})}(\tilde{p}_2 - 1)\tilde{p}_2 q_2^3]}{[B^R_{2222}(q_{\min})\tilde{p}_2^4 + \sigma_{11}\tilde{p}_1^2][B^R_{2222}(q_{\min})(\tilde{p}_2 - 1)^4 + \sigma_{11}(\tilde{p}_1 - \tilde{q}_1)^2]} \\ + \frac{[(\tilde{p}_2 - \tilde{q}_2)\tilde{p}_2 q_2^2 + \frac{C^R_{1111}(\mathbf{q})}{C^R_{1122}(\mathbf{q})}(\tilde{p}_1 - \tilde{q}_1)\tilde{p}_1 q_2^4][(\tilde{p}_1 - \tilde{q}_1)\tilde{p}_1 q_2^4 + \frac{C_{2222}}{C_{1122}}(\tilde{p}_2 - 1)\tilde{p}_2 q_2^2]}{[B^R_{2222}(q_{\min})\tilde{p}_2^4 + \sigma_{11}\tilde{p}_1^2][B^R_{2222}(q_{\min})(\tilde{p}_2 - 1)^4 + \sigma_{11}(\tilde{p}_1 - \tilde{q}_1)^2]}\Bigg] \end{aligned} \tag{2.54}$$

and

$$\frac{1}{C_{1111}} = \frac{1}{C^R_{1111}(\mathbf{q})} - \frac{k_B T}{2(2\pi)^2} \int \frac{d\tilde{p}_1 d\tilde{p}_2}{q_2^5} \frac{[\frac{C^R_{1122}(\mathbf{q})}{C^R_{1111}(\mathbf{q})}(\tilde{p}_2 - 1)\tilde{p}_2 q_2^2 + (\tilde{p}_1 - \tilde{q}_1)\tilde{p}_1 q_2^4][\frac{C_{1122}}{C_{1111}}(\tilde{p}_2 - 1)\tilde{p}_2 q_2^2 + (\tilde{p}_1 - \tilde{q}_1)\tilde{p}_1 q_2^4]}{[B^R_{2222}(q_{\min})(\tilde{p}_2 - 1)^4 + \sigma_{11}(\tilde{p}_1 - \tilde{q}_1)^2][B^R_{2222}(q_{\min})\tilde{p}_2^4 + \sigma_{11}\tilde{p}_1^2]} \tag{2.55}$$

By only paying attention to the most divergent powers of q_2 (as $q_2 \to 0$) and using the fact that the engineering dimension of $\Delta_{C_{1111}} < \Delta_{C_{1122}}$ so that we may ignore the $\frac{C^R_{1111}(\mathbf{q})}{C^R_{1122}(\mathbf{q})}$ term in Eq. 2.54, we may replace these finite integrals with the symbols $I^{(i)}$ and write that:

$$\frac{1}{C_{1122}} = \frac{1}{C^R_{1122}(\mathbf{q})} + q_2 I^{(1)}_{1122} + \frac{1}{C^R_{1122}(\mathbf{q})} I^{(2)}_{1122} + \frac{1}{q_2} I^{(3)}_{1122} \tag{2.56}$$

and

$$\begin{aligned}\frac{1}{C_{1111}} =& \frac{1}{C^R_{1111}(\mathbf{q})} + \frac{C^R_{1122}(\mathbf{q})}{C^R_{1111}(\mathbf{q})}\frac{1}{q_2}I^{(1)}_{1111} + \frac{C^R_{1122}(\mathbf{q})}{C^R_{1111}(\mathbf{q})}q_2 I^{(2)}_{1111} \\ &+ q_2 I^{(3)}_{1111} + q_2^3 I^{(4)}_{1111}\end{aligned} \tag{2.57}$$

where the integrals $I^{(i)}_{1122}$ and $I^{(i)}_{1111}$ can be found in the appendix in Sec. 2.6.3. Collecting the most divergent powers (divergent as $q_2 \to 0$) in each equation gives:

$$\frac{1}{C_{1122}} \approx \frac{1}{C^R_{1122}(\mathbf{q})}(1 + I^{(2)}_{1122}) + \frac{1}{q_2}I^{(3)}_{1122} \tag{2.58}$$

$$\frac{1}{C_{1111}} \approx \frac{1}{C^R_{1111}(\mathbf{q})} + \frac{C^R_{1122}(\mathbf{q})}{C^R_{1111}(\mathbf{q})}\frac{1}{q_2}I^{(1)}_{1111} \tag{2.59}$$

Eq. (2.58) directly shows that it can be solved by the ansatz:

$$C^R_{1122}(\mathbf{q}) \approx C^R_{1122}(q_{\min})\frac{q_2}{q_{\min}}\Omega^1_C(1/\sqrt{q\tilde{q}_1}) \tag{2.60}$$

where we have once again a pre-factor that ensures the the correlation functions for $q < q_{\min}$ and $q > q_{\min}$ match. And the homogeneous function has the following scaling:

$$\Omega^1_C(s) \sim \begin{cases} \text{constant} & s \to \infty \\ s^{-1} & s \to 0 \end{cases} \tag{2.61}$$

and hence, $C_{1122}(\mathbf{q}) \sim q_2$ and thus that:

$$C^R_{1122}(\mathbf{q}) \sim \begin{cases} q_2 & q_2 \gg \sqrt{q_1} \\ \sqrt{q_1} & q_2 \ll \sqrt{q_1} \end{cases} \tag{2.62}$$

When we insert this ansatz into Eq. (2.59) we obtain also that C^R_{1111} becomes a constant and thus must be approximately $C^R_{1111}(q_{\min})$. These can be found in Tables 2.3 and 2.4.

Scaling Behavior of C_{1212}. Lastly, we can check how the shear modulus should scale via its corresponding self-consistent equation:

$$\begin{aligned}\frac{1}{C_{1212}} =& \frac{1}{C^R_{1212}(\mathbf{q})} \\ &- \frac{2k_BT}{(2\pi)^2}\int dp_1 dp_2 \frac{p_1p_2}{[B^R_{2222}(q_{\min})p_2^4 + \sigma_{11}p_1^2]} \\ &\quad \times \frac{(p_1-q_1)(p_2-q_2)}{[B^R_{2222}(q_{\min})(p_2-q_2)^4 + \sigma_{11}(p_1-q_1)^2]}.\end{aligned} \tag{2.63}$$

Repeating a similar analysis as above gives that all contributions of the integral are irrelevant in the limit that $q_2 \to 0$. Hence $C^R_{1212}(\mathbf{q})$ becomes a constant below $\min\{q_\sigma, q_{\text{th}}\}$.

Scaling Behavior of B_{1122} and B_{1111}. Whereas for C^R_{ijkl}, we could conduct a scaling analysis corresponding to SCSA equations for the anharmonic f^4 interaction, B^R_{1111} and B^R_{1122} are coefficients of harmonic terms and will be masked by the stress in the correlation function, $\mathcal{G}_{ff}$, when $q < q_\sigma$.

With our theoretical results we now move on to verify the scaling of this theory via simulations that measure the in-plane and flexural correlation functions in these regimes.

2.3.5 Discussion of Correlation functions from Simulations

The scaling of these moduli should be reflected in the correlation functions in Eqs. (2.11) and (2.12). Molecular dynamics simulations of square-shape systems with spring-mass system arranged in a triangular lattice. Two system sizes were used, one with 2900 masses (amounting to a square sheet of size $50a \times 50a$ where a

Scaling Exponents $q_{\text{th}} > q_\sigma, \sigma_{11} > 0, \sigma_{22} = 0$			
Scale	$q > q_{\text{th}}$	$q_{\text{th}} > q > q_\sigma$	$q_\sigma > q$
C^R_{1111}/C_{1111}	1	$\left(\frac{q}{q_{\text{th}}}\right)^{\eta_u}$	$\left(\frac{q_\sigma}{q_{\text{th}}}\right)^{\eta_u}$
C^R_{1212}/C_{1212}	1	$\left(\frac{q}{q_{\text{th}}}\right)^{\eta_u}$	$\left(\frac{q_\sigma}{q_{\text{th}}}\right)^{\eta_u}$
C^R_{1122}/C_{1122}	1	$\left(\frac{q}{q_{\text{th}}}\right)^{\eta_u}$	$\left(\frac{q_\sigma}{q_{\text{th}}}\right)^{\eta_u} \frac{q_2}{q_\sigma}\Omega^1_C(1/\sqrt{\tilde{q}_1})$
C^R_{2222}/C_{2222}	1	$\left(\frac{q}{q_{\text{th}}}\right)^{\eta_u}$	$\left(\frac{q_\sigma}{q_{\text{th}}}\right)^{\eta_u} \frac{1}{\tilde{q}_\sigma}\Omega^2_C(1/\sqrt{\tilde{q}_1})$
B^R_{1111}/B_{1111}	1	$\left(\frac{q}{q_{\text{th}}}\right)^{-\eta}$	Masked
B^R_{1122}/B_{1122}	1	$\left(\frac{q}{q_{\text{th}}}\right)^{-\eta}$	Masked
B^R_{2222}/B_{2222}	1	$\left(\frac{q}{q_{\text{th}}}\right)^{-\eta}$	$\left(\frac{q_\sigma}{q_{\text{th}}}\right)^{-\eta}$

Table 2.3: The scaling of the elastic moduli is shown when stress is small enough such that $q_{\text{th}} > q_\sigma$. When $q_\sigma > q$, the bending rigidities $B^R_{1111}q_1^4$ and $B^R_{1122}q_1^2q_2^2$ are dominated by the stress term $\sigma_{11}q_1^2$ in Eq. 2.11 and we term them as masked.

Scaling Exponents $q_\sigma > q_{\text{th}}, \sigma_{11} > 0, \sigma_{22} = 0$			
Scale	$q > q_\sigma$	$q_\sigma > q > q_{\text{th}}$	$q_{\text{th}} > q$
C^R_{1111}/C_{1111}	1	1	1
C^R_{1212}/C_{1212}	1	1	1
C^R_{1122}/C_{1122}	1	1	$\frac{q_2}{q_{\text{th}}}\Omega^1_C(1/\sqrt{\tilde{q}_1})$
C^R_{2222}/C_{2222}	1	1	$\frac{q_2}{q_{\text{th}}}\Omega^2_C(1/\sqrt{\tilde{q}_1})$
B^R_{1111}/B_{1111}	1	Masked	Masked
B^R_{1122}/B_{1122}	1	Masked	Masked
B^R_{2222}/B_{2222}	1	1	1

Table 2.4: The scaling of the elastic moduli is shown when stress is large enough such that $q_{\text{th}} < q_\sigma$. When $q_\sigma > q$, the bending rigidities $B^R_{1111}q_1^4$ and $B^R_{1122}q_1^2q_2^2$ are dominated by the stress term $\sigma_{11}q_1^2$ in Eq. 2.11 and we term them as masked.

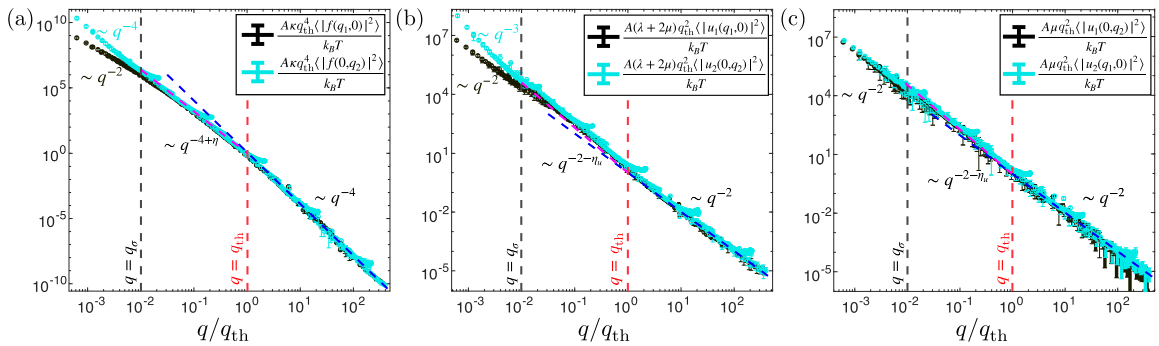

Figure 2.3: Displayed are some simulation results for the (a) flexural correlation functions (b) orthogonal in-plane correlation functions and (c) transverse in-plane correlation function along orthogonal axes. Plots show the changes in these correlation function at the thermal transition, $q = q_{\rm th}$, and when stress becomes dominant, $q = q_\sigma$. Blue dashed lines show the continuation of the harmonic scaling to aid seeing the change in slope when $q = q_{\rm th}$. In (a) and (b), the anisotropy of the correlation functions can be observed when $q < q_\sigma$. The magenta lines show the anomalous thermal exponents η, η_u when $q_{\rm th} > q > q_\sigma$. As one can see in (c), the transverse shear modes are always of the same stiffness.

is the lattice spacing) and 11600 masses ($100a \times 100a$) to show that finite size effects in the correlation functions are negligible. Further details of the simulations can be found in the appendix in Sec. 2.6.1. What gives the dimensional sense of system size are the parameters such as bending rigidity, Young's modulus and temperature, all of which enter into the formula for $q_{\rm th}$. To make this clearer, in the low-stress limit:

$$q_{\rm th} \sim \sqrt{\frac{k_B T Y}{\kappa^2}} \tag{2.64}$$

where $k_B T, \kappa$ have units of energy but Y has units of energy/m^2. Understanding this, the temperature, bending rigidity and Young's modulus were varied in order to piece together data from simulations across a large scale change, which allowed us to be more computationally effective. Specific parameters can be obtained in the appendix. Simulations were only done in the low stress limit. This is because replication of similar results in the large stress case were rendered difficult to obtain due to the non-linear responses of the lattice of springs as well.

Looking at Fig. 2.3, all simulations had a stress value such that $q_\sigma/q_{\text{th}} = 10^{-2}$ (using Eq. (2.13)) while κ, Y, T were varied independently. The Fig. 2.3(a) shows the transition from the harmonic regime to an anomalous thermally renormalized regime where the bending rigidities diverge isotropically with exponent $\eta \approx .8$. At q_σ a second transition can be observed from the isotropic anomalous exponents η, η_u to a regime where anisotropies develop and the scaling takes the form in Table 2.3. In-plane phonon correlation functions associated with normal strains are plotted in Fig. 2.3(b), using the same simulations. Similarly, they show the scaling expected from the theory with a strong anisotropy that develops below q_σ. Finally, we also observed the isotropy of the shear modulus in Fig. 2.3(c) which also matched the scaling we found via our SCSA equations. The shear modulus ceases to renormalize once stress becomes relevant.

Having confirmed our theoretical results with simulations we can now move on to measuring the stress-strain theory that follow from this scaling theory.

2.4 Simulations of Stress-Strain and Poisson's Ratio

The stress-strain relationship of thermalized 2D sheets of dimensions $L \times L$ under uni-axial stress along axis 1 can be theoretically calculated:

$$\left\langle \frac{\delta L_1}{L} \right\rangle_\sigma \approx \frac{\sigma_{11}}{Y} - \frac{1}{2} \sum_{\frac{2\pi}{L} < |\mathbf{q}| < \Lambda} q_1^2 \mathcal{G}_{ff}^R(\mathbf{q}) \tag{2.65}$$

where L is the system size, Λ is again the UV cutoff and δL_1 is the change in length along axis 1 [28]. The first term in the equation reflects the bare response of the material whereas the second term involves the effect of temperature. It is this latter term that gives rise to the tendency of elastic membranes to shrink [28, 74, 46].

Similarly, strains along the axis orthogonal to the stress can be calculated as:

$$\left\langle \frac{\delta L_2}{L} \right\rangle_\sigma \approx \frac{-\nu\sigma_{11}}{Y} - \frac{1}{2} \sum_{\frac{2\pi}{L}<|\mathbf{q}|<\Lambda} q_2^2 \mathcal{G}_{ff}^R(\mathbf{q}) \tag{2.66}$$

where ν is the bare Poisson ratio, in our case $+1/3$ for a triangular lattice. δL_2 is the change of system length along axis 2. The strains are then defined as:

$$\begin{aligned} \epsilon_{11} &= \left\langle \frac{\delta L_1}{L} \right\rangle_\sigma - \left\langle \frac{\delta L_1}{L} \right\rangle_0 \\ \epsilon_{22} &= \left\langle \frac{\delta L_2}{L} \right\rangle_\sigma - \left\langle \frac{\delta L_2}{L} \right\rangle_0 \end{aligned} \tag{2.67}$$

where the subtracted terms express the reference system size in the absence of stress. These terms are necessary to subtract in order to obtain a strain from the un-stressed state where thermal fluctuations naturally induce a shrinking of the membrane. By plugging in our theoretical scaling ansatz for the correlation functions, found in Table 2.5, we can analytically calculate the stress-strain relation. Typically, for real materials such as graphene at room temperature, $a < \ell_{\text{th}} < L$ ($\ell_{\text{th}} \approx 2$nm at 300K). However, since we can only effectively simulate system sizes of the order of $50a \times 50a$, we tuned parameters to generally obtain a large separation of length scales L/ℓ_{th}. We therefore examine the scaling of the stress-strain relation when $\ell_{\text{th}} < a < L$ ($2\pi/L < \Lambda < q_{\text{th}}$) with the stress length scale being variable. The scaling ansatz of the correlation function for the length scales that fall between a and L depends on the magnitude of stress and is shown in Table 2.5. We may show an example of how to obtain one of the scaling functions of $\epsilon_{11}, \epsilon_{22}$ observed in Table 2.5.

For example, in the case $\ell_{\text{th}} < a < \ell_\sigma < L(2\pi/L < q_\sigma < \Lambda < q_{\text{th}})$. Beginning with Eq. (2.67):

Expressions of $\mathcal{G}^R_{ff}$ for $2\pi/L < \Lambda < q_{\rm th}$				
Scale	$q_\sigma < 2\pi/L$	$2\pi/L < q_\sigma < \Lambda$	$\Lambda < q_\sigma < q_{\rm th}$	$q_{\rm th} < q_\sigma$
$q_\sigma < q$	$\frac{k_BT}{A[\kappa q_{\rm th}^\eta q^{4-\eta}+\sigma_{11}q_1^2]}$	$\frac{k_BT}{A[\kappa q_{\rm th}^\eta q^{4-\eta}+\sigma_{11}q_1^2]}$	NA	NA
$q < q_\sigma$	NA	$\frac{k_BT}{A[\kappa q_\sigma^{-\eta} q_{\rm th}^\eta q_2^4+\sigma_{11}q_1^2]}$	$\frac{k_BT}{A[\kappa q_\sigma^{-\eta} q_{\rm th}^\eta q_2^4+\sigma_{11}q_1^2]}$	$\frac{k_BT}{A[\kappa q_2^4+\sigma_{11}q_1^2]}$
Scaling of Strains				
Strain	$q_\sigma < 2\pi/L$	$2\pi/L < q_\sigma < \Lambda$	$\Lambda < q_\sigma < q_{\rm th}$	$q_{\rm th} < q_\sigma$
ϵ_{11}	$\sigma_{11}/(4(1-\eta)Y_R(L))$	$\sim \sigma_{11}^{\eta/(2-\eta)}$	transition to σ_{11}/Y	σ_{11}/Y
ϵ_{22}	$-\sigma_{11}/(12(1-\eta)Y_R(L))$	$\sim \sigma_{11}^{\eta/(2-\eta)}$	transition to $-\nu\sigma_{11}/Y$	$-\nu\sigma_{11}/Y$

Table 2.5: In this table we show the scaling of the flexural correlation functions derived from Sec.2.3. We then write down the corresponding stress-strain behaviors of the strains ϵ_{11} and ϵ_{22}. In the table $Y_R(L) = Y(2\pi/q_{\rm th}L)^{\eta_u}$.

$$
\begin{aligned}
&\epsilon_{11}(\sigma_{11}, T|2\pi/L < q_\sigma < \Lambda < q_{\rm th})\\
=&\frac{\sigma_{11}}{Y} - \frac{1}{2}\sum_{q_\sigma<|\mathbf{q}|<\Lambda} q_1^2\left[\frac{k_BT}{A[\kappa q_{\rm th}^\eta q^{4-\eta}+\sigma_{11}q_1^2]} - \frac{k_BT}{A\kappa q_{\rm th}^\eta q^{4-\eta}}\right]\\
&-\frac{1}{2}\sum_{\frac{2\pi}{L}<|\mathbf{q}|<q_\sigma} q_1^2\left[\frac{k_BT}{A[\kappa q_\sigma^{-\eta}q_{\rm th}^\eta q_2^4+\sigma_{11}q_1^2]} - \frac{k_BT}{A\kappa q_{\rm th}^\eta q^{4-\eta}}\right]\\
=&\frac{\sigma_{11}}{Y} - \frac{1}{2}\sum_{q_\sigma<|\mathbf{q}|<\Lambda} q_1^2\left[-\frac{k_BT\sigma_{11}q_1^2}{A\kappa^2 q_{\rm th}^{2\eta} q^{8-2\eta}}\right]\\
&-\frac{1}{2}\sum_{\frac{2\pi}{L}<|\mathbf{q}|<q_\sigma} q_1^2\left[\frac{k_BT}{A[\kappa q_\sigma^{-\eta}q_{\rm th}^\eta q_2^4+\sigma_{11}q_1^2]} - \frac{k_BT}{A\kappa q_{\rm th}^\eta q^{4-\eta}}\right]
\end{aligned}
\tag{2.68}
$$

and in the first summation we may Taylor expand the correlation function to first order (since for those wave vectors the stress term is not dominant in the denominator of the flexural correlation function). We can then convert these terms to integrals:

$$\epsilon_{11}(\sigma_{11}, T|2\pi/L < q_\sigma < \Lambda < q_{\mathrm{th}}) \approx$$
$$\frac{\sigma}{Y} - \frac{1}{2(2\pi)^2}\int_0^{2\pi} d\theta \int_{q_\sigma}^{\Lambda} dq q^3 \cos^2\theta \left[- \frac{k_B T \sigma_{11} q^2 \cos^2\theta}{\kappa^2 q_{\mathrm{th}}^{2\eta} q^{8-2\eta}} \right]$$
$$- \frac{1}{2(2\pi)^2}\int_0^{2\pi} d\theta \int_{2\pi/L}^{q_\sigma} dq q^3 \cos^2\theta \left[\frac{k_B T}{[\kappa q_\sigma^{-\eta} q_{\mathrm{th}}^{\eta} q^4 \sin^4\theta + \sigma_{11} q^2 \cos^2\theta]} - \frac{k_B T}{\kappa q_{\mathrm{th}}^{\eta} q^{4-\eta}} \right] \tag{2.69}$$

To make the latter integral tractable we approximate the denominator in the following manner:

$$\frac{k_B T}{A[\kappa q_\sigma^{-\eta} q_{\mathrm{th}}^{\eta} q^4 \sin^4\theta + \sigma_{11} q^2 \cos^2\theta]} \approx \frac{k_B T}{A[\kappa q_\sigma^{-\eta} q_{\mathrm{th}}^{\eta} q^4 + \sigma_{11} q^2 \cos^2\theta]} \tag{2.70}$$

This approximation is justified since the stress is dominant when $\theta \neq \pi/2$ in the domain of the integral, $q \in [2\pi/L, q_\sigma]$. These integrals can now be analytically integrated giving rise to:

$$\epsilon_{11}\left(\sigma_{11}, T \middle| \frac{2\pi}{L} < q_\sigma < \Lambda < q_{\mathrm{th}}\right) = \frac{\sigma_{11}}{Y} - \left[\frac{3 k_B T \sigma_{11}}{64\pi(1-\eta)\kappa^2 q_{\mathrm{th}}^{2\eta}} q^{2\eta-2} \right] \Bigg|_{q_\sigma}^{\Lambda}$$
$$- \frac{k_B T}{8\pi\sigma_{11}} \left[q^2 - q\sqrt{q^2 + \left(\frac{q_\sigma}{q_{\mathrm{th}}}\right)^{\eta} \frac{\sigma_{11}}{\kappa}} + \left(\frac{q_\sigma}{q_{\mathrm{th}}}\right)^{\eta} \frac{\sigma_{11}}{\kappa} \sinh^{-1}\left[q \left(\frac{q_\sigma}{q_{\mathrm{th}}}\right)^{-\eta/2} \sqrt{\frac{\kappa}{\sigma_{11}}} \right] \right] \Bigg|_{\frac{2\pi}{L}}^{q_\sigma}$$
$$+ \frac{k_B T}{8\pi\eta\kappa} \left(\frac{q}{q_{\mathrm{th}}}\right)^{\eta} \Bigg|_{\frac{2\pi}{L}}^{q_\sigma} \tag{2.71}$$

By taking the infinite system size limit (which is appropriate since we are also assuming large separation of length scales that $\frac{2\pi}{L}\left(\frac{q_\sigma}{q_{\mathrm{th}}}\right)^{-\eta/2}\sqrt{\frac{\kappa}{\sigma}} \ll 1$ when $2\pi/L < q_\sigma < \Lambda < q_{\mathrm{th}}$), we can obtain a simpler expression:

$$\begin{aligned}
&\lim_{L\to\infty} \epsilon_{11}\left(\sigma_{11}, T\middle|\frac{2\pi}{L} < q_\sigma < \Lambda < q_{\text{th}}\right) \\
&= \frac{\sigma_{11}}{Y}\left[1 - \frac{1}{2(1-\eta)}\left(\frac{\Lambda}{q_{\text{th}}}\right)^{2\eta-2}\right] \\
&- \frac{k_B T}{8\pi\kappa}\left(\frac{q_\sigma}{q_{\text{th}}}\right)^{\eta}\left[(1-\sqrt{2}) + \sinh^{-1}(1) - \eta^{-1} - \frac{3}{8(1-\eta)}\right] \\
&= \frac{\sigma_{11}}{Y}\left[1 - \frac{1}{2(1-\eta)}\left(\frac{\Lambda}{q_{\text{th}}}\right)^{2\eta-2}\right] \\
&- \frac{k_B T}{8\pi\kappa}\left(\frac{16\pi\sigma_{11}\kappa}{3k_BTY}\right)^{\frac{\eta}{2-\eta}}\left[(1-\sqrt{2}) + \sinh^{-1}(1) - \eta^{-1} - \frac{3}{8(1-\eta)}\right]
\end{aligned} \tag{2.72}$$

A similar calculation gives:

$$\begin{aligned}
&\lim_{L\to\infty} \epsilon_{22}\left(\sigma_{11}, T\middle|\frac{2\pi}{L} < q_\sigma < \Lambda < q_{\text{th}}\right) = \frac{\sigma_{11}}{Y}\left[-\nu - \frac{1}{6(1-\eta)}\left(\frac{\Lambda}{q_{\text{th}}}\right)^{2\eta-2}\right] \\
&- \frac{k_B T}{8\pi\kappa}\left(\frac{16\pi\sigma_{11}\kappa}{3k_BTY}\right)^{\frac{\eta}{2-\eta}}\left[(-1+\sqrt{2}) + \sinh^{-1}(1) - \eta^{-1} - \frac{1}{8(1-\eta)}\right]
\end{aligned} \tag{2.73}$$

The rest of the strains for other regimes can be obtained in a similar manner and are shown in Table 2.5 with explicit solutions in Sec. 2.6.4, and these scalings become more accurate with a large separation of length scales (in other words if $2\pi/L$, q_{th} and q_σ being all different orders of magnitude) [36].

Within Eqs. (2.72) and (2.73), pre-factors of each of the power laws can be compared using the fact that $\sigma_{q_{\text{th}}} \gg \sigma$ (where $\sigma_{q_{\text{th}}}$ is defined as the stress such that $q_\sigma = q_{\text{th}}$). The comparison shows that the last terms in each equation, which exhibit the scaling $\sigma^{\eta/(2-\eta)}$, is the dominant power law. Thus, when $2\pi/L < q_\sigma < \Lambda$ with $L \to \infty$ and holding stress fixed, a non-linear stress-strain regime appears for both ϵ_{11} and ϵ_{22}. This scaling for the strains was already known in Ref. [28, 13]. Therefore in the same stress regime, we expect to have a universal absolute Poisson ratio value since:

$$\nu^R = -\frac{\epsilon_{22}}{\epsilon_{11}} \approx -\frac{-1+\sqrt{2}+\text{arcsinh}^{-1}(1)-\eta^{-1}-\frac{1}{8(1-\eta)}}{1-\sqrt{2}+\text{arcsinh}^{-1}(1)-\eta^{-1}-\frac{3}{8(1-\eta)}} \tag{2.74}$$

Theoretically this is expected when the separation of length scales is sufficiently large [36]. Like [36, 37], our value, plugging in $\eta \approx .8$, would not match with the linear response value of $-1/3$. Previous theoretical investigations that have obtained this linear response of $-1/3$, have calculated it via the elastic moduli $\lambda_R/(\lambda_R+2\mu_R)$ which is governed by the Aronovitz-Lubensky fixed point [19, 75, 76]. [36, 37, 77] find that the differential Poission ratio is $-1/3$ in the non-linear regime only when $d_c \to \infty$ whereas the absolute Poisson ratio is never $-1/3$. In addition, the Poisson ratio is sensitive to the type of boundary condition that is used [36].

With simulations we first sought to confirm the non-linear stress strain relation, which can be observed in Fig. 2.4 (a). Between σ_L and $\sigma_{q_{\text{th}}}$ we observed this non-linear relation. For large stresses, the classical response absent of any effects of thermal fluctuations (when $\sigma > \sigma_{q_{\text{th}}}$ in other words when $q_{\text{th}} < q_\sigma$) is obtained. For very small stresses, and with a large separation of length scales, one should observe a linear response that follows from $\epsilon_{11} \approx \sigma/4(1-\eta)Y_R(L)$, where $Y_R(L) = Y(2\pi/q_{\text{th}}L)^{\eta_u}$ (explained in Sec. 2.6.4). We were not able to numerically verify this slope, however we do observe a linear theory where the Young's modulus is softened by thermal fluctuations.

From the same simulations, we can obtain the Poisson ratio by taking the negative ratio of strains ϵ_{11} and ϵ_{22}. In Fig. 2.4(b), the Poisson ratio is plotted against the stress. The Poisson ratio shows potentially a universal flat regime for stress values such that $2\pi/L < q_\sigma < q_{\text{th}}$ [36]. For very small stresses, errors became very difficult to control. Stress free Monte-Carlo simulations in the past, [35], did measure a Poisson ratio via correlations functions and found a linear response of $-1/3$ predicted by [19].

Evidence from other simulations is much more scattered however. In [78], the Poisson ratio was measured to be $-.15$. More recent simulations in [79] may show that the linear response Poisson ratio may be positive. Further simulations done by [38] also found a disagreement with the value of the $-1/3$ in the thermodynamic limit. Thus it is unclear as to what should be the precise value of both the linear response of the Poisson ratio as well as its behavior in the non-linear regime.

Returning to our own data, for large stresses such that $\sigma > \sigma_{q_{\text{th}}}$, the bare Poisson ratio of the triangular lattice of masses connected by springs, 1/3, could not be achieved due to the immediate cross-over to the non-linear elastic regime (in the simulations the data showing the box length along the axis of stress begins to become very large at these stresses leading to a decrease in the Poisson ratio with further application of stress).

2.5 Conclusions

We examined the effects of uni-axial stress on thermally fluctuating sheets. In particular, we see that anomalous scaling due to thermal fluctuations at scales where the uni-axial stress is dominant still appears in the in-plane moduli orthogonal to the stress, such as C^R_{2222}. Furthermore the presence of the two length scales q_σ and q_{th} provides an interesting foreground for various regimes of the scaling of moduli. We verified these scalings via simulations, in particular the anomalous scaling of C^R_{2222} as well as the transition at q_σ, beyond which the correlation functions becomes anisotropic. These results match with previous investigations of tubules [66].

We furthermore verified the existence of a non-linear stress-strain regime $\epsilon \sim \sigma^{\eta/(2-\eta)}$ in our simulations with a numerically accurate exponent. However our results measuring the Poisson ratio were less conclusive and require further investigation.

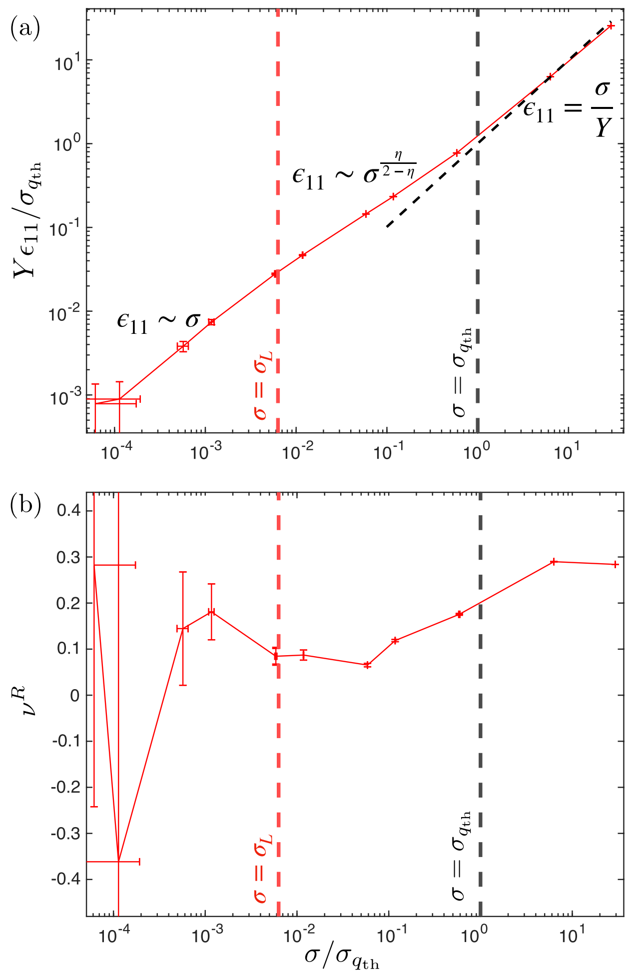

Figure 2.4: The stress strain curve (a) and Poisson strain curve (b) are plotted for simulations with system size $50a \times 50a$, only changing the value of stress. $\sigma_{q_{\rm th}}$ is defined as the stress at which $q_\sigma = q_{\rm th}$. The red dashed vertical line marks when $\sigma = \sigma_L$ (which is when $q_\sigma = 2\pi/L$, a non-linear regime where $\epsilon_{11} \sim \sigma^{.72}$ appears. The angled dashed line marks the $y = x$ line and shows that for large stresses, a classical response is regained.

2.6 Supplementary Information

2.6.1 Methods of Simulation

Simulations were performed on a cluster using 2.4 GHz Broadwell CPUs using molecular dynamics package LAMMPS in the NPT ensemble using a Nosé-Hoover thermostat. The simulations were of a 2D isotropic spring-mass triangular lattice embedded in 3 dimensions and under periodic boundary conditions. The elastic bending energy of such a spring mass system can be formulated as:

$$\text{E}_{\text{bend}} = \frac{\hat{\kappa}}{2} \sum_{\langle IJ \rangle} [1 + \cos\theta_{IJ}] \tag{2.75}$$

where $\hat{\kappa}$ is the microscopic dihedral spring stiffness and θ_{IJ} is the dihedral angle between two triangular faces (which can also be seen as the angle differences between normals of faces). The stretching energy is instead:

$$\text{E}_{\text{stretch}} = \frac{\hat{Y}}{2} \sum_{\langle ij \rangle} (r_{ij} - a)^2 \tag{2.76}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the Euclidean distance between two neighbors i and j and a is the lattice spacing. The bare continuum moduli of such a system can be derived from the discrete spring stiffnesses [55]:

$$\kappa = \frac{\sqrt{3}}{2}\hat{\kappa}, \lambda = \mu = \frac{\sqrt{3}}{4}\hat{Y} \tag{2.77}$$

The parameters were generally varied and hence the time step had to be chosen carefully to be less or equal to the following reduced times and periods:

$$\tau_T = a\sqrt{\frac{m}{k_B T}}, \tau_{\hat{Y}} = \sqrt{\frac{m}{\hat{Y}}}, \tau_{\hat{\kappa}} = a\sqrt{\frac{m}{\hat{\kappa}}} \tag{2.78}$$

The simulations were done non-dimensionally so k_B, the Boltzmann constant, and mass and lattice spacing were set to 1. A simulation generally ran for approximately $1.6 \times 10^8 - 10^9$ time steps each of length Min$\{\tau_T, \tau_{\hat{Y}}, \tau_{\hat{\kappa}}\}$. In computation time this equates to 6-60 hours on the cluster. The system size was mostly kept constant around 50×50 and 100×100 for the molecular dynamics correlations.

Table 2.6:

Data Sets for Fig. 2.3, $q_\sigma/q_{\rm th} = 10^{-2}$			
L/a	$\hat{\kappa}/k_BT$	$\hat{Y}/(k_BT/a^2)$	$\hat{\sigma}/(k_BT/a^2)$
50	10^3	220	1.4×10^{-4}
50	10^2	220	1.4×10^{-3}
50	10^2	2.2×10^4	.14
50	1	220	.14
50	1	2.2×10^4	14
50	1	2.2×10^5	140
100	10^3	2.2×10^3	5.6×10^{-3}
100	10^2	2.2×10^3	5.6×10^{-2}
100	10^2	2.2×10^5	5.6
100	1	2.2×10^3	5.6
100	1	2.2×10^5	560

2.6.2 Data Sets

2.6.3 Homogeneous Integrals For SCSA Analysis of C^R_{1111}, C^R_{1122}

$$
\begin{aligned}
I^{(1)}_{1122} &= -\frac{k_BT}{4(2\pi)^2}\int d\tilde{p}_1 d\tilde{p}_2 \frac{(\tilde{p}_2-1)\tilde{p}_2(\tilde{p}_1-\tilde{q}_1)\tilde{p}_1}{[B^R_{2222}(q_{\rm min})(\tilde{p}_2-1)^4+\sigma_{11}(\tilde{p}_1-\tilde{q}_1)^2][B^R_{2222}(q_{\rm min})\tilde{p}_2^4+\sigma_{11}\tilde{p}_1^2]}\\
I^{(2)}_{1122} &= -\frac{2k_BT}{4(2\pi)^2}\int d\tilde{p}_1 d\tilde{p}_2\\
&\frac{C_{2222}(q_{\rm min})\Omega_C^2(\frac{1}{\sqrt{\tilde{q}_1}})}{q_{\rm min}} \frac{(\tilde{p}_2-1)^2\tilde{p}_2^2}{[B^R_{2222}(q_{\rm min})(\tilde{p}_2-1)^4+\sigma_{11}(\tilde{p}_1-\tilde{q}_1)^2][B^R_{2222}(q_{\rm min})\tilde{p}_2^4+\sigma_{11}\tilde{p}_1^2]}\\
I^{(3)}_{1122} &= -\frac{k_BT}{4(2\pi)^2}\int d\tilde{p}_1 d\tilde{p}_2 \frac{C_{2222}}{C_{1122}}\frac{(\tilde{p}_2-1)^2\tilde{p}_2^2}{[B^R_{2222}(q_{\rm min})(\tilde{p}_2-1)^4+\sigma_{11}(\tilde{p}_1-\tilde{q}_1)^2][B^R_{2222}(q_{\rm min})\tilde{p}_2^4+\sigma_{11}\tilde{p}_1^2]}\\
I^{(1)}_{1111} &= -\frac{k_BT}{4(2\pi)^2}\int d\tilde{p}_1 d\tilde{p}_2 \frac{C_{1122}}{C_{1111}}\frac{(\tilde{p}_2-1)^2\tilde{p}_2^2}{[B^R_{2222}(q_{\rm min})(\tilde{p}_2-1)^4+\sigma_{11}(\tilde{p}_1-\tilde{q}_1)^2][B^R_{2222}(q_{\rm min})\tilde{p}_2^4+\sigma_{11}\tilde{p}_1^2]}\\
I^{(2)}_{1111} &= -\frac{k_BT}{4(2\pi)^2}\int d\tilde{p}_1 d\tilde{p}_2 \frac{(\tilde{p}_2-1)\tilde{p}_2(\tilde{p}_1-\tilde{q}_1)\tilde{p}_1}{[B^R_{2222}(q_{\rm min})(\tilde{p}_2-1)^4+\sigma_{11}(\tilde{p}_1-\tilde{q}_1)^2][B^R_{2222}(q_{\rm min})\tilde{p}_2^4+\sigma_{11}\tilde{p}_1^2]}\\
I^{(3)}_{1111} &= -\frac{k_BT}{4(2\pi)^2}\int d\tilde{p}_1 d\tilde{p}_2 \frac{C_{1122}}{C_{1111}}\frac{(\tilde{p}_2-1)\tilde{p}_2(\tilde{p}_1-\tilde{q}_1)\tilde{p}_1}{[B^R_{2222}(q_{\rm min})(\tilde{p}_2-1)^4+\sigma_{11}(\tilde{p}_1-\tilde{q}_1)^2][B^R_{2222}(q_{\rm min})\tilde{p}_2^4+\sigma_{11}\tilde{p}_1^2]}\\
I^{(4)}_{1111} &= -\frac{k_BT}{4(2\pi)^2}\int d\tilde{p}_1 d\tilde{p}_2 \frac{C_{1122}}{C_{1111}}\frac{(\tilde{p}_1-\tilde{q}_1)^2\tilde{p}_1^2}{[B^R_{2222}(q_{\rm min})(\tilde{p}_2-1)^4+\sigma_{11}(\tilde{p}_1-\tilde{q}_1)^2][B^R_{2222}(q_{\rm min})\tilde{p}_2^4+\sigma_{11}\tilde{p}_1^2]}
\end{aligned}
\tag{2.79}
$$

2.6.4 Stress Strain Relations

In this section we summarize the results of how the strains scale with respect to an applied stress for the other two relevant regimes of stress. For small stresses, when $q_\sigma < \frac{2\pi}{L} < \Lambda < q_{\text{th}}$ we obtain:

$$
\begin{aligned}
&\epsilon_{11}(\sigma_{11}, T | q_\sigma < 2\pi/L < \Lambda < q_{\text{th}}) \\
&\approx \frac{\sigma_{11}}{Y} - \frac{1}{2}\sum_{2\pi/L<|\mathbf{q}|<\Lambda} q_1^2 \left[\frac{k_BT}{A[\kappa q_{\text{th}}^{\eta} q^{4-\eta} + \sigma_{11}q_1^2]} - \frac{k_BT}{A[\kappa q_{\text{th}}^{\eta} q^{4-\eta}]}\right] \\
&\approx \frac{\sigma_{11}}{Y} - \frac{1}{2(2\pi)^2}\int_0^{2\pi} d\theta \int_{2\pi/L}^{\Lambda} dq q^3 \cos^2\theta \left[-\frac{k_BT\sigma_{11}q^2\cos^2\theta}{\kappa^2 q_{\text{th}}^{2\eta} q^{8-2\eta}}\right] \\
&= \frac{\sigma_{11}}{Y} - \left[\frac{3k_BT\sigma_{11}}{64\pi(1-\eta)\kappa^2 q_{\text{th}}^{2\eta}} q^{2\eta-2}\right]\Bigg|_{2\pi/L}^{\Lambda} \\
&\approx \frac{3k_BT\sigma_{11}}{64\pi(1-\eta)\kappa^2 q_{\text{th}}^{2\eta}}\left(\frac{2\pi}{L}\right)^{2\eta-2} \\
&\approx \frac{\sigma_{11}}{4(1-\eta)Y_R(L)}
\end{aligned} \tag{2.80}
$$

$$
\begin{aligned}
&\epsilon_{22}\left(\sigma_{11}, T \middle| q_\sigma < \frac{2\pi}{L} < \Lambda < q_{\text{th}}\right) \\
&\approx \frac{-\nu\sigma_{11}}{Y} - \left[\frac{k_BT\sigma_{11}}{64\pi(1-\eta)\kappa^2 q_{\text{th}}^{2\eta}} q^{2\eta-2}\right]\Bigg|_{2\pi/L}^{\Lambda} \\
&\approx \frac{k_BT\sigma_{11}}{64\pi(1-\eta)\kappa^2 q_{\text{th}}^{2\eta}}\left(\frac{2\pi}{L}\right)^{2\eta-2} \\
&\approx \frac{\sigma_{11}}{12(1-\eta)Y_R(L)}
\end{aligned}
$$

At low stresses we can ignore the bare response term σ/Y for ϵ_{11} or $-\nu\sigma/Y$ for ϵ_{22} and since we are not interested in the effects of microscopic physics, the dominant term in the above expressions is the one that involves the system size. This term can then be reformulated in terms of the renormalized Young's modulus, $Y_R(L) = Y(2\pi/q_{\text{th}}L)^{\eta_u}$. In addition, one can immediately see from the definition of the Poisson ratio, $\nu^R = -1/3$ in the linear response. We do not see this linear response value in our simulations however and there is theory that supports other values [36]. Instead, at large stresses when $\frac{2\pi}{L} < \Lambda < q_{\text{th}} < q_\sigma$ we obtain:

$$
\begin{aligned}
&\epsilon_{11}(\sigma_{11}, T|2\pi/L < \Lambda < q_{\text{th}} < q_\sigma) \approx \\
&\frac{\sigma_{11}}{Y} - \frac{1}{2}\sum_{2\pi/L<|\mathbf{q}|<\Lambda} q_1^2\left[\frac{k_BT}{A[\kappa q^4 + \sigma_{11}q_1^2]} - \frac{k_BT}{A[\kappa q_{\text{th}}^{\eta} q^{4-\eta}]}\right] \\
&= \frac{\sigma_{11}}{Y} - \frac{1}{2(2\pi)^2}\int_0^{2\pi} d\theta \int_{2\pi/L}^{\Lambda} dq q^3\cos^2\theta\left[\frac{k_BT}{\kappa q^4 + \sigma_{11}q^2\cos^2\theta} - \frac{k_BT}{\kappa q_{\text{th}}^{\eta} q^{4-\eta}}\right] \\
&= \frac{\sigma_{11}}{Y} - \frac{k_BT}{8\pi\sigma_{11}}\left[q^2 - q\sqrt{q^2 + \frac{\sigma_{11}}{\kappa}} + \frac{\sigma_{11}}{\kappa}\sinh^{-1}\left[q\sqrt{\frac{\kappa}{\sigma_{11}}}\right]\right]\Bigg|_{\frac{2\pi}{L}}^{\Lambda} + \frac{k_BT}{8\pi\eta\kappa}\left(\frac{q}{q_{\text{th}}}\right)^{\eta}\Bigg|_{\frac{2\pi}{L}}^{\Lambda} \\
&\approx \frac{\sigma_{11}}{Y} \\
&\epsilon_{22}\left(\sigma_{11}, T\middle|\frac{2\pi}{L} < \Lambda < q_{\text{th}} < q_\sigma\right) \\
&\approx \frac{-\nu\sigma_{11}}{Y} - \frac{k_BT}{8\pi\sigma_{11}}\left[-q^2 + q\sqrt{q^2 + \frac{\sigma_{11}}{\kappa}} + \frac{\sigma_{11}}{\kappa}\sinh^{-1}\left[q\sqrt{\frac{\kappa}{\sigma_{11}}}\right]\right]\Bigg|_{\frac{2\pi}{L}}^{\Lambda} + \frac{k_BT}{8\pi\eta\kappa}\left(\frac{q}{q_{\text{th}}}\right)^{\eta}\Bigg|_{\frac{2\pi}{L}}^{\Lambda} \\
&\approx \frac{-\nu\sigma_{11}}{Y}
\end{aligned}
\tag{2.81}
$$

where we have approximated that when stress is quite high, the terms from the integrals can be ignored and the bare material properties can be used to obtain the effective mechanical response. Thus the Poisson ratio we should observe should also be that of the bare material. For our simulations with triangular lattices, $\nu = 1/3$.

Chapter 3

$\epsilon-$Expansion of Elastic Modulus Anisotropies

3.1 Introduction

Graphene is a 2-d material with carbon atoms arranged in a triangular lattice which renders it thus an isotropic elastic material. However, there are a wider range of 2-d materials such as chalcogenides and black phosphorus and not all such 2-d systems are elastically isotropic [80]. For example, black phosphorus has a Young's modulus of $41GPa$ along the axis perpendicular to its puckering and $106GPa$ along the axis parallel to its puckering [80]. Examining the crystal structure as well, one can observe that its elastic symmetry class is orthorhombic (p2mm). Thus, anisotropic perturbations to the Aronovitz-Lubensky fixed point and its associated anomalous exponents η, η_u must be considered and fully understood: in other words, do anisotropies bring us to a different set of critical exponents or do they effectively wash away at larger length scales.

Of course, we consider these anisotropic symmetry-class perturbations far away from any melting where the Kosterlitz-Thouless-Halperin-Nelson-Young theory must once again be considered [53, 54]. We also consider these perturbation at low enough temperatures far from the crumpling transition or even potential tubule-formation [66, 69]. Indeed, Ref. [69] numerically obtained the existence of a tubule phase with merely a sufficiently anisotropic bending rigidity and at intermediate temperatures not high enough for full crumpling, nor low enough such that the flat state was the reference state.

In this chapter we consider briefly discuss again the generalization of the free energy for anisotropic systems. We then explain the failure of the un-controlled $D = 2$ perturbative renormalization group in characterizing the behavior of anisotropic materials in subsection 3.2.1. This will motivate us to perform an ϵ expansion near the upper critical dimension of the system, $D_{uc} = 4$, done in subsection 3.2.2 and perform simulations to confirm results in subsection 3.2.3.

3.2 Monoclinic ϵ-expansion and Correction to Toner's Orthorhombic and cubic ϵ-expansion

In [81], Toner performed an ϵ-expansion in the vicinity of $D_{uc} = 4$ showing that perturbative cubic (p4mm) and orthorhombic (p2mm) anisotropies are irrelevant and thus wash away in the thermodynamic limit. Thus, thermal fluctuations would restore $p3$ rotational symmetry not present in the microscopic lattice.

In our own analysis, we seek to replicate and extend these results to that of elastic systems in the monoclinic symmetry class which has no symmetries other than a π-rotation. The free energy of an isotropic system in general D-dimensions is once again:

$$\mathcal{F} = \frac{1}{2} \int d^D\mathbf{r}[\lambda u_{ii}^2 + 2\mu u_{ij}^2 + \kappa K_{ii}^2 - 2\kappa_G \det(K_{ij})] \tag{3.1}$$

For general anisotropic materials the free energy can be generalized to [44]:

$$\mathcal{F} = \frac{1}{2}\int d^D\mathbf{r}[C_{ijkl}u_{ij}u_{kl} + B_{ijkl}K_{ij}K_{kl}] \tag{3.2}$$

where, the bare elastic moduli tensors have the fundamental major and minor symmetries: $C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$ and $B_{ijkl} = B_{klij} = B_{jikl} = B_{ijlk}$ [70]. In addition, each symmetry class will induce further constraints on the modulus tensors C_{ijkl} and B_{ijkl}. The group of symmetry transformations (reflections and rotations in this case) can find representation in the orthogonal group $O(D)$ (orthogonal is meant in the mathematical sense here and D is the intrinsic dimension of the membrane). Supposing some symmetry transformation belonging to some symmetry class has an orthogonal-matrix representation R_{ij}, further constrains the modulus tensor via the formula: $C_{ijkl} = R_{im}R_{jn}R_{kp}R_{lq}C_{mnpq}$ and likewise for B_{ijkl} (given that our metric is just the flat reference metric, we shall omit the use of covariant index notation associated with more general geometric transformations). Thus, for example, the orthorhombic symmetries will enforce that $C_{iiij} = C_{iijk} = C_{ijkl} = 0$ where each distinct index is taken to be a distinct number between 1 and D. For a system with cubic symmetry C_{iiii} no longer needs to equal $C_{iijj} + 2C_{ijij}$ (as it is for isotropic systems $C_{1111} = C_{1122} + 2C_{1212} = \lambda + 2\mu$). For a monoclinic system, the fundamental major and minor symmetries are the only constraints. One can observe then that the number of elastic moduli for each symmetry class will vary with D unlike for isotropic elastic systems characterized solely by λ, μ, κ (for periodic boundary conditions such that we may ignore κ_G). Indeed the free energy can then become extremely complicated with many varying indices.

3.2.1 Failure of Un-Controlled Renormalization Group for $D = 2$

Given this potential complexity, we first attempted to calculate an un-controlled perturbative renormalization of anisotropic materials for $D = 2$. By un-controlled we intend that it is not done with the ϵ-expansion, where the renormalization group is applied in dimension $D_{uc} - \epsilon$. Given that $D_{uc} = 4$ for our system, $\epsilon = 2$ if $D = 2$ and is thus not a small parameter. For $D = 2$, one can integrate out the in-plane phonons in Eq. (3.2) and obtain the following effective free energy:

$$\frac{\mathcal{F}_{eff}}{A} = \frac{1}{2} \sum_{|\mathbf{q}|<\Lambda} [B_{ijkl}\ q_i q_j q_k q_l + \sigma_{ij} q_i q_j] f(\mathbf{q}) f(-\mathbf{q})$$
$$+ \frac{1}{8} \sum_{\substack{\mathbf{q}_1+\mathbf{q}_2= \\ -\mathbf{q}_3-\mathbf{q}_4=\mathbf{q}\neq 0, |\mathbf{q}_i|_{i=1,\ldots,4}<\Lambda}} q^4 [q_{1i} P^T_{ij}(\mathbf{q}) q_{2j}][q_{3i} P^T_{ij}(\mathbf{q}) q_{4j}] \frac{N}{E(\mathbf{q})} f(\mathbf{q}_1) f(\mathbf{q}_2) f(\mathbf{q}_3) f(\mathbf{q}_4) \tag{3.3}$$

where N and $E(\mathbf{q})$ are now generalized for monoclinic system in 2 dimensions:

$$\begin{aligned} N =& [2C_{1112}C_{1122}C_{2221} - C_{1122}^2 C_{1212} - C_{1112}^2 C_{2222} - C_{1111}C_{2221}^2 + C_{1212}C_{2222}C_{1111}] \\ E(\mathbf{q}) =& \mathrm{Det}[C_{ijkl} q_i q_k] \\ =& [(C_{1111}C_{1212} - C_{1112}^2)\hat{p}_1^4 + 2(C_{1111}C_{2221} - C_{1112}C_{1122})\hat{p}_1^3 \hat{p}_2 \\ &+ (2C_{1112}C_{2221} + C_{1111}C_{2222} - C_{1122}^2 - 2C_{1122}C_{1212})\hat{p}_1^2 \hat{p}_2^2 \\ &+ 2(C_{1112}C_{2222} - C_{1122}C_{2221})\hat{p}_1 \hat{p}_2^3 + (C_{1212}C_{2222} - C_{2221}^2)\hat{p}_2^4] \end{aligned} \tag{3.4}$$

Using these free energies, one can perform calculations of Feynman diagrams to obtain the Self-Consistent-Screening-Analysis equations (SCSA) seen in Fig. 3.1, which are the same as those calculated in Fig. 2.2. We can similarly obtain the renormalization group equations as was done in Eq. (3.5) to obtain:

$$\partial_s C^R_{ijkl}(s) = 2(2\Delta_f - 1)C^R_{ijkl}(s) \\ - \frac{k_B T \Lambda^{-2}}{2(2\pi)^2}\int d\hat{\mathbf{p}} \frac{[C^R_{ijmn}(s)\hat{p}_m\hat{p}_n][C^R_{abkl}(s)\hat{p}_a\hat{p}_b]}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l + \frac{\sigma_{ij}}{\Lambda^2}\hat{p}_i\hat{p}_j]^2} \tag{3.5}$$

and

$$\partial_s B^R_{1111}(s) = 2(\Delta_f - 1)B^R_{1111}(s) + \frac{k_B T}{4\pi^2\Lambda^2}N\int d\hat{\mathbf{p}} \frac{\hat{p}_2^4}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l]E[\hat{p}]} \tag{3.6}$$

$$\partial_s B^R_{1112}(s) = 2(\Delta_f - 1)B^R_{1112}(s) - \frac{k_B T}{4\pi^2\Lambda^2}N\int d\hat{\mathbf{p}} \frac{\hat{p}_1\hat{p}_2^3}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l]E[\hat{p}]} \tag{3.7}$$

$$\partial_s B^R_{1122}(s) = 2(\Delta_f - 1)B^R_{1122}(s) + \frac{k_B T}{4\pi^2\Lambda^2}N\int d\hat{\mathbf{p}} \frac{3\hat{p}_1^2\hat{p}_2^2}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l]E[\hat{p}]} \tag{3.8}$$

$$\partial_s B^R_{2221}(s) = 2(\Delta_f - 1)B^R_{2221}(s) + \frac{k_B T}{4\pi^2\Lambda^2}N\int d\hat{\mathbf{p}} \frac{\hat{p}_1^3\hat{p}_2}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l]E[\hat{p}]} \tag{3.9}$$

$$\partial_s B^R_{2222}(s) = 2(\Delta_f - 1)B^R_{2222}(s) + \frac{k_B T}{4\pi^2\Lambda^2}N\int d\hat{\mathbf{p}} \frac{\hat{p}_1^4}{[B^R_{ijkl}(s)\hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l]E[\hat{p}]} \tag{3.10}$$

as our corresponding RG equations. Unfortunately, even for orthorhombic materials, one can check via a numerical integration that these differential equations give rise to strongly anisotropic correlations with critical exponents that depend on the orien-

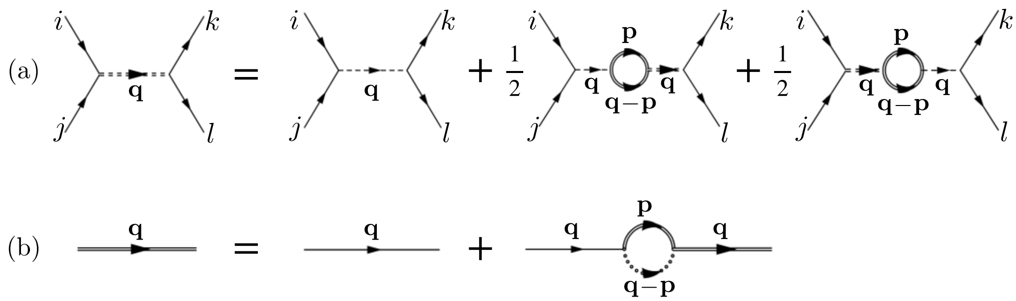

Figure 3.1: (a) The SCSA equation is shown graphically using the $C_{ijkl}\partial_i f \partial_j f \partial_k f \partial_l f$ vertex. This equation is used to obtain a scaling of the C_{ijkl} via a self-consistent analysis. The symmetrization is due to the major symmetry of the free energy. The the dashed line indicates C_{ijkl} and the doubled dashed line C^R_{ijkl}. The solid lines indicate $\mathcal{G}_{ff}$ whereas the doubled solid lines indicate $\mathcal{G}^R_{ff}$. (b) The SCSA equation corresponding to the flexural correlation function is shown using the effective f^4 vertex in Eq. (3.3). The renormalized structure of the vertex is marked by the doubled dotted line.

tation. This is clearly a dramatic departure from the results given by the more sure ϵ-expansion and thus are indicative of the dangers of applying the renormalization group without a controlled parameter.

3.2.2 Application of ϵ-expansion for General Anisotropic Perturbations

Having seen that an uncontrolled renormalization group scheme can lead to erroneous results, we return to applying an ϵ-expansion. However, as noted before, for general D intrinsic dimensions and non-isotropic symmetry classes, the form of the free energy becomes quite complicated and untenable for the orthorhombic and monoclinic classes (for the cubic class of perturbations it is still simple and can be obtained in Ref. [81]). Toner resolved this in his paper, Ref. [81], by examining particular anisotropic perturbations. More specifically, $D_{uc} = 4$ and thus Toner treated 3 dimensions as isotropic between themselves but the fourth dimension to possess anisotropies with respect to the first 3. Thus rather than writing down a free energy in the general form of

Eq. (3.3), orthorhombic free energies were, for example, written down as:

$$\begin{aligned}\mathcal{F} = \frac{1}{2}\int d^D\mathbf{r}[\lambda u_{ii}^2 + 2\mu u_{ij}^2 + \delta C_{1111}u_{11}^2 + \delta C_{11\alpha\alpha}u_{11}u_{\alpha\alpha}\delta C_{1\alpha1\alpha}u_{1\alpha}^2 \\ + \delta B_{1111}(\partial_1^2\mathbf{f})^2 + \delta B_{11\alpha\alpha}\partial_1^2\mathbf{f}\partial_\alpha^2\mathbf{f}]\end{aligned} \tag{3.11}$$

where we use Einstein summation and $\alpha \in \{2, ..., D\}$, thus axis "1" is the special axis with respect to which all anisotropies originate and the other $D-1$ axes are isotropic within themselves. Furthermore $\mathbf{f}$ is a vector of length $d_c = d - D$, the co-dimension of the elastic membrane. This exact free energy can also be found in Toner's paper. Furthermore, one need not treat all $\delta C_{11\alpha\alpha}$ and $\delta C_{11\beta\beta}$ differently for $\alpha \neq \beta$ and likewise for $\delta C_{1\alpha1\alpha}$ and $\delta B_{11\alpha\alpha}$. Despite that $D = 4 - \epsilon$, this is because we are interested in the anisotropy presented by the physical orthorhombic $D = 2$ system and thus breaking isotropy such that $\delta C_{11\alpha\alpha} = \delta C_{11\beta\beta} \forall \alpha, \beta \in \{2, ..., D-1\}$ is sufficient. Thus, by symmetry, all dimensions $\alpha \in \{2, ..., D\}$ are symmetric between themselves. Toner then calculates the an-harmonic renormalizations of $\lambda, \mu, \kappa, \delta C_{1111}, \delta C_{11\alpha\alpha}, \delta C_{1\alpha1\alpha}, \delta B_{1111}, \delta B_{11\alpha\alpha}$ due to the Feynman diagrams in Fig. 3.1. This gives rise to a set of ODEs (β equations) for those parameters. One can then non-dimensionalize the system such that we instead examine the flows of $\hat{\lambda} = \lambda/\kappa^2, \hat{\mu} = \mu/\kappa^2, \delta C_{11\alpha\alpha}/\mu, \delta C_{1\alpha1\alpha}\mu, \delta C_{1111}/\mu, \delta B_{11\alpha\alpha}/\kappa, \delta B_{1111}/\kappa$. In this non-dimensionalized form $\hat{\mu}, \hat{\lambda}$ take on the Aronovitz-Lubensky fixed point values in Ref. [12] and one can determine whether anisotropy is important with respect to the perturbations in $\delta C_{11\alpha\alpha}/\mu, \delta C_{1\alpha1\alpha}\mu, \delta C_{1111}/\mu, \delta B_{11\alpha\alpha}/\kappa, \delta B_{1111}/\kappa$ by linearizing their respective β equations and performing a stability analysis. These calculations are done for $D = 4 - \epsilon$ and thus, despite taking simplifying steps, it is still difficult to show them by hand. However, they can be done in Mathematica. A Mathematica code has been provided (found in Sec. A and Sec. B in the Appendix) which performs the ϵ-expansion in the case of both the cubic perturbations (identical to those Ref. [81] considered) as well as monoclinic perturbations which encompasses the orthorhombic class of perturbations (orthorhombic being what Ref. [81] considered as well). In the case of the cubic ϵ-expansion, our results differ quantitatively but not qualitatively from Ref. [81]. That is, we also obtain negative eigenvalues with respect to cubic perturbative eigen-vectors but their specific values are different and are:

$$\epsilon\frac{-306 - 25d_c \pm \sqrt{93636 - 8556d_c + 625d_c^2}}{50(24 + d_c)} \tag{3.12}$$

One may check that for all d_c, these eigen-values are always negative. By running the Mathematica code one can find that the eigen-vectors are of the cubic class of perturbations and are thus irrelevant. Thus, Ref. [81] is still qualitatively correct. However, the reason for the quantitative difference is that $\mathcal{G}_{ff}$ was not Taylor expanded with respect to the perturbative term it contains: δB_{1111}.

This lack of Taylor expanding $\mathcal{G}_{ff}$ also leads to both quantitative and qualitative differences in the case of the orthorhombic class of perturbations. Not only do we obtain a differing set of negative eigen-values, but we also obtain a single 0 eigen-value to 1-loop order in the ϵ-expansion and for general d_c. The 0 eigen-value corresponds to the eigen-vector:

$$\begin{bmatrix} \delta\hat{C}_{1111}/\mu \\ \delta\hat{C}_{1112}/\mu \\ \delta\hat{C}_{1122}/\mu \\ \delta\hat{C}_{1212}/\mu \\ \delta\hat{C}_{2221}/\mu \\ \delta\hat{B}_{1111}/\kappa \\ \delta\hat{B}_{1112}/\kappa \\ \delta\hat{B}_{1122}/\kappa \\ \delta\hat{B}_{2221}/\kappa \end{bmatrix} = \begin{bmatrix} 10/3 \\ 0 \\ -1/3 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{3.13}$$

This eigen-vector clearly breaks isotropy in the orthorhombic class and thus indicates that a 2-loop calculation is necessary. Eigen-vectors that introduced monoclinic perturbations corresponded to negative eigen-values. However, the presence of this 0 eigen-value indicates that a more careful 2-loop analysis is essential.

3.2.3 Simulations of $D = 2$ Monoclinic Elastic Systems

To make sure that indeed anisotropies were not de-stabilizing the Aronovitz-Lubensky fixed point, we performed simulations using LAMMPS. The methods we used were the same as those found in section 2.6.1. However, to simulate monoclinic systems using a triangular lattice we had to assign differing stiffness to dihedral and in-plane bonds that broke as many symmetries as possible. Triangular lattices can be characterized by the in-plane bonds oriented at 3 angles: $60, 120, 180$ degrees. To achieve a monoclinic system, one may simply assign differing stiffness to bonds along each of these angles. An analogous procedure was done in [69]. Doing so does not produce a mechanically unstable system and furthermore creates an elastic material in the monoclinic class. A similar such idea can be implemented for the dihedral bonds. Continuum moduli corresponding to a lattice with certain bond stiffness can be obtained by a coarse graining procedure found in [55]. With this knowledge, we simulated a monoclinic system such that $B_{1111} = 5B_{2222}$ and $C_{1111} = 18C_{2222}$. Below the thermal length scale, we expect the bare anisotropy to appear in our plot of the flexural and in-plane correlations. Beyond this thermal length scale, we are interested in whether this anisotropy does not change, grows or if the propagators become isotropic. We can check this by observing the propagators $\langle f(q,0)f(-q,0)\rangle$ vs. $\langle f(0,q)f(0,-q)\rangle$ and $\langle u_1(q,0)u_1(-q,0)\rangle$ vs. $\langle u_2(0,q)u_2(0,-q)\rangle$. The propagators discussed for the discrete monoclinic system we chose can be observed in Fig. 3.2. As can be seen, the anisotropy washes away. Thus, it is once again confirmed that the controlled ϵ-expansion is indeed qualitatively correct despite being performed in

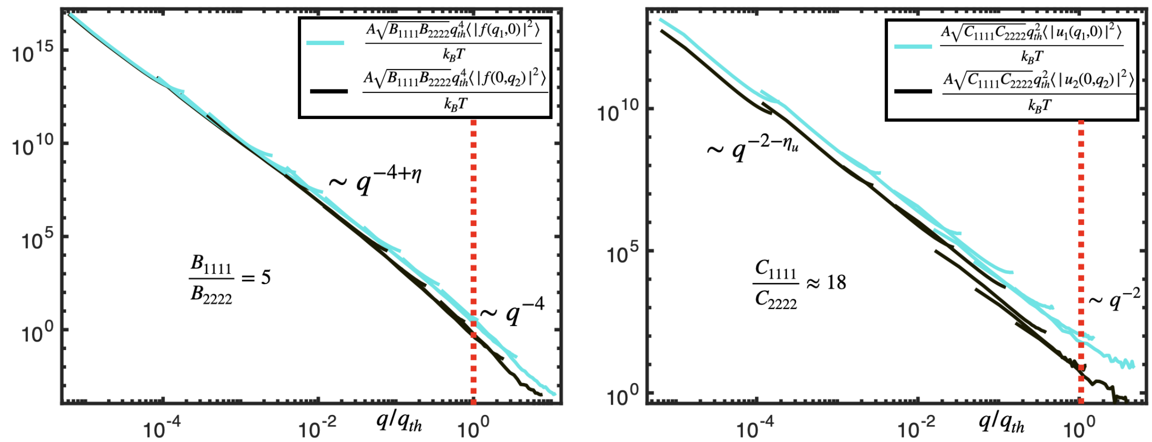

Figure 3.2: Above are shown simulations for a ball-spring system with bare moduli possessing monoclinic symmetries (only a π rotation). The x-axis of the plot is non-dimensionalized by the geometric mean thermal length scale q_{th}. As can be seen, beyond a thermal length scale both (a) the flexural Green's functions along orthogonal axes and (b) the in-plane Green's functions, which are plotted along orthogonal axes, begin to converge and become effectively isotropic. This implies that temperature washes out the bare symmetries of the crystal and renders the Green's functions effectively isotropic.

dimension $D = 4 - \epsilon$. Thus, our un-controlled renormalization attempt at $D = 2$ showed erroneous strong anisotropy which is inconsistent with both $D = 2$ simulations done in LAMMPS and the ϵ-expansion.

Chapter 4

Fluctuations of Odd Elastic Membranes

4.1 Introduction To Odd Elasticity

To repeat, the effect of thermal fluctuations on equilibrated d-dimensional elastic membranes embedded in $D + d_c$ dimensions (where d_c is the codimension) has been the subject of study since the 80s [11, 13, 46]. It was noted that unlike $D = 1$ polymers which perform random walks in any embedding dimension satisfying $D + d_c \geq 2$ [16], elastic membranes undergo a de-crumpling phase transition to an ordered phase as the temperature is decreased below $T_c \sim \kappa/k_B$ (where κ is the bare bending rigidity of the elastic membrane and k_B is the Boltzmann constant) [10]. In their low temperature symmetry-broken phase, such elastic systems are characterized by anomalous scaling of their elastic moduli beyond a thermal length scale, ℓ_{th}; these exponents being determined by the non-Gaussian Aronovitz-Lubensky fixed point [11, 12].

Variations of this theory have been studied in the presence of weak crystal anisotropy [81], homogeneous disorder [56] and more [46]. Typically the Aronovitz-Lubensky fixed point is stable to such perturbations with a few exceptions [66, 46, 34].

However, a less explored direction is the impact of non-equilibrium effects which force us away from a Boltzmann-measure utilizing a free energy. This would be exemplified, for example, by the KPZ equation in which a term of the form $\lambda(\nabla h)^2$ is present and cannot be derived from a free energy [82]. Exploring non-equilibrium effects requires resorting to the Langevin equation. Interestingly though, Nelson and Frey did study the dynamical Langevin equations associated with such elastic membranes [83]. Although, they also studied the long-range mediated forces via hydrodynamics for membranes that were not completely permeable, they did also study the dynamical renormalization of these free draining membranes (Rouse dynamics). With this formalism, they replicated the same Aronovitz-Lubensky anomalous exponents as for static elastic membranes. Thus it appears that the dissipative Langevin equation does describe the phenomenological theory associated with the Boltzmann weight.

Returning to a motivation for studying thermally fluctuating elastic membranes with non-equilibrium effects, we resort to the field of active matter. With the development of the Vicsek model and the Toner-Tu equations, the study of active systems has relatively recently exploded into a new and fascinating field where one may often break many of these laws or symmetries (conservation of energy as an example) [84, 85]. For an interesting discussion of the history of active systems, which does go earlier than the aforementioned papers, see [86].

Within the field of active matter, elastic systems are currently being explored [86, 87, 88, 89, 37, 90, 91, 92, 93]. Within natural phenomena, active elasticity may be especially relevant for biological systems. A recent paper found potential signatures of odd elasticity in spontaneously formed crystals consisting of starfish embryos [88]. One could also, for example, study mechanics of the actin cortex, a layer of cross-linked actin that lies beneath the plasma membrane of animal cells [91, 89]. The activity of this mesh arises from the myosin motors that exert contractile forces. Due to their length scale, such systems are concomitantly under the influence of temperature [46]. We are thus interested in investigating the behavior of thermalized active elastic membranes.

Recently, Scheibner et al. have extended the behavior of elastic systems to those which may not possess conservation of energy or angular momentum [93]. Such systems are referred to as being odd elastic since they possess some active moduli. For typical elastic solids that can be described by a free energy, the generalized elastic modulus tensor C_{ijkl} possesses two minor symmetries and one major symmetry [44, 93]: $C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$. If we further restrict our interest to that of isotropic solids then the tensor takes the form: $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$. The minor symmetry $C_{ijkl} = C_{ijlk}$ is associated with deformation dependence which makes sense to keep. The major symmetry $C_{ijkl} = C_{klij}$ is associated with removing conservation of energy. The minor symmetry $C_{ijkl} = C_{jikl}$ is associated with the conservation of angular momentum which one can also remove. The removal of conservation of angular momentum and energy, in general dimensions, cannot be done while preserving isotropy. However, for the special physical case of $D = 2$, this turns out to be possible. The reason for this is invariance under π-rotations,R^{π}, have no consequences on C_{ijkl} in $D = 2$ via $C_{ijkl} = R^{\pi}_{im}R^{\pi}_{jn}R^{\pi}_{kp}R^{\pi}_{lq}C_{mnpq}$. Whereas in higher dimensions, such rotations force odd elastic parameters to be zero as the elastic tensor forcibly satisfies: $C_{ijkl} = -C_{ijkl}$ as long as an index is repeated only an odd number of times (for example $C_{1222}, C_{3213}, C_{1232}, C_{1234}, C_{4412}$ for $D = 4$). Isotropy is thus much more restrictive in higher dimensions [93, 94]. Hence 3D and 4D odd elastic systems must be anisotropic to be odd.

Thus for $D = 4 - \epsilon$ membranes, odd elastic membranes must also be anisotropic. This therefore obstructs any attempt at performing a controlled ϵ-expansion as was done in [12] to obtain new fixed points. Though one may conduct a study evaluating the stability of the Aronovitz-Lubensky fixed point to odd perturbations as was done in [81], the number of perturbations to consider becomes quite arduous and must consider anisotropy, which we are not interest in for the time being. We thus restrict our attention to $D = 2$ membranes where isotropy can still be maintained. From here on, we will assume that $D = 2$ and that we are working with isotropic systems. In this special dimension, there are two odd elastic constants which are introduced by the removal of the above mentioned minor and major symmetry and the elastic tensor takes the form:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] + K_{odd}E_{ijkl} - A_{odd}\epsilon_{ij}\delta_{kl} \tag{4.1}$$

where $E_{ijkl} = \frac{1}{2}[\epsilon_{ik}\delta_{jl} + \epsilon_{il}\delta_{jk} + \epsilon_{jk}\delta_{il} + \epsilon_{jl}\delta_{ik}]$. As discussed in [93], both of the new odd elastic constants are chiral since they both break reflection symmetry. Furthermore, K_{odd} couples simple shear and pure shear whereas A_{odd} couples dilations with torques. Both A_{odd} and K_{odd} break the major symmetry, however only A_{odd} breaks the minor symmetry associated with conservation of angular momentum.

In this chapter, we will be studying the effect of thermal fluctuations on permeable D-dimensional odd elastic membranes fluctuating in a larger d-dimensional space. We will conduct a $1/d_c$ expansion for $D = 2$ to obtain the scaling behavior of the theory. Furthermore we show simulation results for comparison.

4.2 Langevin Equations of Permeable Odd Elastic Membranes

We desire to write down the Rouse-dynamic Langevin equations for $D = 2$ odd elastic membranes embedded in $d = D + d_c$-dimensions and freely suspended in a heat bath that is in thermal equilibrium. In a field-theoretic language, we seek to understand the perturbations of A_{odd} and K_{odd} to the Aronovitz-Lubensky fixed point. Our first step in conducting this analysis is to set up the proper Langevin equations. Since we cannot write down a free energy derived force, we must turn to the elastodynamic equations, which can then be converted to stochastic equations. The full form of the deterministic dynamic inertial equations with damping of in-plane deformations take the form [93]:

$$\rho\partial_t^2 u_j(\mathbf{r},t) + \partial_t u_j(\mathbf{r},t) = D_{jm}\partial_i\sigma_{im}(\mathbf{r},t) = D_{jm}C_{imkl}\partial_i u_{kl}(\mathbf{r},t) \tag{4.2}$$

where u_{kl} is the form of the strain tensor with the out-of-plane geometric non-linearity:

$$u_{ij} = \frac{1}{2}[\partial_i u_j + \partial_j u_i + \partial_i f^\alpha \partial_j f^\alpha] \tag{4.3}$$

Where $i, j \in 1, 2$ and α sums over the remaining d_c dimensions. ρ is defined as the density of the membrane. By adding a noise term we arrive to a form of a Langevin equation similar to [83]:

$$\rho\partial_t^2 u_j(\mathbf{r},t) + \partial_t u_j(\mathbf{r},t) = D_{jm}C_{imkl}\partial_i u_{kl}(\mathbf{r},t) + \eta_j(\mathbf{r},t) \tag{4.4}$$

We take the simplest Gaussian distribution for the noise such that its mean is zero, $\langle\eta_j(\mathbf{r},t)\rangle = 0$ and it has zero-memory, $\langle\eta_j(\mathbf{r},t)\eta_i(\mathbf{r}',t')\rangle = 2L_{ij}k_BT\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$. As done in [83], we shall take $k_BT = 1$ for simplicity. However, unlike in [83], we shall take the diffusion coefficient matrices D_{ij}, L_{ij} to be diagonal (and thus symmetric which is allowed since odd elastic systems do satisfy time-reversal symmetry) and isotropic. As one can note, we need not necessarily assume that the fluctuation-dissipation holds as in the case of the KPZ equation [82]. We shall return to this point further on.

With a Langevin equation for the in-plane deformations, we must perform an analogous procedure for the out-of-plane deformation field $f(\mathbf{r},t)$. The dynamic equation of undulation fields takes the form:

$$\rho\partial_t^2 f^\alpha(\mathbf{r},t) + \partial_t f^\alpha(\mathbf{r},t) = -\partial_i\partial_j M_{ij}^\alpha(\mathbf{r},t) + \partial_i[\sigma_{ij}(\mathbf{r},t)\partial_j f^\alpha(\mathbf{r},t)] \tag{4.5}$$

where the moment tensor satisfies $M_{ij}^\alpha = B_{ijkl}\partial_k\partial_l f^\alpha$. Here B_{ijkl} is the analogous bending rigidity tensor. We shall see shortly that even inserting odd moduli in such a tensor is futile. Returning to the form of the Langevin equation we obtain:

$$\rho\partial_t^2 f^\alpha(\mathbf{r},t) + \partial_t f^\alpha(\mathbf{r},t) = -D_f\partial_i\partial_j M_{ij}^\alpha(\mathbf{r},t) + D_f\partial_i[\sigma_{ij}(\mathbf{r},t)\partial_j f^\alpha(\mathbf{r},t)] + \eta_f^\alpha(\mathbf{r},t) \tag{4.6}$$

where D_f is a diffusion coefficient and the noise term here satisfies similarly: $\langle\eta_f^\alpha\rangle = 0$ and $\langle\eta_f^\alpha(\mathbf{r},t)\eta_f^\beta(\mathbf{r}',t')\rangle = 2L_f\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$. We furthermore assume that $\langle\eta_f^\alpha\eta_j\rangle = 0$. By inserting the form of the strain and moment tensor we obtain:

$$\rho\partial_t^2 f^\alpha(\mathbf{r},t) + \partial_t f^\alpha(\mathbf{r},t) = -D_f B_{ijkl}\partial_i\partial_j\partial_k\partial_l f^\alpha(\mathbf{r},t) + D_f C_{ijkl}\partial_i[u_{kl}(\mathbf{r},t)\partial_j f^\alpha(\mathbf{r},t)] + \eta_f^\alpha(\mathbf{r},t) \tag{4.7}$$

As we can see from the $B_{ijkl}\partial_i\partial_j\partial_k\partial_l$ term, all indices are contracted and thus any anti-symmetric terms of the odd type will indeed disappear and thus we take $B_{ijkl}\partial_i\partial_j\partial_k\partial_l = \kappa\Delta^2$ under assumptions of isotropy:

$$\rho\partial_t^2 f^\alpha(\mathbf{r},t) + \partial_t f^\alpha(\mathbf{r},t) = -D_f\kappa\Delta^2 f^\alpha(\mathbf{r},t) + D_f C_{ijkl}\partial_i[u_{kl}(\mathbf{r},t)\partial_j f^\alpha(\mathbf{r},t)] + \eta_f^\alpha(\mathbf{r},t) \tag{4.8}$$

Next we want to take the over-damped limit and omit the inertial terms from the above Langevin equations. However, before doing so we must perform a linear stability analysis to examine whether there are any instabilities of the full dynamic equations. This will inform us of phenomenological limits of our model.

4.2.1 Stability Analysis of Linearized Equations

Given the elastodynamic equations we have derived, one can ask if in their linearized form, there are any stability conditions to be wary of, particularly in the presence of inertia. The dynamic equation Eq. (4.8), holds no non-trivial stability constraints once non-linearities are removed so we are interested more so in the stability of Eq. (4.4).

We begin from the noiseless Langevin equation Eq. (4.2) and perform a linear stability analysis. Thus, interactions between in-plane phonons and flexural modes are omitted and we can examine the purely 2-D system described by:

$$\rho\partial_t^2 u_j + \partial_t u_j = \Gamma C_{ijkl}\partial_i\partial_k u_l \tag{4.9}$$

where we have used the assumed diagonal and isotropic properties of $D_{ij} = \Gamma\delta_{ij}$ (we use Γ since we have assigned D to the dimension of the membrane). Via a Fourier transform, $f(\mathbf{r}, t) = \frac{1}{AT}\sum_{\mathbf{q},\omega} f(\mathbf{q}, \omega)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$ (where A is the area of the system and T is the volume of time), it is easy to solve the quadratic equation for:

$$(-\rho\omega^2 - i\omega)\tilde{u}_j = \Gamma q^2 \begin{pmatrix} B+\mu & K \\ -K-A & \mu \end{pmatrix} \begin{bmatrix} \tilde{u}_\parallel \\ \tilde{u}_\perp \end{bmatrix} \tag{4.10}$$

where as in [93]: $\tilde{u}_\parallel = \hat{q}_i\tilde{u}_i$ and $\tilde{u}_\perp = \epsilon_{ij}\hat{q}_i\tilde{u}_j$. By doing the eigenvalue analysis of the matrix we obtain:

$$(-\rho\omega^2 - i\omega) = -\Gamma\left[\frac{B}{2} + \mu \pm \sqrt{\left(\frac{B}{2}\right)^2 - A_{odd}K_{odd} - K_{odd}^2}\right]q^2 \tag{4.11}$$

For simplicity let us redefine $J = A_{odd}K_{odd} + K_{odd}^2$. We can then solve this quadratic equation to obtain:

$$\omega = -\frac{i}{2\rho} \pm \frac{1}{2\rho}\sqrt{-1 + \rho\Gamma\left[\frac{B}{2} + \mu \pm \sqrt{\left(\frac{B}{2}\right)^2 - J}\right]q^2} \tag{4.12}$$

We do not want ω to have a positive imaginary branch. Thus when the second term in the above equation becomes more "positive" than the first, then we have obtained an instability. We can follow an analysis very similar to [93] and we find 3 regimes:

$J < -\mu(B + \mu)$

If J satisfies this condition then we obtain that:

$$\sqrt{\left(\frac{B}{2}\right)^2 - J} > \frac{B}{2} + \mu \tag{4.13}$$

and thus:

$$i\omega = \frac{1}{2\rho} - \frac{i}{2\rho}\sqrt{-1 + \rho\Gamma\left[\frac{B}{2} + \mu - \sqrt{\left(\frac{B}{2}\right)^2 - J}\right]q^2} < 0 \tag{4.14}$$

which thus gives rise to an instability for all q.

$-\mu(B + \mu) < J < (B/2)^2$

In this case a careful analysis of all 4 eigenvalues shows that there is no possibility for ω to have a positive imaginary branch since:

$$|Im\left[\sqrt{-1 + \rho\Gamma\left[\frac{B}{2} + \mu \pm \sqrt{\left(\frac{B}{2}\right)^2 - J}\right]q^2}\right]| < 1 \tag{4.15}$$

Thus we have a stable system for all q.

$J > (B/2)^2$

In this case we can rewrite the eigenvalues as:

$$\omega = -\frac{i}{2\rho} \pm \frac{1}{2\rho}\sqrt{-1 + \rho\Gamma\left[\frac{B}{2} + \mu \pm i\sqrt{J - \left(\frac{B}{2}\right)^2}\right]q^2} \tag{4.16}$$

By rewriting:

$$Re^{-i\theta_\pm} \equiv 1 - \rho\Gamma\left[\frac{B}{2} + \mu \pm i\sqrt{J - \left(\frac{B}{2}\right)^2}\right]q^2 \tag{4.17}$$

And taking the principal branch cut in the complex plane to be the non-positive x-axis (so that $-\pi < \theta_\pm < \pi$), we can take the square root and write:

$$\omega = -\frac{i}{2\rho} \pm \frac{i}{2\rho}\sqrt{R}e^{-i\theta_\pm/2} \tag{4.18}$$

And thus:

$$Im(\omega) = -\frac{1}{2\rho} \pm \frac{1}{2\rho}\sqrt{R}\cos\theta_\pm/2 \tag{4.19}$$

Since $-\pi < \theta_\pm < \pi$, then $\cos\frac{\theta_\pm}{2} > 0$. Therefore we need only examine:

$$Im(\omega) = -\frac{1}{2\rho} + \frac{1}{2\rho}\sqrt{R}\cos\theta_\pm/2 \tag{4.20}$$

Once $\sqrt{R}\cos(\theta_\pm/2) > 1$ we have an instability. This is equivalent to checking when $R(\cos(\theta_\pm/2))^2 > 1$. Via the half-angle formula this becomes:

$$R(1 + \cos\theta_\pm) > 2 \tag{4.21}$$

Thus we obtain:

$$\sqrt{(1 - \rho q^2\Gamma\left[\frac{B}{2} + \mu\right])^2 + \rho^2\Gamma q^4\left[J - \left(\frac{B}{2}\right)^2\right]} + 1 - \rho\Gamma q^2\left[\frac{B}{2} + \mu\right] > 2\gamma^2 \tag{4.22}$$

which leads us to:

$$\sqrt{(1-\rho\Gamma q^2\left[\frac{B}{2}+\mu\right])^2+\rho^2\Gamma^2 q^4\left[J-\left(\frac{B}{2}\right)^2\right]} > 1+\rho\Gamma q^2\left[\frac{B}{2}+\mu\right] > 0 \tag{4.23}$$

By squaring the inequality once more and solving we see that we obtain instabilities when:

$$\rho^2\Gamma^2 q^4\left[J-\left(\frac{B}{2}\right)^2\right] > 4\rho\Gamma q^2\left[\frac{B}{2}+\mu\right] \tag{4.24}$$

resulting in:

$$q > \sqrt{\frac{4}{\rho\Gamma}\frac{\left[\frac{B}{2}+\mu\right]}{\left[J-\left(\frac{B}{2}\right)^2\right]}} \equiv q_c \tag{4.25}$$

Hence we see that in this regime, at length scales small enough to observe, we have instabilities. If we were to explore over-damping via zero-inertia Brownian method ($\rho = 0$), we see that $q_c = \infty$ and thus we should not be able to observe this instability. In the Langevin method where we have under-damping, if $q_c < a$ where a is the lattice spacing of our discrete system, then we should once again not see this instability. Otherwise, the under-damped Langevin case should show this instability.

4.2.2 Why Choose Over-Damping?

Given the above stability analysis, we have observed that as long as we consider $J < -\mu(B+\mu)$ and we permit sufficient over-damping, we can avoid any stabilities of the linearized in-plane equations. This stability is, of course, a pre-requisite for studying odd elastic membranes that can fluctuate out of plane.

We are also interested in the over-damped case for two other reasons. Firstly, general high-frequency phenomena are un-important to the scaling analysis and phenomenological behavior associated with long-range and long-time behaviors. One can easily see this more rigorously via a power-counting analysis often done in field theory [58], in other words $\partial_t^2 f, \partial_t^2 u$ scale to zero in the low frequency limit.

Secondly, in the case where over-damping is not assumed, it is well known that one must consider an active heat flow [95]. This active heat flow is a form of work done by non-equilibrium or active forces that are not derived from a free energy. However, in the over-damped limit, one may disregard such terms. This will help us to simplify simulations that we performed where we use barostats and thermostats which in general should contain an active heat flow term.

With this choice we write down the final form of our Langevin equations, as derived from a physical picture:

$$\partial_t u_j(\mathbf{r}, t) = D_{jm} C_{imkl} \partial_i u_{kl}(\mathbf{r}, t) + \eta_j(\mathbf{r}, t) \tag{4.26}$$

$$\partial_t f^\alpha(\mathbf{r}, t) = -D_f \kappa \Delta^2 f^\alpha(\mathbf{r}, t) + D_f C_{ijkl} \partial_i [u_{kl}(\mathbf{r}, t) \partial_j f^\alpha(\mathbf{r}, t)] + \eta_f^\alpha(\mathbf{r}, t) \tag{4.27}$$

4.2.3 Field Theoretic Set Up Of Transition Probability Measure

From where we have left off in the last section one may use Eq. (4.27) and Eq. (4.26) to perform a renormalization group calculation in Ma's formalism [96, 97]. Instead though, we will use the Martin-Siggia-Rose-Janssen-DeDominicis formalism which is most analogous to the field theoretic approach [98, 99, 100, 72]. The formalism adopts the path-integral formulation by taking advantage of the form of the distribution of the thermal noises. Thus the transition probability is of the form [72, 83]:

$$\mathcal{W}(\eta_j, \eta_f^\alpha) \propto e^{-\frac{1}{4}\int dt \int d^d\mathbf{r}[L^{-1}\eta_i(\mathbf{r},t)^2 + L_f^{-1}\eta_f^\alpha(\mathbf{r},t)^2]} \tag{4.28}$$

In this form, inserting in the Langevin equations renders the expression into a complicated set of terms with high-degree non-linearities. Thus, via an imaginary Hubbard-Stratonovich transformation we introduce the following response non-physical variables and linearize the Langevin noises:

$$\mathcal{W}(\eta_j, \eta_f^\alpha) \propto \int \prod_i D[i\Upsilon_i] \prod_\alpha D[i\Phi^\alpha] e^{\int dt \int d^d\mathbf{r}[L\Upsilon_i(\mathbf{r},t)^2 - \Upsilon_i(\mathbf{r},t)\eta_i(\mathbf{r},t) + L_f\Phi^\alpha(\mathbf{r},t)^2 - \Phi^\alpha(\mathbf{r},t)\eta_f^\alpha(\mathbf{r},t)]} \tag{4.29}$$

Via a Fourier Transformation we finally arrive to:

$$\begin{aligned}\mathcal{W}(\eta_j, \eta_f^\alpha, \Upsilon_i, \Phi^\alpha) &\propto e^{-\mathcal{A}_{MSRJD}} \\ &\equiv e^{\int d\omega A\cdot T \sum_{\mathbf{q}}[L|\Upsilon_i(\mathbf{q},\omega)|^2 - \Upsilon_i(\mathbf{q},\omega)\eta_i(-\mathbf{q},-\omega) + L_f|\Phi^\alpha(\mathbf{q},\omega)|^2 - \Phi^\alpha(\mathbf{q},\omega)\eta_f^\alpha(-\mathbf{q},-\omega)]}\end{aligned} \tag{4.30}$$

where A is the area of the system and T is the volume of time. For completeness, the corresponding Fourier equations of the noises are the following:

$$\eta_j(\mathbf{q},\omega) = DC_{ijkl}q_i[q_k u_l(\mathbf{q},\omega) - \frac{i}{2}\sum_{\mathbf{p},\gamma} p_k(p_l - q_l)f^\alpha(\mathbf{p},\gamma)f^\alpha(\mathbf{q}-\mathbf{p},\omega-\gamma)] - i\omega u_j(\mathbf{q},\omega) \tag{4.31}$$

$$\begin{aligned}\eta_f^\alpha(\mathbf{q},\omega) &= (D_f[\kappa q^4 + \sigma q^2] - i\omega)f^\alpha(\mathbf{q},\omega) \\ &\quad + D_f C_{ijkl}q_i\Bigg[\sum_{(\mathbf{p},\gamma)\neq\mathbf{0}}\Bigg(ip_k u_l(\mathbf{p},\gamma) - \frac{1}{2}\sum_{\mathbf{z},\xi}(p_k - z_k)z_l f^\beta(\mathbf{p}-\mathbf{z},\gamma-\xi)f^\beta(\mathbf{z},\xi)\Bigg) \\ &\quad (q_j - p_j)f^\alpha(\mathbf{q}-\mathbf{p},\omega-\gamma)\Bigg]\end{aligned} \tag{4.32}$$

The MSRJD action, $\mathcal{A}_{MSRJD}$, possesses the following harmonic terms in matrix form:

$$\mathcal{A}^{harm.}_{MSRJD} = \frac{1}{2}\begin{bmatrix}\Upsilon_j(\mathbf{q},\omega)\\ u_j(\mathbf{q},\omega)\end{bmatrix}^T \begin{bmatrix}-2L\delta_{jl} & i\omega\delta_{jl}+DC_{ijkl}q_iq_k\\ -i\omega\delta_{jl}+DC_{ilkj}q_iq_k & 0\end{bmatrix}\begin{bmatrix}\Upsilon_l(-\mathbf{q},-\omega)\\ u_l(-\mathbf{q},-\omega)\end{bmatrix}$$
$$+\frac{\delta_{\alpha\beta}}{2}\begin{bmatrix}\Phi^\alpha(\mathbf{q},\omega)\\ f^\alpha(\mathbf{q},\omega)\end{bmatrix}^T \begin{bmatrix}-2L_f & i\omega+D_f\kappa q^4+D_f\sigma q^2\\ -i\omega+D_f\kappa q^4+D_f\sigma q^2 & 0\end{bmatrix}\begin{bmatrix}\Phi^\beta(-\mathbf{q},-\omega)\\ f^\beta(-\mathbf{q},-\omega)\end{bmatrix} \tag{4.33}$$

We can further simplify these equations via two observations. Firstly, we can scale out κ via the following transformation:

$$\begin{aligned}&\{\Phi^\alpha, f^\alpha, C_{ijkl}, u_j, \Upsilon_j, D, L, D_f, L_f, \sigma\}\rightarrow\\ &\{\sqrt{\kappa}\Phi^\alpha, \frac{1}{\sqrt{\kappa}}f^\alpha, C_{ijkl}\kappa^2, u_j/\kappa, \kappa\Upsilon_j, \frac{1}{\kappa^2}D, \frac{1}{\kappa^2}L, D_f/\kappa, L_f/\kappa, \sigma\kappa\}\end{aligned} \tag{4.34}$$

The second simplifying step can be taken by recognizing that the absolute value of the critical temperature of the theory, L_f/D_f, is not important [72]. Thus one can rescale all the order parameters and diffusivities such that L_f disappears from the equations:

$$\begin{aligned}&\{\Phi^\alpha, f^\alpha, C_{ijkl}, u_j, \Upsilon_j, D, L\} =\\ &\{\sqrt{\frac{D_f}{L_f}}\tilde{\Phi}^\alpha, \sqrt{\frac{L_f}{D_f}}\tilde{f}^\alpha, \frac{D_f}{L_f}\tilde{C}_{ijkl}, \frac{L_f}{D_f}\tilde{u}_j, \frac{D_f}{L_f}\tilde{\Upsilon}_j, \frac{L_f}{D_f}\tilde{D}, \left(\frac{L_f}{D_f}\right)^2\tilde{L}\}\end{aligned} \tag{4.35}$$

Hence, the harmonic theory adjusts as follows:

$$\mathcal{A}_{MSRJD}^{harm.} = \frac{1}{2}\begin{bmatrix}\tilde{\Upsilon}_j(\mathbf{q},\omega)\\ \tilde{u}_j(\mathbf{q},\omega)\end{bmatrix}^T \begin{bmatrix}-2\tilde{L}\delta_{jl} & i\omega\delta_{jl}+\tilde{D}\tilde{C}_{ijkl}q_iq_k\\ -i\omega\delta_{jl}+\tilde{D}\tilde{C}_{ilkj}q_iq_k & 0\end{bmatrix}\begin{bmatrix}\tilde{\Upsilon}_l(-\mathbf{q},-\omega)\\ \tilde{u}_l(-\mathbf{q},-\omega)\end{bmatrix}$$
$$+\frac{\delta_{\alpha\beta}}{2}\begin{bmatrix}\tilde{\Phi}^\alpha(\mathbf{q},\omega)\\ \tilde{f}^\alpha(\mathbf{q},\omega)\end{bmatrix}^T\begin{bmatrix}-2D_f & i\omega+D_fq^4+D_f\sigma q^2\\ -i\omega+D_fq^4+D_f\sigma q^2 & 0\end{bmatrix}\begin{bmatrix}\tilde{\Phi}^\beta(-\mathbf{q},-\omega)\\ \tilde{f}^\beta(-\mathbf{q},-\omega)\end{bmatrix} \tag{4.36}$$

Despite having introduced these extra Hubbard-Stratonovich variables, a key simplifying feature of the MSRJD approach can be seen from the form of these matrices: Feynman diagrams which contain the contraction of two response variables together will be null [72, 59]. Thus, such Feynman diagrams need not be considered. But before obtaining information about the renormalization factors and the Feynman diagrams, we first review the scaling of the harmonic theory.

4.2.4 Scaling of Harmonic Theory

We briefly discuss the scaling of the harmonic theory. Since we have two order parameters with different dispersion relations, we are first confronted with how to rescale frequencies. To resolve this we must go to the propagators in the linear theory:

$$\langle\tilde{\Phi}^\alpha(\mathbf{q},\omega)\tilde{f}^\alpha(-\mathbf{q},-\omega)\rangle \sim \frac{1}{A[-i\omega+D_f(q^4+\sigma q^2)]} \tag{4.37}$$

where we will make the assumptions of vanishing stress, $\sigma q^2 << \kappa q^4$, and

$$\langle\tilde{\Upsilon}_j(\mathbf{q},\omega)\tilde{u}_l(-\mathbf{q},-\omega)\rangle \sim \frac{1}{A[-i\omega\delta_{jl}+\tilde{D}\tilde{C}_{ijkl}q_iq_k]} \tag{4.38}$$

From this we see that in the small stress limit, $\omega \sim q^4$ in the flexural response propagator and $\omega \sim q^2$ for the in-plane phonon response propagator. Thus in the small frequency limit, the flexural modes are much slower and thus in-plane phonons should be considered as "fast" variables in the field theoretic sense. Thus, the terms that establish the harmonic theory in $\mathcal{A}_{MSRJD}$ are:

$$\tilde{\Phi}^{\alpha}\partial_t\tilde{f}^{\alpha}, \tilde{\Phi}^{\alpha}\nabla^4\tilde{f}^{\alpha}, \tilde{\Phi}^{\alpha 2} \tag{4.39}$$

This makes the coefficients of these terms automatically scale-invariant. We now perform a power counting procedure to obtain the scaling of the theory and render the action, $\mathcal{A}_{MSRJD}$, massless [57]. Thus if we assign scale powers in a momentum-shell sense [57]:

$$\mathbf{r} \to b\mathbf{r}, t \to b^{\zeta_t}t, \tilde{f}^{\alpha} \to b^{\zeta_f}\tilde{f}^{\alpha}, \tilde{\Phi}^{\alpha} \to b^{\zeta_\Phi}\tilde{\Phi}^{\alpha} \tag{4.40}$$

then we obtain that:

$$\zeta_t = 4, \zeta_f = \frac{-D+\zeta_t}{2}, \zeta_\Phi = \frac{-D-\zeta_t}{2} \tag{4.41}$$

Though we are taking $D = 2$, we leave D un-inserted to show the general scaling. If we also assign the following rescaling factors:

$$\tilde{u}_i \to b^{\zeta_u}\tilde{u}_i, \tilde{\Upsilon}_i \to b^{\zeta_\Upsilon}\tilde{\Upsilon}_i \tag{4.42}$$

then via the form of the strain tensor we also obtain:

$$\zeta_u = 2\zeta_f - 1 = -D + \zeta_t - 1 \tag{4.43}$$

and thus via observing that $\tilde{\Upsilon}_i\partial_t\tilde{u}_i$ must also be scale invariant we obtain:

$$\zeta_\Upsilon = -D - 1 \tag{4.44}$$

The linear scaling of the theory thus results in the following mass dimensions:

$$\tilde{C}_{ijkl} \to b^{4-D}\tilde{C}_{ijkl}, \tilde{L}, \tilde{D} \to b^{D-2}\tilde{L}, \tilde{D}, \sigma \to b^2\sigma \tag{4.45}$$

This establishes that the upper critical dimension for $\tilde{C}_{ijkl}$ is 4.

4.2.5 Absence of Ward Identity

An important tool in the renormalization group are the use of symmetries of the action. In the case of the free energy associated with equilibrium fluctuating elastic membranes, 3.2, Guitter et al. [13] established a symmetry of both the strain tensor:

$$\begin{aligned} f^\alpha(\mathbf{r}) &\to f^\alpha(\mathbf{r}) + A_i^\alpha r_i \\ u_i(\mathbf{r}) &\to u_i(\mathbf{r}) - A_i^\alpha f^\alpha(\mathbf{r}) - \frac{1}{2}A_i^\alpha A_j^\alpha r_j \end{aligned} \tag{4.46}$$

which helps to establish the Ward identity associated with the effective action, Γ [60]:

$$\int d^D\mathbf{r}\left[r_i\frac{\delta\Gamma}{\delta f^\alpha} - f^\alpha\frac{\delta\Gamma}{\delta u_i}\right] = 0 \tag{4.47}$$

This Ward identity spares extra calculations as it enforces that the coefficient of the $\partial_i u_j \partial_k u_l$ vertex will renormalize exactly as $\partial_i u_j \partial_k f^\beta \partial_l f^\beta$ and $\partial_i f^\alpha \partial_j f^\alpha \partial_k f^\beta \partial_l f^\beta$. Thus one need only renormalize one of these vertices to obtain the renormalization of the in-plane elastic constants, C_{ijkl}.

Because the symmetry [13] obtained is a symmetry of the strain tensor, as soon as one incorporates components that prevent an elastic action from being entirely formulated in u_{ij} and $\nabla^2 f$, we lose the symmetry and its associated Ward identity. Indeed the only dynamic term that could be added to a free energy should be of the form $(\partial_t u_{ij})^2$, which would lead to non-physical forces and equations that we wouldn't derive kinematically. One need not the formalism of the Ward identity to observe this either. From Eq. (4.27) and Eq. (4.26), since the kinematic condition is not necessarily satisfied, $\partial_i \sigma_{ij} = C_{ijkl}\partial_i u_{kl} \neq 0$, Eq. 4.46 is no longer a symmetry of the dynamic equations. This holds whether we are working with the odd elastic equations or not. Thus [83] is also missing the Ward identity. In the case of equilibrium Langevin elastic membranes, Feynman diagrams will retain the same effective structure as in the case of [13] and thus they derive the matching results. However, for odd elastic systems where there is no analogue and the structure of elastic tensors has been generalized, this is no longer a guarantee. Thus, without a Ward identity, further care will be required because many more Feynman diagrams must be calculated.

One immediate consequence of the lack of the Ward identity means that renormalization may not preserve equality between the coefficients of vertices in Eq. (4.26) and Eq. (4.27), potentially resulting in breaking any microscopic fluctuation-dissipation. Thus these equations must be generalized into the following form:

$$\partial_t \tilde{u}_j(\mathbf{r},t) = \tilde{C}^{u,D}_{ijkl}\partial_i\partial_k \tilde{u}_l(\mathbf{r},t) + \frac{1}{2}\tilde{C}^{f,D}_{ijkl}\partial_i(\partial_k \tilde{f}^\alpha(\mathbf{r},t)\partial_l \tilde{f}^\alpha(\mathbf{r},t)) + \eta_j(\mathbf{r},t) \tag{4.48}$$

$$\begin{aligned}\partial_t \tilde{f}^\alpha(\mathbf{r},t) = [-D_f\Delta^2 + \tilde{\sigma}\Delta]\tilde{f}^\alpha(\mathbf{r},t) + \tilde{C}^{u,D_f}_{ijkl}\partial_i[\partial_k \tilde{u}_l(\mathbf{r},t)\partial_j \tilde{f}^\alpha(\mathbf{r},t)] \\ + \tilde{C}^{f,D_f}_{ijkl}\partial_i[\frac{1}{2}\partial_k \tilde{f}^\beta(\mathbf{r},t)\partial_l \tilde{f}^\beta(\mathbf{r},t)\partial_j \tilde{f}^\alpha(\mathbf{r},t)] + \eta^\alpha_f(\mathbf{r},t)\end{aligned} \tag{4.49}$$

where one can observe that the diffusivities and elastic tensor have been combined and $\tilde{\sigma} = D_f\sigma$. Consequently the MSRJD action must also be generalized accordingly. We re-state them in their final form before commencing the renormalization scheme:

$$\eta_j(\mathbf{q},\omega) = q_i[\tilde{C}^{u,D}_{ijkl} q_k \tilde{u}_l(\mathbf{q},\omega) - \frac{i}{2}\tilde{C}^{f,D}_{ijkl}\sum_{\mathbf{p},\gamma} p_k(p_l - q_l)\tilde{f}^\alpha(\mathbf{p},\gamma)\tilde{f}^\alpha(\mathbf{q}-\mathbf{p},\omega-\gamma)] - i\omega\tilde{u}_j(\mathbf{q},\omega) \tag{4.50}$$

$$\eta_f^\alpha(\mathbf{q},\omega) = ([D_f q^4 + \tilde{\sigma} q^2] - i\omega)\tilde{f}^\alpha(\mathbf{q},\omega)$$
$$+ q_i\Bigg[\sum_{(\mathbf{p},\gamma)\neq\mathbf{0}}\Bigg(i\tilde{C}^{u,D_f}_{ijkl}p_k u_l(\mathbf{p},\gamma) - \frac{\tilde{C}^{f,D_f}_{ijkl}}{2}\sum_{\mathbf{z},\xi}(p_k - z_k)z_l\tilde{f}^\beta(\mathbf{p}-\mathbf{z},\gamma-\xi)\tilde{f}^\beta(\mathbf{z},\xi)\Bigg)$$
$$(q_j - p_j)\tilde{f}^\alpha(\mathbf{q}-\mathbf{p},\omega-\gamma)\Bigg] \tag{4.51}$$

$$\mathcal{W}(\eta_j, \eta_f^\alpha, \Upsilon_i, \tilde{\Phi}^\alpha) \propto e^{-\mathcal{A}_{MSRJD}} = e^{\int d\omega A\cdot T\sum_{\mathbf{q}}[\tilde{L}|\tilde{\Upsilon}_i(\mathbf{q},\omega)|^2 - \tilde{\Upsilon}_i(\mathbf{q},\omega)\eta_i(-\mathbf{q},-\omega) + D_f|\tilde{\Phi}^\alpha(\mathbf{q},\omega)|^2 - \tilde{\Phi}^\alpha(\mathbf{q},\omega)\eta_f^\alpha(-\mathbf{q},-\omega)]} \tag{4.52}$$

with the harmonic portion of the action taking the form:

$$\mathcal{A}^{harm.}_{MSRJD} = \frac{1}{2}\begin{bmatrix}\tilde{\Upsilon}_j(\mathbf{q},\omega)\\ \tilde{u}_j(\mathbf{q},\omega)\end{bmatrix}^T\begin{bmatrix}-2\tilde{L}\delta_{jl} & i\omega\delta_{jl} + \tilde{C}^{u,D}_{ijkl}q_i q_k\\ -i\omega\delta_{jl} + \tilde{C}^{u,D}_{ijkl}q_i q_k & 0\end{bmatrix}\begin{bmatrix}\tilde{\Upsilon}_l(-\mathbf{q},-\omega)\\ \tilde{u}_l(-\mathbf{q},-\omega)\end{bmatrix}$$
$$+ \frac{\delta_{\alpha\beta}}{2}\begin{bmatrix}\tilde{\Phi}^\alpha(\mathbf{q},\omega)\\ \tilde{f}^\alpha(\mathbf{q},\omega)\end{bmatrix}^T\begin{bmatrix}-2D_f & i\omega + D_f q^4 + \tilde{\sigma}q^2\\ -i\omega + D_f q^4 + \tilde{\sigma}q^2 & 0\end{bmatrix}\begin{bmatrix}\tilde{\Phi}^\beta(-\mathbf{q},-\omega)\\ \tilde{f}^\beta(-\mathbf{q},-\omega)\end{bmatrix} \tag{4.53}$$

4.3 Renormalization of Over-Damped Odd Elastic Membranes

4.3.1 Feynman Diagrams and Renormalization Group Equations

With the set up of the equations complete, we can commence the process of renormalization. Each term can be renormalized by the contraction of Taylor-expanded an-harmonicities. We intend to show the calculations diagrammatically with an attached Mathematica code (found in Sec. C in the appendix) which performs the accompanying analytic calculations. We begin by representing an-harmonic terms diagrammatically in isolation. This is done in Fig. 4.1 (a),(b) and (c). By further integrating out the harmonic in-plane field using Eq. (4.52) and Eq. (4.53), one can also obtain effective vertices of the flexural field shown in (d) and (e). This is analytically done in the attached Mathematica code, but is too complicated to put in closed form in text. The effective flexural vertices will aid us by allowing us to calculate less Feynman diagrams in total.

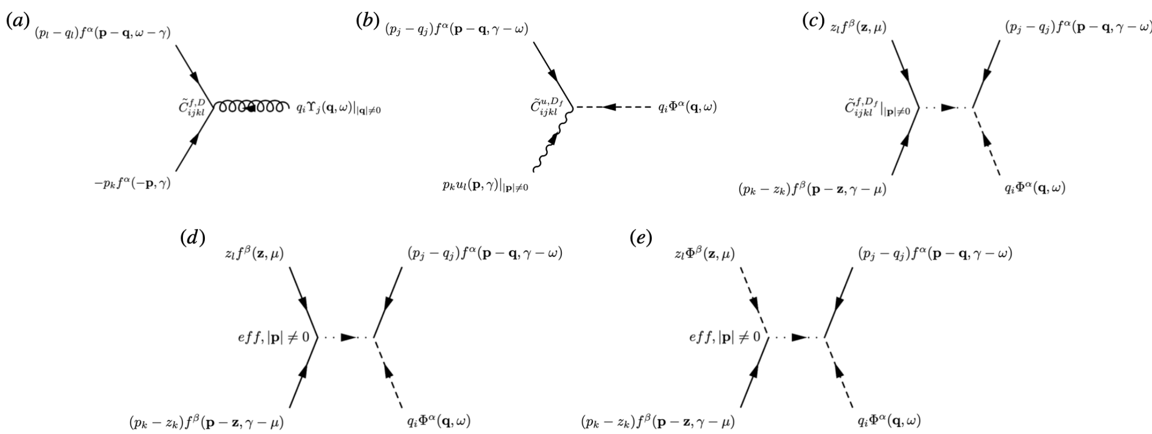

Figure 4.1: The Feynman diagrams corresponding to the linear terms in equations Eq. (4.50) and Eq. (4.51) are shown in (a), (b) and (c). If one integrates out the in-plane order parameters $\tilde{u}_j$, $\tilde{\Upsilon}_j$, then one obtains in effective flexural vertices shown in (d) and (e).

We perform a 1-loop perturbative momentum-shell renormalization group scheme to leading order in d_c. The 1-particle-irreducible diagrams included in such a scheme are given in Fig. 4.2 [60, 72, 97, 83]. Other diagrams in a 1-loop scheme are ignored as they are lower in order d_c, such as that found in Fig. 4.3(c). However, such diagrams would have to be hypothetically included in an ϵ-expansion analysis, as all 1-loop diagrams are of order ϵ. Thus further investigation is merited in exploring these other potential Feynman diagrams that we have ignored here. Other diagrams such as Fig. 4.3(a,b) can be ignored on the grounds that they produce only higher-order-wavelength contributions to the original vertices and thus are respectively irrelevant.

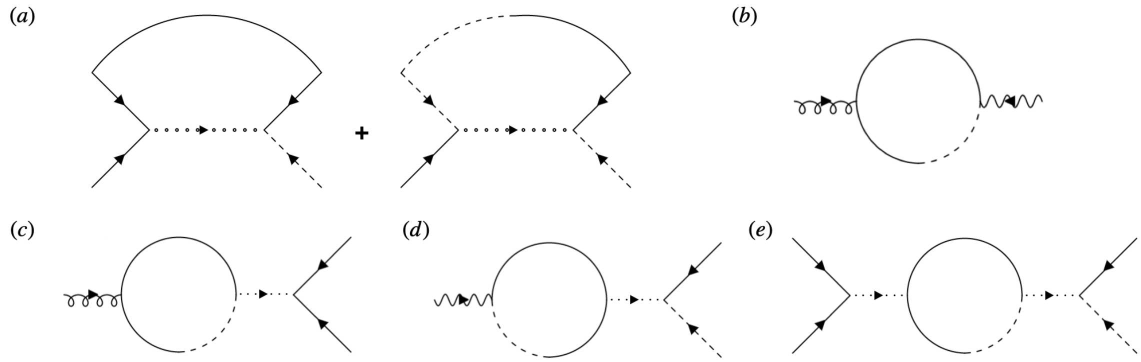

Figure 4.2: The Feynman diagrams corresponding to a 1-loop renormalization group to leading order in d_c are shown. Diagrams in (a) renormalize D_f and $\tilde{\sigma}$, (b) $\tilde{C}^{u,D}_{ijkl}$, (c) $\tilde{C}^{f,D}_{ijkl}$, (d) $\tilde{C}^{u,D_f}_{ijkl}$, (e) $\tilde{C}^{f,D_f}_{ijkl}$.

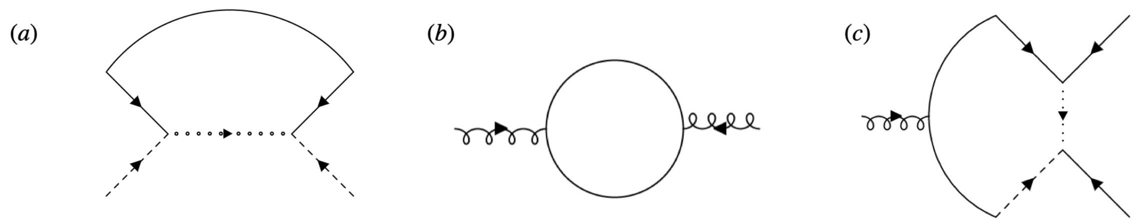

Figure 4.3: Examples of numerous Feynman diagrams that have been ignored are shown. (a) renormalizes $(\tilde{\Phi}^{\alpha})^2$ to non-zero orders of the wave-vector and thus would be irrelevant with respect to its scalar coefficient. (b) is ignored on basis of the same argument with respect to $\tilde{\Upsilon}_j^2$. (c) Diagrams of this type have been ignored as they contribute to $\tilde{C}^{f,D}_{ijkl}$ to a lower order in d_c than the diagram found in Fig. 4.2(c).

Considering these Feynman diagrams, we can write down the renormalization factors in order to calculate our renormalization group equations. To remove the UV divergences due to an-harmonic Feynman diagrams and reformulate the theory in terms of renormalized variables, we make the following ansatz of how to rescale the theory:

$$\tilde{u}_i^R = Z^{-1}\tilde{u}_i, \tilde{f}^{\alpha,R} = Z_f^{-1/2}\tilde{f}^{\alpha},$$
$$\tilde{\Upsilon}^R = Z_{\Upsilon}^{-1}\tilde{\Upsilon}, \tilde{\Phi}^{\alpha,R} = Z_{\Phi}^{-1/2}\tilde{\Phi}^{\alpha} \tag{4.54}$$

$$D_f^R = Z_{D_f} D_f \tag{4.55}$$

$$\tilde{L}^R = Z_L \tilde{L} \tag{4.56}$$

$$\tilde{C}_{ijkl}^{u,D,R} = Z_{ijkl}^{u,D}\tilde{C}_{ijkl}^{u,D} \tag{4.57}$$

$$\tilde{C}_{ijkl}^{f,D,R} = Z_{ijkl}^{f,D}\tilde{C}_{ijkl}^{f,D} \tag{4.58}$$

$$\tilde{C}_{ijkl}^{u,D_f,R} = Z_{D_f}^{-1} Z_{ijkl}^{u,D_f}\tilde{C}_{ijkl}^{u,D_f} \tag{4.59}$$

$$\tilde{C}_{ijkl}^{f,D_f,R} = Z_{D_f}^{-1} Z_{ijkl}^{f,D_f}\tilde{C}_{ijkl}^{f,D_f} \tag{4.60}$$

$$\tilde{\sigma}^R = Z_\sigma \tilde{\sigma} \tag{4.61}$$

where we have used our scale transformations in Eq. 4.35 to account for how the diffusivity, D_f, has been absorbed into the tensors $\tilde{C}_{ijkl}^{u,D_f}, \tilde{C}_{ijkl}^{f,D_f}$ and the stress $\tilde{\sigma}$. From this ansatz, similar as to [83], we can obtain certain reductions in the number of independent renormalization factors. We obtain symmetries of $\Gamma_{M,\tilde{M},N,\tilde{N}}$ which is the effective vertex function with M f fields, $\tilde{M}$ Φ fields, N u fields and $\tilde{N}$ Υ fields as in [83, 60]. Since:

$$\partial_\omega \Gamma_{1100}(\mathbf{q}=0,\omega) \sim i\sqrt{Z_f Z_\Phi}$$
$$\partial_\omega \Gamma_{0011}(\mathbf{q}=0,\omega) \sim i Z_u Z_\Upsilon \tag{4.62}$$

and since all an-harmonic vertices vanish when $\mathbf{q} = 0$, we obtain that $Z = 1/Z_\Upsilon$ and $Z_f = 1/Z_\Phi$. Furthermore:

$$\Gamma_{0200}(\mathbf{q} = 0, \omega) \sim Z_{D_f} Z_\Phi \tag{4.63}$$

which also implies that $Z_{D_f} = 1/Z_\Phi$. Thus we have established that $Z_f = Z_{D_f}$. However, unlike [83], we cannot use any Ward identity as we have shown that it is not valid for a dynamical action and thus we cannot establish that $Z = Z_f$. This is an important point because without the Ward identity, we cannot establish further simplifications of the following renormalization factors $Z^{u,D}_{ijkl}, Z^{f,D}_{ijkl}, Z^{u,D_f}_{ijkl}, Z^{f,D_f}_{ijkl}$. Thus we must treat these renormalization factors as independent. With renormalization factors established, we can write down the following renormalization group ODEs for the following parameters with the use of our evaluated Feynman diagrams:

$$\beta_{\tilde{\mu}^{u,D,R}} = 2\tilde{\mu}^{u,D,R} - d_c D^R_f \Lambda^2 \frac{\tilde{\mu}^{f,D,R}\tilde{\mu}^{u,D_f,R} - \tilde{K}^{f,D,R}_{odd}\tilde{K}^{u,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2} \tag{4.64}$$

$$\beta_{\tilde{\lambda}^{u,D,R}} = 2\tilde{\lambda}^{u,D,R}$$

$$- d_c D^R_f \Lambda^2 \frac{2\tilde{\lambda}^{f,D,R}(\tilde{\lambda}^{u,D_f,R} + \tilde{\mu}^{u,D_f,R}) + \tilde{\mu}^{f,D,R}(2\tilde{\lambda}^{u,D_f,R} + \tilde{\mu}^{u,D_f,R}) - \tilde{K}^{f,D,R}_{odd}\tilde{K}^{u,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$\beta_{\tilde{K}^{u,D,R}_{odd}} = 2\tilde{K}^{u,D,R}_{odd} - d_c D^R_f \Lambda^2 \frac{\tilde{K}^{f,D,R}_{odd}\tilde{\mu}^{u,D_f,R} + \tilde{\mu}^{f,D,R}\tilde{K}^{u,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$\beta_{\tilde{A}^{u,D,R}_{odd}} = 2\tilde{A}^{u,D,R}_{odd} - d_c D^R_f \Lambda^2 \frac{\tilde{A}^{f,D,R}_{odd}(\tilde{\mu}^{u,D_f,R} + \tilde{\lambda}^{u,D_f,R})}{4\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$\beta_{\tilde{\mu}^{u,D_f,R}} = 2\tilde{\mu}^{u,D_f,R} - d_c D^R_f \Lambda^2 \frac{\tilde{\mu}^{f,D_f,R}\tilde{\mu}^{u,D_f,R} - \tilde{K}^{f,D_f,R}_{odd}\tilde{K}^{u,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2} - \frac{\tilde{\mu}^{u,D_f,R}}{D^R_f}\frac{\partial log Z_{D_f,R}}{\partial b}$$

$$\beta_{\tilde{\lambda}^{u,D_f,R}} = 2\tilde{\lambda}^{u,D_f,R}$$

$$- d_c D^R_f \Lambda^2 \frac{2\tilde{\lambda}^{f,D_f,R}(\tilde{\lambda}^{u,D_f,R} + \tilde{\mu}^{u,D_f,R}) + \tilde{\mu}^{f,D_f,R}(2\tilde{\lambda}^{u,D_f,R} + \tilde{\mu}^{u,D_f,R}) - \tilde{K}^{f,D_f,R}_{odd}\tilde{K}^{u,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$- \frac{\tilde{\lambda}^{u,D_f,R}}{D^R_f}\frac{\partial log Z_{D_f,R}}{\partial b}$$

$$\beta_{\tilde{K}^{u,D_f,R}_{odd}} = 2\tilde{K}^{u,D_f,R}_{odd} - d_c D^R_f \Lambda^2 \frac{\tilde{K}^{f,D_f,R}_{odd}\tilde{\mu}^{u,D_f,R} + \tilde{\mu}^{f,D_f,R}\tilde{K}^{u,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2} - \frac{\tilde{K}^{u,D_f,R}_{odd}}{D^R_f}\frac{\partial log Z_{D_f,R}}{\partial b}$$

$$\beta_{\tilde{A}^{u,D_f,R}_{odd}} = 2\tilde{A}^{u,D_f,R}_{odd} - d_c D^R_f \Lambda^2 \frac{\tilde{A}^{f,D_f,R}_{odd}(\tilde{\mu}^{u,D_f,R} + \tilde{\lambda}^{u,D_f,R})}{4\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2} - \frac{\tilde{A}^{u,D_f,R}_{odd}}{D^R_f}\frac{\partial log Z_{D_f}}{\partial b}$$

$$\beta_{\tilde{\mu}^{f,D,R}} = 2\tilde{\mu}^{f,D,R} - d_c D^R_f \Lambda^2 \frac{\tilde{\mu}^{f,D,R}\tilde{\mu}^{f,D_f,R} - \tilde{K}^{f,D,R}_{odd}\tilde{K}^{f,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$\beta_{\tilde{\lambda}^{f,D,R}} = 2\tilde{\lambda}^{f,D,R}$$

$$- d_c D^R_f \Lambda^2 \frac{2\tilde{\lambda}^{f,D,R}(\tilde{\lambda}^{f,D_f,R} + \tilde{\mu}^{f,D_f,R}) + \tilde{\mu}^{f,D,R}(2\tilde{\lambda}^{f,D_f,R} + \tilde{\mu}^{f,D_f,R}) - \tilde{K}^{f,D,R}_{odd}\tilde{K}^{f,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$\beta_{\tilde{K}^{f,D,R}_{odd}} = 2\tilde{K}^{f,D,R}_{odd} - d_c D^R_f \Lambda^2 \frac{\tilde{K}^{f,D,R}_{odd}\tilde{\mu}^{f,D_f,R} + \tilde{\mu}^{f,D,R}\tilde{K}^{f,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$\beta_{\tilde{A}^{f,D,R}_{odd}} = 2\tilde{A}^{f,D,R}_{odd} - d_c D^R_f \Lambda^2 \frac{\tilde{A}^{f,D,R}_{odd}(\tilde{\mu}^{f,D_f,R} + \tilde{\lambda}^{f,D_f,R})}{4\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$\beta_{\tilde{\mu}^{f,D_f,R}} = 2\tilde{\mu}^{f,D_f,R} - d_c D^R_f \Lambda^2 \frac{\tilde{\mu}^{f,D_f,R}\tilde{\mu}^{f,D_f,R} - \tilde{K}^{f,D_f,R}_{odd}\tilde{K}^{f,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2} - \frac{\tilde{\mu}^{f,D_f,R}}{D^R_f}\frac{\partial log Z_{D_f}}{\partial b}$$

$$\beta_{\tilde{\lambda}^{f,D_f,R}} = 2\tilde{\lambda}^{f,D_f,R}$$

$$- d_c D^R_f \Lambda^2 \frac{2\tilde{\lambda}^{f,D_f,R}(\tilde{\lambda}^{f,D_f,R} + \tilde{\mu}^{f,D_f,R}) + \tilde{\mu}^{f,D_f,R}(2\tilde{\lambda}^{f,D_f,R} + \tilde{\mu}^{f,D_f,R}) - \tilde{K}^{f,D_f,R}_{odd}\tilde{K}^{f,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2}$$

$$- \frac{\tilde{\lambda}^{f,D_f,R}}{D^R_f}\frac{\partial log Z_{D_f}}{\partial b}$$

$$\beta_{\tilde{K}^{f,D_f,R}_{odd}} = 2\tilde{K}^{f,D_f,R}_{odd} - d_c D^R_f \Lambda^2 \frac{\tilde{K}^{f,D_f,R}_{odd}\tilde{\mu}^{f,D_f,R} + \tilde{\mu}^{f,D_f,R}\tilde{K}^{f,D_f,R}_{odd}}{8\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2} - \frac{\tilde{K}^{f,D_f,R}_{odd}}{D^R_f}\frac{\partial log Z_{D_f}}{\partial b}$$

$$\beta_{\tilde{A}^{f,D_f,R}_{odd}} = 2\tilde{A}^{f,D_f,R}_{odd} - d_c D^R_f \Lambda^2 \frac{\tilde{A}^{f,D_f,R}_{odd}(\tilde{\mu}^{f,D_f,R} + \tilde{\lambda}^{f,D_f,R})}{4\pi[D^R_f\Lambda^2 + \tilde{\sigma}^R]^2} - \frac{\tilde{A}^{f,D_f,R}_{odd}}{D^R_f}\frac{\partial log Z_{D_f}}{\partial b}$$

$$\beta_{\tilde{\sigma}^R} = \frac{\partial log Z_\sigma}{\partial b}$$

$$\beta_{\tilde{L}^R} = 2\frac{\tilde{L}^R}{D^R_f}\frac{\partial log Z_{D_f}}{\partial b}$$

$$\beta_{D^R_f} = \frac{\partial log Z_{D_f}}{\partial b} \tag{4.65}$$

where we have used that $Z_L = 1$ since no Feynman diagrams contribute to zero-th order in the wave vectors to Υ^2_j. Furthermore Λ is the UV cutoff, in other words the

Fourier wave-vector corresponding to the microscopic length scale of the theory (for example, the lattice spacing of the system). The parameter $b = e^s$ where s is the re-scaling parameter [58, 60, 57].

Stability Analysis

Upon these above equations, we may perform a stability analysis. We are interested in having only physically relevant parameters. Thus we further reduce the equations so that the only parameters considered are:

$$\begin{aligned}&\{\tilde{C}_{ijkl}^{u,D,R}/\tilde{L}^R, \tilde{C}_{ijkl}^{f,D,R}/\tilde{L}^R, \tilde{C}_{ijkl}^{u,D_f,R}/D_f^R, \tilde{C}_{ijkl}^{f,D_f,R}/D_f^R, \tilde{\sigma}^R/D_f^R\}\\ &= \{\hat{\tilde{C}}_{ijkl}^{u,D,R}, \hat{\tilde{C}}_{ijkl}^{f,D,R}, \hat{\tilde{C}}_{ijkl}^{u,D_f,R}, \hat{\tilde{C}}_{ijkl}^{f,D_f,R}, \hat{\tilde{\sigma}}^R\}\end{aligned} \tag{4.66}$$

The equations derived are then:

$$\beta_{\hat{\tilde{\mu}}^{u,D,R}} = 2\hat{\tilde{\mu}}^{u,D,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\tilde{\mu}}^{f,D,R}\hat{\tilde{\mu}}^{u,D_f,R} - \hat{\tilde{K}}_{odd}^{f,D,R}\hat{\tilde{K}}_{odd}^{u,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\tilde{\sigma}}^R]^2} - 2\frac{\hat{\tilde{\lambda}}^{u,D,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\begin{aligned}\beta_{\hat{\tilde{\lambda}}^{u,D,R}} &= 2\hat{\tilde{\lambda}}^{u,D,R}\\ &- d_c(D_f^R)^2\Lambda^2\frac{2\hat{\tilde{\lambda}}^{f,D,R}(\hat{\tilde{\lambda}}^{u,D_f,R} + \hat{\tilde{\mu}}^{u,D_f,R}) + \hat{\tilde{\mu}}^{f,D,R}(2\hat{\tilde{\lambda}}^{u,D_f,R} + \hat{\tilde{\mu}}^{u,D_f,R}) - \hat{\tilde{K}}_{odd}^{f,D,R}\hat{\tilde{K}}_{odd}^{u,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\tilde{\sigma}}^R]^2}\\ &- 2\frac{\hat{\tilde{\lambda}}^{u,D,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}\end{aligned}$$

$$\beta_{\hat{\tilde{K}}_{odd}^{u,D,R}} = 2\hat{\tilde{K}}_{odd}^{u,D,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\tilde{K}}_{odd}^{f,D,R}\hat{\tilde{\mu}}^{u,D_f,R} + \hat{\tilde{\mu}}^{f,D,R}\hat{\tilde{K}}_{odd}^{u,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\tilde{\sigma}}^R]^2} - 2\frac{\hat{\tilde{K}}_{odd}^{u,D,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\tilde{A}}_{odd}^{u,D,R}} = 2\hat{\tilde{A}}_{odd}^{u,D,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\tilde{A}}_{odd}^{f,D,R}(\hat{\tilde{\mu}}^{u,D_f,R} + \hat{\tilde{\lambda}}^{u,D_f,R})}{4\pi[D_f^R\Lambda^2 + \hat{\tilde{\sigma}}^R]^2} - 2\frac{\hat{\tilde{A}}_{odd}^{u,D,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\tilde{\mu}}^{u,D_f,R}} = 2\hat{\tilde{\mu}}^{u,D_f,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\tilde{\mu}}^{f,D_f,R}\hat{\tilde{\mu}}^{u,D_f,R} - \hat{\tilde{K}}_{odd}^{f,D_f,R}\hat{\tilde{K}}_{odd}^{u,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\tilde{\sigma}}^R]^2} - 2\frac{\hat{\tilde{\mu}}^{u,D_f,R}}{D_f^R}\frac{\partial logZ_{D_f,R}}{\partial b}$$

$$\begin{aligned}\beta_{\hat{\tilde{\lambda}}^{u,D_f,R}} &= 2\hat{\tilde{\lambda}}^{u,D_f,R}\\ &- d_c(D_f^R)^2\Lambda^2\frac{2\hat{\tilde{\lambda}}^{f,D_f,R}(\hat{\tilde{\lambda}}^{u,D_f,R} + \hat{\tilde{\mu}}^{u,D_f,R}) + \hat{\tilde{\mu}}^{f,D_f,R}(2\hat{\tilde{\lambda}}^{u,D_f,R} + \hat{\tilde{\mu}}^{u,D_f,R}) - \hat{\tilde{K}}_{odd}^{f,D_f,R}\hat{\tilde{K}}_{odd}^{u,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\tilde{\sigma}}^R]^2}\end{aligned}$$

$$- 2\frac{\hat{\hat{\lambda}}^{u,D_f,R}}{D_f^R}\frac{\partial logZ_{D_f,R}}{\partial b}$$

$$\beta_{\hat{\hat{K}}_{odd}^{u,D_f,R}} = 2\hat{\hat{K}}_{odd}^{u,D_f,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\hat{K}}_{odd}^{f,D_f,R}\hat{\hat{\mu}}^{u,D_f,R} + \hat{\hat{\mu}}^{f,D_f,R}\hat{\hat{K}}_{odd}^{u,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2} - 2\frac{\hat{\hat{K}}_{odd}^{u,D_f,R}}{D_f^R}\frac{\partial logZ_{D_f,R}}{\partial b}$$

$$\beta_{\hat{\hat{A}}_{odd}^{u,D_f,R}} = 2\hat{\hat{A}}_{odd}^{u,D_f,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\hat{A}}_{odd}^{f,D_f,R}(\hat{\hat{\mu}}^{u,D_f,R} + \hat{\hat{\lambda}}^{u,D_f,R})}{4\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2} - 2\frac{\hat{\hat{A}}_{odd}^{u,D_f,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{\mu}}^{f,D,R}} = 2\hat{\hat{\mu}}^{f,D,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\hat{\mu}}^{f,D,R}\hat{\hat{\mu}}^{f,D_f,R} - \hat{\hat{K}}_{odd}^{f,D,R}\hat{\hat{K}}_{odd}^{f,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2} - 2\frac{\hat{\hat{\mu}}^{f,D,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{\lambda}}^{f,D,R}} = 2\hat{\hat{\lambda}}^{f,D,R}$$

$$- d_c(D_f^R)^2\Lambda^2\frac{2\hat{\hat{\lambda}}^{f,D,R}(\hat{\hat{\lambda}}^{f,D_f,R} + \hat{\hat{\mu}}^{f,D_f,R}) + \hat{\hat{\mu}}^{f,D,R}(2\hat{\hat{\lambda}}^{f,D_f,R} + \hat{\hat{\mu}}^{f,D_f,R}) - \hat{\hat{K}}_{odd}^{f,D,R}\hat{\hat{K}}_{odd}^{f,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2}$$

$$- 2\frac{\hat{\hat{\lambda}}^{f,D,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{K}}_{odd}^{f,D,R}} = 2\hat{\hat{K}}_{odd}^{f,D,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\hat{K}}_{odd}^{f,D,R}\hat{\hat{\mu}}^{f,D_f,R} + \hat{\hat{\mu}}^{f,D,R}\hat{\hat{K}}_{odd}^{f,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2} - 2\frac{\hat{\hat{K}}_{odd}^{f,D,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{A}}_{odd}^{f,D,R}} = 2\hat{\hat{A}}_{odd}^{f,D,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\hat{A}}_{odd}^{f,D,R}(\hat{\hat{\mu}}^{f,D_f,R} + \hat{\hat{\lambda}}^{f,D_f,R})}{4\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2} - 2\frac{\hat{\hat{A}}_{odd}^{f,D,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{\mu}}^{f,D_f,R}} = 2\hat{\hat{\mu}}^{f,D_f,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\hat{\mu}}^{f,D_f,R}\hat{\hat{\mu}}^{f,D_f,R} - \hat{\hat{K}}_{odd}^{f,D_f,R}\hat{\hat{K}}_{odd}^{f,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2} - 2\frac{\hat{\hat{\mu}}^{f,D_f,R}}{D_f, R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{\lambda}}^{f,D_f,R}} = 2\hat{\hat{\lambda}}^{f,D_f,R}$$

$$- d_c(D_f^R)^2\Lambda^2\frac{2\hat{\hat{\lambda}}^{f,D_f,R}(\hat{\hat{\lambda}}^{f,D_f,R} + \hat{\hat{\mu}}^{f,D_f,R}) + \hat{\hat{\mu}}^{f,D_f,R}(2\hat{\hat{\lambda}}^{f,D_f,R} + \hat{\hat{\mu}}^{f,D_f,R}) - \hat{\hat{K}}_{odd}^{f,D_f,R}\hat{\hat{K}}_{odd}^{f,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2}$$

$$- 2\frac{\hat{\hat{\lambda}}^{f,D_f,R}}{D_f, R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{K}}_{odd}^{f,D_f,R}} = 2\hat{\hat{K}}_{odd}^{f,D_f,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\hat{K}}_{odd}^{f,D_f,R}\hat{\hat{\mu}}^{f,D_f,R} + \hat{\hat{\mu}}^{f,D_f,R}\hat{\hat{K}}_{odd}^{f,D_f,R}}{8\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2} - 2\frac{\hat{\hat{K}}_{odd}^{f,D_f,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{A}}_{odd}^{f,D_f,R}} = 2\hat{\hat{A}}_{odd}^{f,D_f,R} - d_c(D_f^R)^2\Lambda^2\frac{\hat{\hat{A}}_{odd}^{f,D_f,R}(\hat{\hat{\mu}}^{f,D_f,R} + \hat{\hat{\lambda}}^{f,D_f,R})}{4\pi[D_f^R\Lambda^2 + \hat{\hat{\sigma}}^R]^2} - 2\frac{\hat{\hat{A}}_{odd}^{f,D_f,R}}{D_f^R}\frac{\partial logZ_{D_f}}{\partial b}$$

$$\beta_{\hat{\hat{\sigma}}^R} = \frac{\partial logZ_\sigma}{\partial b} \tag{4.67}$$

Formulated in terms of these variables, one in principle should solve for the fixed point/manifold. However, as we know $\frac{\partial log Z_{D_f}}{\partial b}$ is a complicated expression and thus renders obtainment of these solutions analytically intractable. In addition we note that, $\frac{\partial log Z_{\sigma}}{\partial b}$ is also quite complicated as an expression, however it indicates that stress is generated when fluctuation dissipation is broken or when odd elastic parameters are present. Despite the complexity of these expressions, if we define the Aronovitz-Lubensky fixed point as:

$$\{\hat{\tilde{\mu}}^{\gamma,\delta,R}, \hat{\tilde{\lambda}}^{\gamma,\delta,R}, \hat{\tilde{K}}_{odd}^{\gamma,\delta,R}, \hat{\tilde{A}}_{odd}^{\gamma,\delta,R}, \hat{\tilde{\sigma}}^{R}\} = \{\frac{16\pi\Lambda^2}{4+d_c}, \frac{-8\pi\Lambda^2}{4+d_c}, 0, 0, 0\} \tag{4.68}$$

where $\gamma \in \{u, f\}, \delta \in \{D, D_f\}$, one can check that it is indeed a fixed point of the equations and we can analyze its stability. Performing a stability analysis on these 16 variables gives the following eigen-values:

$$\text{Eigen-values} = \{0,0,0,0,0,0,0,0, \frac{-2d_c}{4+d_c}, \frac{-2d_c}{4+d_c}, -2, \frac{2d_c}{4+d_c}, \frac{2d_c}{4+d_c}, \frac{2d_c}{4+d_c}, \frac{2d_c}{4+d_c}, \frac{2d_c}{4+d_c}, \frac{2(2+d_c)}{4+d_c}\} \tag{4.69}$$

The eight zero eigen-values indicate that the Aronovitz-Lubensky fixed point potentially belongs to a higher-dimensional fixed manifold, although perhaps not a stable one. Four of these eight zero eigen-values are perturbations purely in $\{\hat{\tilde{\mu}}^{\gamma,\delta,R}, \hat{\tilde{\lambda}}^{\gamma,\delta,R}\}$ and three of them are somewhat analytically complicated, however they indicate that there are potentially a broader set of fixed points where fluctuation-dissipation may not necessarily hold. The one simple eigen-vector that is a pure perturbation in

$\{\hat{\hat{\mu}}^{\gamma,\delta,R}, \hat{\hat{\lambda}}^{\gamma,\delta,R}\}$ is:

$$[\delta\hat{\hat{A}}^{u,D,R}_{odd}, \delta\hat{\hat{A}}^{f,D,R}_{odd}, \delta\hat{\hat{A}}^{u,D_f,R}_{odd}, \delta\hat{\hat{A}}^{f,D_f,R}_{odd}, \delta\hat{\hat{K}}^{u,D,R}_{odd}, \delta\hat{\hat{K}}^{f,D,R}_{odd}, \delta\hat{\hat{K}}^{u,D_f,R}_{odd}, \delta\hat{\hat{K}}^{f,D_f,R}_{odd},$$
$$\delta\hat{\hat{\lambda}}^{u,D,R}, \delta\hat{\hat{\lambda}}^{f,D,R}, \delta\hat{\hat{\lambda}}^{u,D_f,R}, \delta\hat{\hat{\lambda}}^{f,D_f,R}, \delta\hat{\hat{\mu}}^{u,D,R}, \delta\hat{\hat{\mu}}^{f,D,R}, \delta\hat{\hat{\mu}}^{u,D_f,R}, \delta\hat{\hat{\mu}}^{f,D_f,R}, \delta\hat{\hat{\sigma}}^{R}]$$
$$= [0,0,0,0,0,0,0,0,-\frac{-4-3d_c}{16}, -\frac{-3(4+d_c)}{32}, -\frac{-3(4+d_c)}{32}, -1/2, -1, 0, 0, 1, 0] \tag{4.70}$$

The other 4 zero eigen-values are perturbations in $\{\hat{\hat{K}}^{\gamma,\delta,R}_{odd}, \hat{\hat{A}}^{\gamma,\delta,R}_{odd}\}$ and include:

$$[\delta\hat{\hat{A}}^{u,D,R}_{odd}, \delta\hat{\hat{A}}^{f,D,R}_{odd}, \delta\hat{\hat{A}}^{u,D_f,R}_{odd}, \delta\hat{\hat{A}}^{f,D_f,R}_{odd}, \delta\hat{\hat{K}}^{u,D,R}_{odd}, \delta\hat{\hat{K}}^{f,D,R}_{odd}, \delta\hat{\hat{K}}^{u,D_f,R}_{odd}, \delta\hat{\hat{K}}^{f,D_f,R}_{odd},$$
$$\delta\hat{\hat{\lambda}}^{u,D,R}, \delta\hat{\hat{\lambda}}^{f,D,R}, \delta\hat{\hat{\lambda}}^{u,D_f,R}, \delta\hat{\hat{\lambda}}^{f,D_f,R}, \delta\hat{\hat{\mu}}^{u,D,R}, \delta\hat{\hat{\mu}}^{f,D,R}, \delta\hat{\hat{\mu}}^{u,D_f,R}, \delta\hat{\hat{\mu}}^{f,D_f,R}, \delta\hat{\hat{\sigma}}^{R}]$$
$$= [1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],$$
$$[0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0], \tag{4.71}$$
$$[0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0],$$
$$[0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0]$$

These four marginal perturbations indicate that perturbations in $\hat{A}^{\gamma,\delta,R}_{odd}$ that preserve fluctuation dissipation are marginal. Furthermore the two eigen-vectors corresponding to the eigen-value $\frac{-2d_c}{4+d_c}$ are:

$$[\delta\hat{\hat{A}}^{u,D,R}_{odd}, \delta\hat{\hat{A}}^{f,D,R}_{odd}, \delta\hat{\hat{A}}^{u,D_f,R}_{odd}, \delta\hat{\hat{A}}^{f,D_f,R}_{odd}, \delta\hat{\hat{K}}^{u,D,R}_{odd}, \delta\hat{\hat{K}}^{f,D,R}_{odd}, \delta\hat{\hat{K}}^{u,D_f,R}_{odd}, \delta\hat{\hat{K}}^{f,D_f,R}_{odd},$$
$$\delta\hat{\hat{\lambda}}^{u,D,R}, \delta\hat{\hat{\lambda}}^{f,D,R}, \delta\hat{\hat{\lambda}}^{u,D_f,R}, \delta\hat{\hat{\lambda}}^{f,D_f,R}, \delta\hat{\hat{\mu}}^{u,D,R}, \delta\hat{\hat{\mu}}^{f,D,R}, \delta\hat{\hat{\mu}}^{u,D_f,R}, \delta\hat{\hat{\mu}}^{f,D_f,R}, \delta\hat{\hat{\sigma}}^{R}] \tag{4.72}$$
$$= [0,0,0,0,0,0,0,0,-5/4,-5/4,-5/4,-5/4,1,1,1,1,0],$$
$$[0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0]$$

and the eigen-value -2 corresponds to the eigen-vector:

$$
\begin{aligned}
&[\delta\hat{\hat{A}}^{u,D,R}_{odd},\delta\hat{\hat{A}}^{f,D,R}_{odd},\delta\hat{\hat{A}}^{u,D_f,R}_{odd},\delta\hat{\hat{A}}^{f,D_f,R}_{odd},\delta\hat{\hat{K}}^{u,D,R}_{odd},\delta\hat{\hat{K}}^{f,D,R}_{odd},\delta\hat{\hat{K}}^{u,D_f,R}_{odd},\delta\hat{\hat{K}}^{f,D_f,R}_{odd},\\
&\delta\hat{\hat{\lambda}}^{u,D,R},\delta\hat{\hat{\lambda}}^{f,D,R},\delta\hat{\hat{\lambda}}^{u,D_f,R},\delta\hat{\hat{\lambda}}^{f,D_f,R},\delta\hat{\hat{\mu}}^{u,D,R},\delta\hat{\hat{\mu}}^{f,D,R},\delta\hat{\hat{\mu}}^{u,D_f,R},\delta\hat{\hat{\mu}}^{f,D_f,R},\delta\hat{\hat{\sigma}}^{R}] \\
&=[0,0,0,0,0,0,0,0,-1/2,-1/2,-1/2,-1/2,1,1,1,1,0]
\end{aligned} \tag{4.73}
$$

One of these negative eigen-values and their respective eigen-vectors tells us that perturbations in $\hat{K}^{\gamma,\delta,R}_{odd}$ that preserve fluctuation-dissipation are irrelevant. Meanwhile, the other two correspond to eigen-vectors found in a fixed point analysis using Boltzmann weights for equilibrium elastic membranes (using Boltzmann weights [12]), indeed they indicate that if we restrict ourselves to the domain of phase space where fluctuation-dissipation holds and $\hat{A}^{\gamma,\delta,R}_{odd},\hat{K}^{\gamma,\delta,R}_{odd}$ are odd, then the Aronovitz-Lubensky fixed point is the globally stable fixed point. The eigen-vectors corresponding to the eigen-value $2d_c/(2+d_c)$ correspond to:

$$
\begin{aligned}
&[\delta\hat{\hat{A}}^{u,D,R}_{odd},\delta\hat{\hat{A}}^{f,D,R}_{odd},\delta\hat{\hat{A}}^{u,D_f,R}_{odd},\delta\hat{\hat{A}}^{f,D_f,R}_{odd},\delta\hat{\hat{K}}^{u,D,R}_{odd},\delta\hat{\hat{K}}^{f,D,R}_{odd},\delta\hat{\hat{K}}^{u,D_f,R}_{odd},\delta\hat{\hat{K}}^{f,D_f,R}_{odd},\\
&\delta\hat{\hat{\lambda}}^{u,D,R},\delta\hat{\hat{\lambda}}^{f,D,R},\delta\hat{\hat{\lambda}}^{u,D_f,R},\delta\hat{\hat{\lambda}}^{f,D_f,R},\delta\hat{\hat{\mu}}^{u,D,R},\delta\hat{\hat{\mu}}^{f,D,R},\delta\hat{\hat{\mu}}^{u,D_f,R},\delta\hat{\hat{\mu}}^{f,D_f,R},\delta\hat{\hat{\sigma}}^{R}] \\
&=[0,0,0,0,0,0,0,0,\frac{-31-9dc}{26},-1/2,-1/2,-1/2,\frac{32+3dc}{26},1,1,1,0],\\
&[0,0,0,0,0,0,0,0,\frac{8\pi(34+9d_c)}{13(4+d_c)},0,0,0,-\frac{8\pi(20+3d_c)}{13(4+d_c)},0,0,0,1],\\
&[0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0],\\
&[0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0],\\
&[1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]
\end{aligned} \tag{4.74}
$$

The last eigen-vector corresponding to eigen-value $2(2+d_c)/(4+d_c)$ is:

$$\begin{aligned}
&[\delta\hat{\tilde{A}}_{odd}^{u,D,R}, \delta\hat{\tilde{A}}_{odd}^{f,D,R}, \delta\hat{\tilde{A}}_{odd}^{u,D_f,R}, \delta\hat{\tilde{A}}_{odd}^{f,D_f,R}, \delta\hat{\tilde{K}}_{odd}^{u,D,R}, \delta\hat{\tilde{K}}_{odd}^{f,D,R}, \delta\hat{\tilde{K}}_{odd}^{u,D_f,R}, \delta\hat{\tilde{K}}_{odd}^{f,D_f,R},\\
&\delta\hat{\tilde{\lambda}}^{u,D,R}, \delta\hat{\tilde{\lambda}}^{f,D,R}, \delta\hat{\tilde{\lambda}}^{u,D_f,R}, \delta\hat{\tilde{\lambda}}^{f,D_f,R}, \delta\hat{\tilde{\mu}}^{u,D,R}, \delta\hat{\tilde{\mu}}^{f,D,R}, \delta\hat{\tilde{\mu}}^{u,D_f,R}, \delta\hat{\tilde{\mu}}^{f,D_f,R}, \delta\hat{\tilde{\sigma}}^{R}]\\
&= [0,0,0,0,0,0,0,0,-\frac{8(2+d_c)\pi}{(3+d_c)(4+d_c)}, -\frac{8(2+d_c)\pi}{(3+d_c)(4+d_c)},\\
&-\frac{8(2+d_c)\pi}{(3+d_c)(4+d_c)}, -\frac{8(2+d_c)\pi}{(3+d_c)(4+d_c)}, \frac{16(2+d_c)\pi}{(3+d_c)(4+d_c)}, \frac{16(2+d_c)\pi}{(3+d_c)(4+d_c)},\\
&\frac{16(2+d_c)\pi}{(3+d_c)(4+d_c)}, \frac{16(2+d_c)\pi}{(3+d_c)(4+d_c)}, 1]
\end{aligned} \tag{4.75}$$

All in all, we can then summarize our findings via the following statements:

1. If we enforce fluctuation dissipation and insist upon the odd elastic constants being zero, then indeed the Aronovitz-Lubensky fixed point is the globally stable fixed point.

2. If we enforce fluctuation dissipation but do not insist upon the odd elastic constants being zero, then indeed the Aronovitz-Lubensky fixed point is stable to perturbations in $\hat{\tilde{K}}_{odd}^{\gamma,\delta,R}$ but is marginally stable with respect to perturbations in $\hat{\tilde{A}}_{odd}^{\gamma,\delta,R}$. Stress however is always an unstable direction.

3. If we do not enforce fluctuation dissipation and insist upon the odd elastic constants being zero, then the Aronovitz-Lubensky fixed point potentially belongs to a larger manifold and the fixed point is no longer globally stable. Relevant unstable directions could potentially take us to a new fixed point to be obtained. Indeed this has been better explored in [34].

4. If we do not enforce fluctuation dissipation and do not insist upon the odd elastic constants being zero, then the Aronovitz-Lubensky fixed point is not globally stable. A globally stable fixed point has not been established.

5. All the above analysis has been only done to 1-loop order and thus a 2-loop expansion is in principle necessary to establish whether the zero eigen-value directions stay zero or become non-zero. This would be important for comprehending how these other perturbative eigen-vectors impact the stability of the Aronovitz-Lubensky fixed point.

4.3.2 Numerical Integration of Renormalization Group Equations

With our stability analysis done, we can perform a numerical perturbation analysis to obtain all the exponents associated with these equations. We aim to first formulate the propagators and then plot them since they are the relevant observables of the theory. We will eventually focus on the physical case of interest: $d_c = 1$ so that we later compare with simulations. Another complication concerns that one may still ask of course whether we must re-introduce all the Feynman diagrams that are valid for $d_c = 1$ (which we ignored in using a large d_c limit argument) and as stated previously, a future investigation would have to be done to understand this. At the moment, we believe that our Feynman diagrams are the ones necessary for a controlled calculation, but this is not a sufficiently strong argument.

Equations Of Propagators

As previously establish, odd elastic parameters do not enter into the harmonic terms relevant for the flexural modes. Thus the analytic form of their propagators remain the same regardless of whether odd elastic parameters are present or not and take the value:

$$\langle \tilde{f}^{\alpha}(\mathbf{q},\omega)\tilde{f}^{\beta}(-\mathbf{q},-\omega)\rangle = \frac{2D_f\delta_{\alpha\beta}}{A^2[D_f q^4 + D_f\sigma q^2]+\omega^2} \tag{4.76}$$

where A is once again the area of the system. This corresponds to a static propagator, which is obtained by integrating over all ω, thus applying the Cauchy residue theorem [97, 72]:

$$\langle \tilde{f}^{\alpha}(\mathbf{q})\tilde{f}^{\beta}(-\mathbf{q})\rangle = \frac{\delta_{\alpha\beta}}{A[q^4+\sigma q^2]} \tag{4.77}$$

though keep in mind that we have set $k_BT = 1$ and re-scaled such that L_f does not appear in the theory. The form of the in-plane propagators contains odd elastic parameters explicitly and takes the form:

$$\begin{aligned}
&\langle \tilde{u}_j(\mathbf{q},\omega)\tilde{u}_j(-\mathbf{q},-\omega)\rangle = \\
&\tilde{L}^R\Bigg[2\omega^2 + [(\tilde{A}^{u,D,R}_{odd})^2 + 2\tilde{A}^{u,D,R}_{odd}\tilde{K}^{u,D,R}_{odd} + 2(\tilde{K}^{u,D,R}_{odd})^2 + (\tilde{\lambda}^{u,D,R})^2 \\
&+ 4\tilde{\lambda}^{u,D,R}\tilde{\mu}^{u,D,R} + 5(\tilde{\mu}^{u,D,R})^2 - \\
&(-1)^{1+j}\cos 2\theta[(\tilde{A}^{u,D,R}_{odd})^2 + 2\tilde{A}^{u,D,R}_{odd}\tilde{K}^{u,D,R}_{odd} + (\tilde{\lambda}^{u,D,R}+\tilde{\mu}^{u,D,R})(\tilde{\lambda}^{u,D,R}+3\tilde{\mu}^{u,D,R})] \\
&+ (-1)^{1+j}\sin 2\theta[-\tilde{A}^{u,D,R}_{odd}\tilde{\mu}^{u,D,R} + \tilde{K}^{u,D,R}_{odd}(\tilde{\lambda}^{u,D,R}+\tilde{\mu}^{u,D,R})]]q^4\Bigg] \\
&/\Bigg[\omega^4 + q^4\omega^2[-2\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd}+\tilde{K}^{u,D,R}_{odd}) + (\tilde{\lambda}^{u,D,R})^2 + 4\tilde{\lambda}^{u,D,R}\tilde{\mu}^{u,D,R} + 5(\tilde{\mu}^{u,D,R})^2] \\
&+ q^8[\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd}+\tilde{K}^{u,D,R}_{odd}) + \tilde{\mu}^{u,D,R}(\tilde{\lambda}^{u,D,R}+2\tilde{\mu}^{u,D,R})]\Bigg]
\end{aligned} \tag{4.78}$$

and

$$
\begin{aligned}
\langle \tilde{u}_j(\mathbf{q},\omega)\tilde{u}_l(-\mathbf{q},-\omega)\rangle_{j\neq l} = \tilde{L}^R q^2 \Bigg[& 2i(-1)^j\omega[\tilde{K}^{u,D,R}_{odd} + \tilde{A}^{u,D,R}_{odd}] \\
& + q^2\cos 2\theta[(2\tilde{A}^{u,D,R}_{odd}\tilde{\mu}^{u,D,R} - 2\tilde{K}^{u,D,R}_{odd}(\tilde{\lambda}^{u,D,R} + \tilde{\mu}^{u,D,R}))] \\
& - q^2\sin 2\theta[(\tilde{A}^{u,D,R}_{odd})^2 + 2\tilde{A}^{u,D,R}_{odd}\tilde{K}^{u,D,R}_{odd} + (\tilde{\lambda}^{u,D,R} + \tilde{\mu}^{u,D,R})(\tilde{\lambda}^{u,D,R} + 3\tilde{\mu}^{u,D,R})]\Bigg] \\
& /\Bigg[\omega^4 + q^4\omega^2[-2\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd} + \tilde{K}^{u,D,R}_{odd}) + (\tilde{\lambda}^{u,D,R})^2 + 4\tilde{\lambda}^{u,D,R}\tilde{\mu}^{u,D,R} + 5(\tilde{\mu}^{u,D,R})^2] \\
& + q^8[\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd} + \tilde{K}^{u,D,R}_{odd}) + \tilde{\mu}^{u,D,R}(\tilde{\lambda}^{u,D,R} + 2\tilde{\mu}^{u,D,R})]\Bigg]
\end{aligned}
\tag{4.79}
$$

which correspond to the static propagators:

$$
\begin{aligned}
\langle \tilde{u}_j(\mathbf{q})\tilde{u}_j(-\mathbf{q})\rangle = \tilde{L}^R \Bigg[& [(\tilde{A}^{u,D,R}_{odd} + 2\tilde{K}^{u,D,R}_{odd})^2 + (\tilde{\lambda}^{u,D,R} + 3\tilde{\mu}^{u,D,R})^2 \\
& - (-1)^{1+j}\cos 2\theta[(\tilde{A}^{u,D,R}_{odd})^2 + 2\tilde{A}^{u,D,R}_{odd}\tilde{K}^{u,D,R}_{odd} + (\tilde{\lambda}^{u,D,R} + \tilde{\mu}^{u,D,R})(\tilde{\lambda}^{u,D,R} + 3\tilde{\mu}^{u,D,R})] \\
& + (-1)^{1+j}\sin 2\theta[-\tilde{A}^{u,D,R}_{odd}\tilde{\mu}^{u,D,R} + \tilde{K}^{u,D,R}_{odd}(\tilde{\lambda}^{u,D,R} + \tilde{\mu}^{u,D,R})]]\Bigg] \\
& /\Bigg[2q^2(\tilde{\lambda}^{u,D,R} + 3\tilde{\mu}^{u,D,R})[\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd} + \tilde{K}^{u,D,R}_{odd}) + \tilde{\mu}^{u,D,R}(\tilde{\lambda}^{u,D,R} + 2\tilde{\mu}^{u,D,R})]\Bigg]
\end{aligned}
\tag{4.80}
$$

and

$$
\begin{aligned}
\langle \tilde{u}_j(\mathbf{q})\tilde{u}_l(-\mathbf{q}])\rangle_{j\neq l} = \tilde{L}^R \Bigg[& \cos 2\theta[(2\tilde{A}^{u,D,R}_{odd}\tilde{\mu}^{u,D,R} - 2\tilde{K}^{u,D,R}_{odd}(\tilde{\lambda}^{u,D,R} + \tilde{\mu}^{u,D,R}))] \\
& - \sin 2\theta[(\tilde{A}^{u,D,R}_{odd})^2 + 2\tilde{A}^{u,D,R}_{odd}\tilde{K}^{u,D,R}_{odd} + (\tilde{\lambda}^{u,D,R} + \tilde{\mu}^{u,D,R})(\tilde{\lambda}^{u,D,R} + 3\tilde{\mu}^{u,D,R})]\Bigg] \\
& /\Bigg[2q^2(\tilde{\lambda}^{u,D,R} + 3\tilde{\mu}^{u,D,R})[\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd} + \tilde{K}^{u,D,R}_{odd}) + \tilde{\mu}^{u,D,R}(\tilde{\lambda}^{u,D,R} + 2\tilde{\mu}^{u,D,R})]\Bigg]
\end{aligned}
\tag{4.81}
$$

With these equal-time correlations in place, we now have a set of observables that we can calculate whether in our numerical integration or in simulations. Two of the in-plane propagators that we will focus, along with the flexural propagator, on are the following:

$$\langle \tilde{u}_1(q_1,0)\tilde{u}_1(-q_1,0)\rangle = \tilde{L}^R\Bigg[[\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd}+2\tilde{K}^{u,D,R}_{odd})+\tilde{\mu}^{u,D,R}(\tilde{\lambda}^{u,D,R}+3\tilde{\mu}^{u,D,R})]\Bigg] \Big/\Bigg[q^2(\tilde{\lambda}^{u,D,R}+3\tilde{\mu}^{u,D,R})[\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd}+\tilde{K}^{u,D,R}_{odd})+\tilde{\mu}^{u,D,R}(\tilde{\lambda}^{u,D,R}+2\tilde{\mu}^{u,D,R})]\Bigg] \tag{4.82}$$

and

$$\langle \tilde{u}_1(q_1,0)\tilde{u}_2(-q_1,0)\rangle = \tilde{L}^R\Bigg[[\tilde{A}^{u,D,R}_{odd}\tilde{\mu}^{u,D,R}_{odd}-\tilde{K}^{u,D,R}_{odd}(\tilde{\lambda}^{u,D,R}+\tilde{\mu}^{u,D,R})]\Bigg] \Big/\Bigg[q^2(\tilde{\lambda}^{u,D,R}+3\tilde{\mu}^{u,D,R})[\tilde{K}^{u,D,R}_{odd}(\tilde{A}^{u,D,R}_{odd}+\tilde{K}^{u,D,R}_{odd})+\tilde{\mu}^{u,D,R}(\tilde{\lambda}^{u,D,R}+2\tilde{\mu}^{u,D,R})]\Bigg] \tag{4.83}$$

The former propagator gives us the correlations of longitudinal in-plane phonons with which we may compare the theory of elastic membranes that do not possess odd elastic parameters. The latter propagator isolates the effect of odd elastic systems because this propagator would otherwise vanish for a non-odd elastic membrane.

Scaling of Propagators Via Renormalization Group: $\{\tilde{A}^{\gamma,\delta,R}_{odd},\tilde{K}^{\gamma,\delta,R}_{odd}\} \leq \{\tilde{\lambda}^{\gamma,\delta,R},\tilde{\mu}^{\gamma,\delta,R}\}$ And Microscopic Fluctuation-Dissipation

Given our stability analysis was a perturbative analysis of the Aronovitz-Lubensky fixed point, we aim to integrate the renormalization group equations, Eq. (4.64), with microscopic parameters such that microscopic odd elastic coefficients, $\tilde{A}^{\gamma,\delta,o}_{odd},\tilde{K}^{\gamma,\delta,o}_{odd}$, are relatively smaller than $\tilde{\lambda}^{\gamma,\delta,o},\tilde{\mu}^{\gamma,\delta,o}$. This is because for very large perturbations, it is not necessarily true that our stability analysis will be even relevant. Thus, we preface our investigation with this restriction and delegate analysis of systems where odd elastic parameters are very large to the future. We thus restrict ourselves to microscopic parameters that satisfy $\{\tilde{A}^{\gamma,\delta,o}_{odd},\tilde{K}^{\gamma,\delta,o}_{odd}\} \leq \{\tilde{\lambda}^{\gamma,\delta,o},\tilde{\mu}^{\gamma,\delta,o}\}$. We further restrict ourselves to examining cases where $\tilde{K}^{\gamma,\delta,o}_{odd} \neq 0, \tilde{A}^{\gamma,\delta,o}_{odd} = 0$ or $\tilde{K}^{\gamma,\delta,o}_{odd} = 0, \tilde{A}^{\gamma,\delta,o}_{odd} \neq$

0 so that we comprehend the effect of each odd elastic parameter independently. When both are present, one has to be concerned with the stability criteria mentioned previously and in [93]. We furthermore assume that fluctuation-dissipation holds at the microscopic scale, $\tilde{C}^{u,\delta,o}_{ijkl} = \tilde{C}^{f,\delta,o}_{ijkl}$, as we are concerned particularly with this case. Finally, it is important to note that we intend to compare this data with simulations run with a barostat that will tune $\sigma^R(L) = 0$. Since $\partial Z_\sigma/\partial b \neq 0$, stress is generated when fluctuation-dissipation is broken or even odd elastic moduli are present. However, it is important to note that our simulations will be run with a barostat where $\sigma^R(L) = 0$ where L is the linear system size length here. Thus when we numerically integrate, we will have to use a shooting method where stress is microscopically tuned such that $\sigma^R(L) = 0$. This is an approach that is also taken in [34].

1. $\tilde{K}^{\gamma,\delta,R}_{odd} \neq 0, \tilde{A}^{\gamma,\delta,R}_{odd} = 0$ **and** $\tilde{C}^{u,\delta,o}_{ijkl} = \tilde{C}^{f,\delta,o}_{ijkl}$ In this case, when we integrate Eq. (4.64) with microscopic parameters satisfying $\tilde{K}^{\gamma,\delta,o}_{odd} \leq \{\tilde{\lambda}^{\gamma,\delta,o}, \tilde{\mu}^{\gamma,\delta,o}\}$, we obtain the scaling found in Fig. 4.4 where the parameters are plotted vs. the linear dimension size. Though stress is not plotted, we have used the shooting method to tune the stress to be a numerically irrelevant parameter. From the plot one can see that $\tilde{K}^{\gamma,\delta}_{odd}$ is an irrelevant parameter and converges more quickly to zero. Thus with this scaling we may extrapolate the scaling of the propagators:

$$\begin{aligned} \{Aq^4\langle \tilde{f}^\alpha(\mathbf{q})\tilde{f}^\beta(-\mathbf{q})\rangle, &Aq^2\langle \tilde{u}_1(q_1,0)\tilde{u}_1(-q_1,0)\rangle, Aq^2\langle \tilde{u}_1(q_1,0)\tilde{u}_2(-q_1,0)\rangle\} \\ &\sim \{q^{-\eta}, q^{-\eta_u}, 1\} \end{aligned} \tag{4.84}$$

One can of course see a length scale where the scaling exponent appears, which will be derived in 4.3.2. We furthermore find that fluctuation dissipation is not spontaneously broken, which is consistent with our stability analysis.

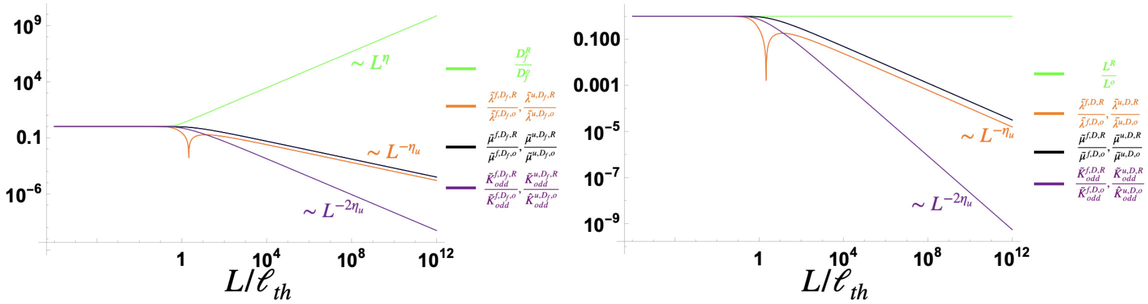

Figure 4.4: Above are plotted the parameters of the theory in the presence of the odd elastic parameter $\tilde{K}^{\gamma,\delta}_{odd}$; they are plotted against a non-dimensionalized linear dimension of the system. $\eta \approx .8$ and $\eta_u \approx .4$. Microscopic initial conditions assume fluctuation-dissipation which is not broken at larger length scales. We thus see that $\tilde{K}^{\gamma,\delta,R}_{odd}$ is an irrelevant perturbation to the Aronovitz-Lubensky fixed point.

2. $\tilde{K}^{\gamma,\delta,R}_{odd} = 0, \tilde{A}^{\gamma,\delta,R}_{odd} \neq 0$ **and** $\tilde{C}^{u,\delta,o}_{ijkl} = \tilde{C}^{f,\delta,o}_{ijkl}$ In this case, when we integrate Eq. (4.64) with microscopic parameters satisfying $\tilde{A}^{\gamma,\delta,o}_{odd} \leq \{\tilde{\lambda}^{\gamma,\delta,o}, \tilde{\mu}^{\gamma,\delta,o}\}$, we obtain the scaling found in Fig. 4.5 where the parameters are plotted vs. the linear dimension size. One can see that $\tilde{A}^{\gamma,\delta}_{odd}$ gives rise to a marginal perturbation that does not affect the exponents of the theory nor the scaling of the other elastic constants. Thus with this scaling we may extrapolate the scaling of the propagators:

$$\begin{aligned} \{Aq^4\langle \tilde{f}^\alpha(\mathbf{q})\tilde{f}^\beta(-\mathbf{q})\rangle, Aq^2\langle \tilde{u}_1(q_1,0)\tilde{u}_1(-q_1,0)\rangle, Aq^2\langle \tilde{u}_1(q_1,0)\tilde{u}_2(-q_1,0)\rangle\} \\ \sim \{q^{-\eta}, q^{-\eta_u}, q^{-\eta_u}\} \end{aligned} \tag{4.85}$$

One can of course see a length scale where the scaling exponent appears, which will be derived in 4.3.2. We furthermore find that fluctuation dissipation is not spontaneously broken, which is consistent with our stability analysis.

Derivation of Thermal Length Scale

As can be seen from the Fig. 4.5 and Fig. 4.4, we have non-dimensionalized the linear system size by a thermal length scale which we have yet to define. Given that the odd elastic parameters are now present, there is no reason that the thermal length

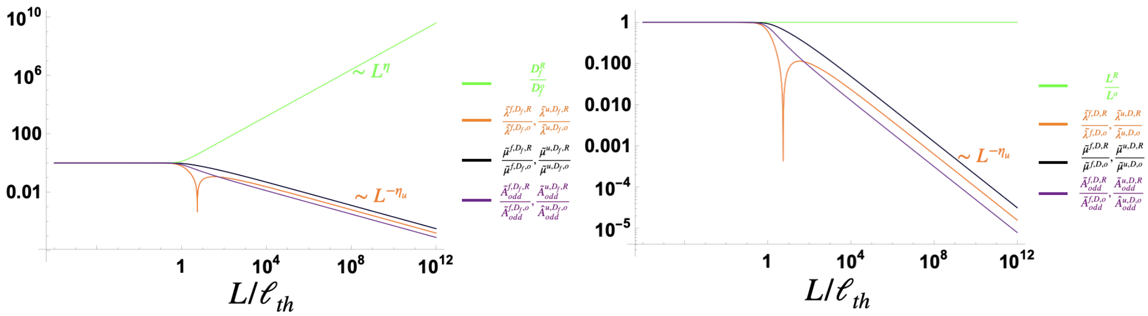

Figure 4.5: Above are plotted the parameters of the theory in the presence of the odd elastic parameter $\tilde{A}^{\gamma,\delta}_{odd}$; they are plotted against a non-dimensionalized linear dimension of the system. $\eta \approx .8$ and $\eta_u \approx .4$. Microscopic initial conditions assume fluctuation-dissipation which is not broken at larger length scales. We thus see that $\tilde{A}^{\gamma,\delta,R}_{odd}$ is a marginal perturbation to the Aronovitz-Lubensky fixed point.

scale should take the value given in the equilibrium theory:

$$\ell_{\text{th}} = \sqrt{\frac{16\pi^3\kappa^2}{3k_BTY}} \tag{4.86}$$

The manner in which one may obtain this length scale in the equilibrium case is by comparing the an-harmonic $\partial logZ_{D_f}/\partial logb$ with D^o_f and obtain for what value of UV-cutoff, Λ, for which the two are comparable. The procedure is no different in the odd elastic case, however, $\partial logZ_{D_f}/\partial logb$ is now significantly more complicated. Without assuming fluctuation-dissipation to hold or even making some simplifications explained in the next sentence, it is difficult to obtain a thermal length scale. Thus we resort to obtaining a provisional form done in the Mathematica code whereby one assumes fluctuation-dissipation to hold and obtains the lowest powers of Λ in $\partial logZ_{D_f}/\partial logb$ (in order to reduce the power of the polynomial in Λ that one has to solve for) and then comparing them to D^o_f. By doing this, we obtain that in terms of the bare elastic moduli of the theory:

$$q_{\text{th}} = \frac{2\pi}{\ell_{\text{th}}} = \left(\frac{3k_BT}{(\kappa^o)^2}(\lambda^o+\mu^o)\left((K^o_{odd})^2+(\mu^o)^2\right)\left(K^o_{odd}(A^o_{odd}+K^o_{odd})+\mu^o(\lambda^o+2\mu^o)\right)\right)^{1/2}$$
$$/\Bigg((\mu^o)^2\left((A^o_{odd})^2+2(8\pi-3)A^o_{odd}K^o_{odd}+4\pi(\lambda^o)^2+2(3+8\pi)(K^o_{odd})^2\right)$$
$$+(K^o_{odd})^2\left((A^o_{odd})^2+4\pi(A^o_{odd}+K^o_{odd})^2+6(\lambda^o)^2\right)$$
$$+2\lambda^oK^o_{odd}\mu^o(4\pi(A^o_{odd}+K^o_{odd})-3A^o_{odd}+6K^o_{odd})+16\pi\lambda^o(\mu^o)^3+16\pi(\mu^o)^4\Bigg)^{1/2} \tag{4.87}$$

which reduces to the form Eq. (4.86) when A_{odd} and K_{odd} are both zero. We use this definition of ℓ_{th} to plot Fig. 4.5 and Fig. 4.4.

4.4 Simulations

We use a GPU-suitable package code PyMembrane developed by Professor Daniel A. Matoz-Fernandez. We simulated a system of atoms arranged into a triangular lattice. We used 2900 atoms with a lattice space of $a = 1$ arranged into a square sheet that measured $50a \times 50a$. We used dihedral springs to replicate the bending rigidity of elastic systems. However, unlike previous studies that have used bolts and springs [64], we implement a different method that allows us to simulate an elastic system associated with any in-plane elastic moduli of one's choosing.

Implementation of Dihedral Forces

The bending forces were determined directly from a set of dihedral springs between the faces of the triangles. The elastic bending energy of such a system can be formulated as:

$$\text{E}_{\text{bend}} = \frac{\hat{\kappa}}{2}\sum_{\langle IJ\rangle}[1+\cos\theta_{IJ}] \tag{4.88}$$

where $\hat{\kappa}$ is the microscopic dihedral spring stiffness and θ_{IJ} is the dihedral angle between two triangular faces (which can also be seen as the angle differences between normals of faces). We can relate the microscopic spring stiffness to the coarse-grained bending rigidity [55]:

$$\kappa = \frac{\sqrt{3}}{2}\hat{\kappa} \tag{4.89}$$

Implementation of In-Plane Forces

In an orthonormal basis $\{\hat{e}_1, \hat{e}_2\}, \hat{e}_1 \cdot \hat{e}_2 = 0$ in $D = 2$, our odd elastic modulus tensor takes the form:

$$\begin{aligned} &C^{1111} = C^{2222} = \lambda + 2\mu, C^{1212} = C^{1221} = C^{2112} = C^{2121} = \mu, \\ &C^{1122} = C^{2211} = \lambda, C^{1112} = C^{1121} = K, C^{2212} = C^{2221} = -K, \\ &C^{2122} = A + K, C^{1222} = -A + K, C^{2111} = A - K, C^{1211} = -A - K \end{aligned} \tag{4.90}$$

where we will now use co-variant and contra-variant notation of differential geometry [101]. In tensor notation this boils down to:

$$C^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu[\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}] + KE^{ijkl} - A\epsilon^{ij}\delta^{kl} \tag{4.91}$$

where

$$E^{ijkl} = \frac{1}{2}[\epsilon^{ik}\delta^{jl} + \epsilon^{il}\delta^{jk} + \epsilon^{jk}\delta^{il} + \epsilon^{jl}\delta^{ik}] \tag{4.92}$$

where $\epsilon_{11} = \epsilon_{22} = 0, \epsilon_{12} = -\epsilon_{21} = 1$ is the Levi-Civita symbol. Within this Euclidean definition in a reference flat frame, we can transform the tensor in a co-variant way to accommodate non-orthogonal bases. Assume the new basis takes the form: $\{\bar{e}'_1, \bar{e}'_2\}$ and define:

$$\bar{g}_{ij} = \bar{e}'_i \cdot \bar{e}'_j \tag{4.93}$$

then in the new non-orthonormal Euclidean frame we have:

$$\mathcal{C}^{ijkl} = (\Lambda)^i_a(\Lambda)^j_b(\Lambda)^k_m(\Lambda)^l_n C^{abmn} \tag{4.94}$$

where $\Lambda^i_j = \partial x'^i/\partial x^j$ is the Jacobian of the transformation. With the knowledge that:

$$(\Lambda^{-1})^k_i(\Lambda^{-1})^l_j\delta_{kl} = \bar{g}_{ij} \tag{4.95}$$

where g is the current metric, then we obtain:

$$\mathcal{C}^{ijkl} = \lambda\bar{g}^{ij}\bar{g}^{kl} + \mu[\bar{g}^{ik}\bar{g}^{jl} + \bar{g}^{il}\bar{g}^{jk}] + \frac{K}{\sqrt{\det[\bar{g}]}}\mathcal{E}^{ijkl} - \frac{A}{\sqrt{\det[\bar{g}]}}\epsilon^{ij}\bar{g}^{kl} \tag{4.96}$$

where still

$$\mathcal{E}^{ijkl} = \frac{1}{2}[\epsilon^{ik}g^{jl}_{ref} + \epsilon^{il}\bar{g}^{jk} + \epsilon^{jk}\bar{g}^{il} + \epsilon^{jl}\bar{g}^{ik}] \tag{4.97}$$

We can use the current metric tensor to calculate the strain:

$$u_{ij} = \frac{1}{2}[g_{ij} - \bar{g}_{ij}] \tag{4.98}$$

where g is again the current metric tensor and $\bar{g}$ is the reference metric tensor. From here, we can use the constitutive formula to obtain the stress in the current frame:

$$\sigma^{ij} = \mathcal{C}^{ijkl}u_{kl} \tag{4.99}$$

This gives us the homogeneous stress that a triangle experiences. All that remains is to calculate the normals to the edges.

We return to the lab frame to first obtain the normal to the face which can be obtained by means of the cross-product:

$$\hat{\mathbf{n}}_{face} = \frac{1}{|\mathbf{r}_{12} \times \mathbf{r}_{13}|}\mathbf{r}_{12} \times \mathbf{r}_{13} \tag{4.100}$$

Where $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{r}_{13} = \mathbf{r}_3 - \mathbf{r}_1$ are vectors in the lab frame. From this we can calculate the normals to any of the edges in the lab frame:

$$\mathbf{n}_{i\to j} = \mathbf{r}_{i\to j} \times \hat{\mathbf{n}}_{face} \tag{4.101}$$

where we assume that $1 \to 2 \to 3 \to 1$ is a counter-clockwise orientation list of the triangle and $\mathbf{r}_{i\to j} = \mathbf{r}_j - \mathbf{r}_i$. Now we must obtain an expression of the normal in the tangent plane. To do this we write down:

$$(N_{i\to j})_k = \mathbf{n}_{i\to j} \cdot \mathbf{r}_{1\to k+1}|_{k=1,2} \tag{4.102}$$

$$\mathbf{N}_{i\to j} = (N_{i\to j})_k \mathbf{R}^k \tag{4.103}$$

$$\mathbf{R}^k = g^{km}\mathbf{r}_{1\to m+1} \tag{4.104}$$

$$\mathbf{R}^k = g^{-1}_{km}\mathbf{r}_{1\to m+1} \tag{4.105}$$

where now $\mathbf{N}_{i\to j}$ is the expression of the normal in the tangent plane.

We can calculate the total traction force:

$$(T_{i\to j})^k = (N_{i\to j})_l \sigma^{lk} \tag{4.106}$$

With this final form, we can apply the force $\vec{F}_{i(j)} = (1/2)\vec{T}_{i\to j} = (1/2)(T_{i\to j})^k \mathbf{r}_{1\to k+1}$ on each of the vertices adjacent to the edge ij. Thus, we have an implementation that gives us the forces we should exert on every vertex.

We can also write down the stress-strain relations more explicitly in terms of the metric tensor. If we define the elastic deformation tensor for the triangular lattice as follows:

$$F = \frac{1}{2}\left(\bar{g}^{-1}g - I\right)$$

then and consider the strain given by:

$$\begin{aligned} u^{\alpha}_{\beta} &= \bar{g}^{\alpha\gamma}u_{\beta\gamma} \\ &= \frac{1}{2}\bar{g}^{\alpha\gamma}\left(g_{\beta\gamma} - \bar{g}_{\beta\gamma}\right) \\ &= \frac{1}{2}\left(\bar{g}^{\alpha\gamma}g_{\beta\gamma} - \bar{g}^{\alpha\gamma}\bar{g}_{\beta\gamma}\right) \end{aligned}$$

using the fact $\bar{g}^{-1}_{ij} = \bar{g}^{ij}$ then we get $u^{\alpha}_{\beta} = F^{\alpha}_{\beta}$ which can be easily calculated in simulations.

Then begin with the contribution of A_{odd} to Eq. 4.99,

$$\begin{aligned} \sigma^{ij}_{A} &= -\frac{A_{odd}}{\sqrt{|\bar{g}|}}\epsilon^{ij}\bar{g}^{kl}u_{kl} \\ &= -\frac{A_{odd}}{\sqrt{|\bar{g}|}}\epsilon^{ij}u^{k}_{k} \\ &= -\frac{A_{odd}}{\sqrt{|\bar{g}|}}\epsilon^{ij}\mathrm{Tr}(F) \end{aligned} \tag{4.107}$$

where Tr is the trace of F. So now:

$$\begin{aligned} \sigma^{11}_{A} &= 0 \\ \sigma^{12}_{A} &= -\frac{A_{odd}}{\sqrt{|\bar{g}|}}\mathrm{Tr}(F) \\ \sigma^{22}_{A} &= 0 \\ \sigma^{21}_{A} &= -\sigma^{12}_{A} \end{aligned}$$

Next, we move the contribution to K,

$$\begin{aligned}
\sigma_K^{ij} &= \frac{K_{odd}}{2\sqrt{|\bar{g}|}}\left(\epsilon^{ik}\bar{g}^{jl} + \epsilon^{il}\bar{g}^{jk} + \epsilon^{jk}\bar{g}^{il} + \epsilon^{jl}\bar{g}^{ik}\right)u_{kl} \\
&= \frac{K_{odd}}{2\sqrt{|\bar{g}|}}\left(\epsilon^{ik}\bar{g}^{jl}u_{kl} + \epsilon^{il}\bar{g}^{jk}u_{kl} + \epsilon^{jk}\bar{g}^{il}u_{kl} + \epsilon^{jl}\bar{g}^{ik}u_{kl}\right) \\
&= \frac{K_{odd}}{2\sqrt{|\bar{g}|}}\left(\epsilon^{ik}u_k^j + \epsilon^{il}u_l^j + \epsilon^{jk}u_k^i + \epsilon^{jl}u_l^i\right) \\
&= \frac{K_{odd}}{2\sqrt{|\bar{g}|}}\left(\epsilon^{ik}F_k^j + \epsilon^{il}F_l^j + \epsilon^{jk}F_k^i + \epsilon^{jl}F_l^i\right) \\
&= \frac{K_{odd}}{2\sqrt{|\bar{g}|}}\left(2\epsilon^{ik}F_k^j + 2\epsilon^{jk}F_k^i\right)
\end{aligned} \tag{4.108}$$

So now we get:

$$\begin{aligned}
\sigma_K^{11} &= \frac{K_{odd}}{\sqrt{|\bar{g}|}}\left(2\epsilon^{12}F_2^1\right) \\
\sigma_K^{12} &= \frac{K_{odd}}{\sqrt{|\bar{g}|}}\epsilon^{12}\left(F_2^2 - F_1^1\right) \\
\sigma_K^{21} &= \frac{K_{odd}}{\sqrt{|\bar{g}|}}\epsilon^{12}\left(F_2^2 - F_1^1\right) \\
\sigma_K^{22} &= \frac{K_{odd}}{\sqrt{|\bar{g}|}}\left(2\epsilon^{21}F_1^2\right)
\end{aligned} \tag{4.109}$$

We can write the same for the normal elastic parameters λ, μ:

$$\begin{aligned}
\sigma_E^{ij} &= \left[\lambda\bar{g}^{ij}\bar{g}^{kl} + \mu(\bar{g}^{ik}\bar{g}^{jl} + \bar{g}^{il}\bar{g}^{jk})\right]u_{kl} \\
&= \lambda\bar{g}^{ij}\bar{g}^{kl}u_{kl} + \mu(\bar{g}^{ik}\bar{g}^{jl}u_{kl} + \bar{g}^{il}\bar{g}^{jk}u_{kl}) \\
&= \lambda\bar{g}^{ij}u_l^l + \mu(\bar{g}^{ik}u_k^j + \bar{g}^{il}u_l^j) \\
&= \lambda\bar{g}^{ij}TrF + 2\mu(\bar{g}^{ik}u_k^j) \\
&= \lambda\bar{g}^{ij}TrF + 2\mu(\bar{g}^{ik}F_k^j) \\
&= \lambda\bar{g}_{ij}^{-1}TrF + 2\mu(\bar{g}_{ik}^{-1}F_k^j)
\end{aligned} \tag{4.110}$$

$$\begin{aligned}
\sigma_E^{11} &= \lambda \bar{g}_{11}^{-1} TrF + 2\mu(\bar{g}_{11}^{-1} F_1^1 + \bar{g}_{12}^{-1} F_2^1) \\
\sigma_E^{12} &= \lambda \bar{g}_{12}^{-1} TrF + 2\mu(\bar{g}_{11}^{-1} F_1^2 + \bar{g}_{12}^{-1} F_2^2) \\
\sigma_E^{21} &= \lambda \bar{g}_{21}^{-1} TrF + 2\mu(\bar{g}_{21}^{-1} F_1^2 + \bar{g}_{22}^{-1} F_2^1) \\
\sigma_E^{22} &= \lambda \bar{g}_{22}^{-1} TrF + 2\mu(\bar{g}_{21}^{-1} F_1^2 + \bar{g}_{22}^{-1} F_2^2)
\end{aligned} \tag{4.111}$$

Thus we have establish stress-strain relations in terms of the metric tensor.

Area Potential In addition, for reasons concerned with the stability of the triangular lattice under large deformations in the presence of odd elastic parameters, we also added in an area potential. To add such a term we have to potentially implement the following energy term:

$$\mathcal{F} = C_1 \log \frac{\det g}{\det \bar{g}} + C_2 \left(\log \frac{\det g}{\det \bar{g}} \right)^2 \tag{4.112}$$

where g and $\bar{g}$ are the current and reference metrics respectively. Of course we want to avoid using the energy term as we don't have any conservation of energy in our system. Thus we are instead interested in deriving a stress strain relation:

$$\sigma^{ij} = \frac{\delta \mathcal{F}}{\delta u_{ij}} \tag{4.113}$$

In order to do this we must express the current metric tensor in terms of the reference basis and the strains. Since the $\det g$ is an invariant, we are free to operate with lower indices. Using the fact that:

$$u_{ij} = \frac{1}{2}[g_{ij} - \bar{g}_{ij}] \tag{4.114}$$

we can write:

$$\begin{aligned}\det g =&(\vec{e}_1 \cdot \vec{e}_1)(\vec{e}_2 \cdot \vec{e}_2) - (\vec{e}_1 \cdot \vec{e}_2)^2 \\ =&[2u_{11} + \bar{e}_1 \cdot \bar{e}_1][2u_{22} + \bar{e}_2 \cdot \bar{e}_2] - [2u_{12} + \bar{e}_1 \cdot \bar{e}_2]^2\end{aligned} \tag{4.115}$$

Using this we can rewrite and define:

$$\mathcal{G} = \log \frac{\det g}{\det \bar{g}} = \log \frac{[2u_{11} + \bar{e}_1 \cdot \bar{e}_1][2u_{22} + \bar{e}_2 \cdot \bar{e}_2] - [2u_{12} + \bar{e}_1 \cdot \bar{e}_2]^2}{(\bar{e}_1 \cdot \bar{e}_1)(\bar{e}_2 \cdot \bar{e}_2) - (\bar{e}_1 \cdot \bar{e}_2)^2} \tag{4.116}$$

Differentiating with respect to the strain we obtain:

$$\frac{\delta \mathcal{G}}{\delta u_{11}} = \frac{2[2u_{22} + \bar{e}_2 \cdot \bar{e}_2]}{[2u_{11} + \bar{e}_1 \cdot \bar{e}_1][2u_{22} + \bar{e}_2 \cdot \bar{e}_2] - [2u_{12} + \bar{e}_1 \cdot \bar{e}_2]^2} \tag{4.117}$$

$$\frac{\delta \mathcal{G}}{\delta u_{22}} = \frac{2[2u_{11} + \bar{e}_1 \cdot \bar{e}_1]}{[2u_{11} + \bar{e}_1 \cdot \bar{e}_1][2u_{22} + \bar{e}_2 \cdot \bar{e}_2] - [2u_{12} + \bar{e}_1 \cdot \bar{e}_2]^2} \tag{4.118}$$

$$\frac{\delta \mathcal{G}}{\delta u_{12}} = -\frac{2[2u_{12} + \bar{e}_1 \cdot \bar{e}_2]}{[2u_{11} + \bar{e}_1 \cdot \bar{e}_1][2u_{22} + \bar{e}_2 \cdot \bar{e}_2] - [2u_{12} + \bar{e}_1 \cdot \bar{e}_2]^2} \tag{4.119}$$

This gives us:

$$\sigma^{ij} = \frac{\delta \mathcal{F}}{\delta u_{ij}} = C_1 \frac{\delta \mathcal{G}}{\delta u_{ij}} + 2C_2 \mathcal{G} \frac{\delta \mathcal{G}}{\delta u_{ij}} \tag{4.120}$$

And thus, having obtained the stress tensor we can use the rest of the implementation above to determine the necessary forces to apply on the vertices. By evaluating these expressions at $u_{ij} = 0$, we immediately obtain that $C_1 = 0$ to have an absent constant response to zero strain. Thus we are left with:

$$\sigma^{ij} = \frac{\delta \mathcal{F}}{\delta u_{ij}} = 2C_2 \mathcal{G} \frac{\delta \mathcal{G}}{\delta u_{ij}} \tag{4.121}$$

To obtain the elastic modulus from this we can differentiate again with respect to u_{kl} and then set $u_{ij} = 0$. This in turn gives us:

$$\frac{\delta\sigma^{ij}}{\delta u_{kl}} = 2C_2\Big[\mathcal{G}\frac{\delta^2\mathcal{G}}{\delta u_{ij}\delta u_{kl}} + \frac{\delta\mathcal{G}}{\delta u_{ij}}\frac{\delta\mathcal{G}}{\delta u_{kl}}\Big]|_{u=0} = 2C_2\frac{\delta\mathcal{G}}{\delta u_{ij}}\frac{\delta\mathcal{G}}{\delta u_{kl}}|_{u=0} \tag{4.122}$$

We thus obtain that:

$$C^{1111} = 2C_2\left[\frac{2\bar{e}_2\cdot\bar{e}_2}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right]^2 \tag{4.123}$$

$$C^{2222} = 2C_2\left[\frac{2\bar{e}_1\cdot\bar{e}_1}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right]^2 \tag{4.124}$$

$$C^{1122} = C^{2211} = 2C_2\left[\frac{2\bar{e}_2\cdot\bar{e}_2}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right]\left[\frac{2\bar{e}_1\cdot\bar{e}_1}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right] \tag{4.125}$$

$$C^{1212} = C^{2112} = C^{2121} = C^{1221} = 2C_2\left[\frac{2\bar{e}_1\cdot\bar{e}_2}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right]^2 \tag{4.126}$$

$$\begin{aligned}&C^{1112} = C^{1121} = C^{1211} = C^{2111} = \\ &2C_2\left[\frac{2\bar{e}_1\cdot\bar{e}_2}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right]\left[\frac{2\bar{e}_2\cdot\bar{e}_2}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right]\end{aligned} \tag{4.127}$$

$$\begin{aligned}&C^{2221} = C^{2212} = C^{2122} = C^{1222} = \\ &2C_2\left[\frac{2\bar{e}_1\cdot\bar{e}_2}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right]\left[\frac{2\bar{e}_1\cdot\bar{e}_1}{(\bar{e}_1\cdot\bar{e}_1)(\bar{e}_2\cdot\bar{e}_2)-(\bar{e}_1\cdot\bar{e}_2)^2}\right]\end{aligned} \tag{4.128}$$

Barostat, Integrators and Simulation Procedure

For the simulations we used a Berendsen barostat to tune the box pressure to zero and to tune the temperature of the system, we utilized a Gaussian distributed noise-force whose variance dictated the magnitude of the temperature. For the integrator, we implemented a BAOAB-limit method [102], thus the noise is not completely memoryless. The friction or diffusivity of the integrated Langevin equations was take to be equal for all forces, thus fluctuation-dissipation was assumed at the microscopic scale.

Furthermore, as mentioned previously, since we are simulating over-damped systems, we need not worry ourselves with the active heat flow that could otherwise arise [95].

Procedure We simulated A_{odd} and K_{odd} separately in an otherwise normal elastic system. We took a variety of values of each of the parameters while varying the temperature. Varying the temperature allows us to access different effective length scales (by effectively changing L/ℓ_{th} without changing the linear dimension of the system L), rather than doing a computationally costly simulation with a large system size. We allowed the each elastic system to "thermally equilibrate" for around $2 \cdot 10^7$ time steps, after which we would begin recording instantaneous snapshots of the configurations of the system. The time step was determined by the limiting factors given by the natural frequencies of the system:

$$\tau_T = a\sqrt{\frac{m}{k_B T}}, \tau_Y = \sqrt{\frac{m}{Y}}, \tau_{A_{odd}} = \sqrt{\frac{m}{A_{odd}}}, \tau_{K_{odd}} = \sqrt{\frac{m}{K_{odd}}}, \tau_{\hat{\kappa}} = a\sqrt{\frac{m}{\hat{\kappa}}} \tag{4.129}$$

where Y is the Young's modulus, m is the mass of a vertex (we took $m = 1$) and a is the lattice spacing. Thus $\tau \leq$Min $\{ \tau_T, \tau_Y, \tau_{A_{odd}}, \tau_{K_{odd}}, \tau_{\hat{\kappa}}\}$.

Each simulation, after thermal equilibration, ran through $8 \cdot 10^8$ time steps, recording snapshots every 10^5 time steps. This gave us a total of 8000 snapshots of data with which we could calculate averaged equal-time correlations. In computation time, this amounted to about 80 hours.

4.4.1 Results

Simulations with K_{odd}: For our simulations with K_{odd}, we ran the parameters given in Table. 4.2 and this resulted in a data collapse seen in Fig. 4.6 around the length scale q_{th} given by Eq. (4.87). In panels Fig. 4.6(a) and (b) we see that the flexural modes and the longitudinal in-plane modes are still associated with the Aronovitz-Lubensky exponents of $\{\eta, \eta_u\} \approx \{.8, .4\}$. In panel Fig. 4.6($c$), we observe that the correlation is non-zero, whereas it would be for a non-odd elastic material, but that it presents no anomalous exponent. One can conclude that since $\lambda^R(q), \mu^R(q) \sim q^{\eta_u}$, then we must necessarily have that $K^R_{odd}(q) \sim q^{2\eta_u}$. This confirms that K_{odd} is an irrelevant parameter of the theory and matches with our theoretical predictions.

Table 4.1:

Data Sets for Fig. 4.6			
L/a	$\hat{\kappa}/k_BT$	$K_{odd}/(\lambda+2\mu)$	$C_2/(\lambda+2\mu)$
50	1	4.76	.048
50	10	4.76	.048
50	$1e3$	4.76	.048
50	$1e5$	4.76	.048
50	1	.73	.007
50	10	.73	.007
50	$1e3$	.73	.007
50	$1e5$	.73	.007
50	1	.076	$8e-4$
50	10	.076	$8e-4$
50	$1e3$	.076	$8e-4$
50	$1e5$	.076	$8e-4$

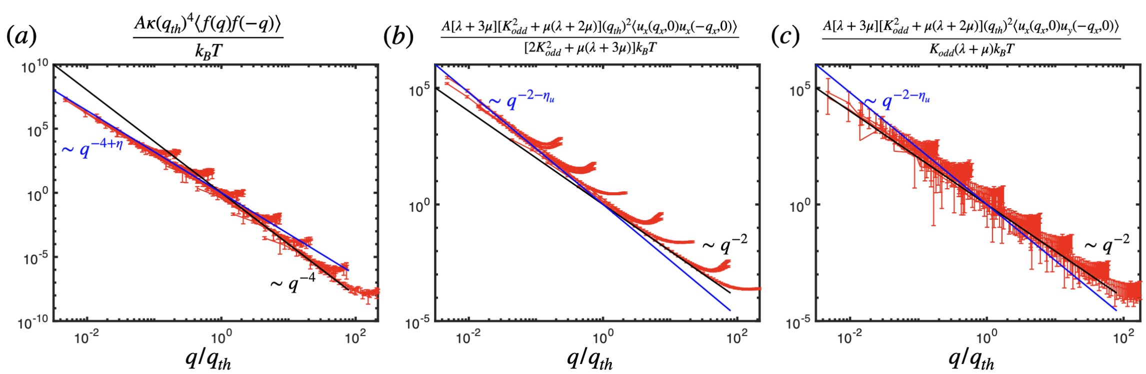

Figure 4.6: Above are plotted the correlation functions, non-dimensionalized by their microscopic parameters, of the (a) flexural modes, (b) longitudinal in-plane modes and (c) transverse-longitudinal in-plane modes. (a) and (b) show that the Aronovitz-Lubensky exponents ($\{\eta, \eta_u\} \approx \{.8, .4\}$) are un-perturbed by K_{odd}. The black curves mark the harmonic approximation and the blue curves mark the slopes with anomalous exponents. It is further confirmed that K_{odd} is irrelevant due to the fact that the scaling of (c) never presents with any anomalous behavior: thus meaning that K^R_{odd} scales with exponent $2\eta_u$.

Simulations with A_{odd}: For our simulations with K_{odd}, we ran the parameters given in Table. 4.2 and this resulted in a data collapse seen in Fig. 4.6 around the length scale q_{th} given by Eq. (4.87). In panels Fig. 4.7(a) and (b) we see that the flexural modes and the longitudinal in-plane modes are still associated with the Aronovitz-Lubensky exponents of $\{\eta, \eta_u\} \approx \{.8, .4\}$. In panel Fig. 4.7($c$), we observe that the correlation is non-zero. It is unclear from the simulations as to what the exact scaling is. It could potentially scale with exponent η_u, which would match with the theory, but more simulations are necessary to confirm this. We can potentially conclude that A_{odd} is at most a marginal perturbation of the theory. More simulations are needed to confirm the theoretical exponent from which one could conclude that since $\lambda^R(q), \mu^R(q) \sim q^{\eta_u}$, then we would necessarily have that $A^R_{odd}(q) \sim q^{\eta_u}$. One may notice from the table of parameters that a smaller system size was taken, this was in order to speed computation time.

Table 4.2:

Data Sets for Fig. 4.7			
L/a	$\hat{\kappa}/k_B T$	$A_{odd}/(\lambda+2\mu)$	$C_2/(\lambda+2\mu)$
35	1	.1	.048
35	10	.1	.048
35	100	.1	.048
35	$1e3$	.1	.048
35	$1e4$	.1	.048
35	$1e5$	.1	.048
35	$1e6$	.1	.048

4.5 Conclusion

During this project, we established new results concerning the behavior of non-equilibrium elastic sheets in particular with regards to the perturbative breaking of fluctuation-dissipation and the presence of odd elastic parameters A_{odd}, K_{odd}. Of the two, we have explored more so the latter and thus more investigation is merited

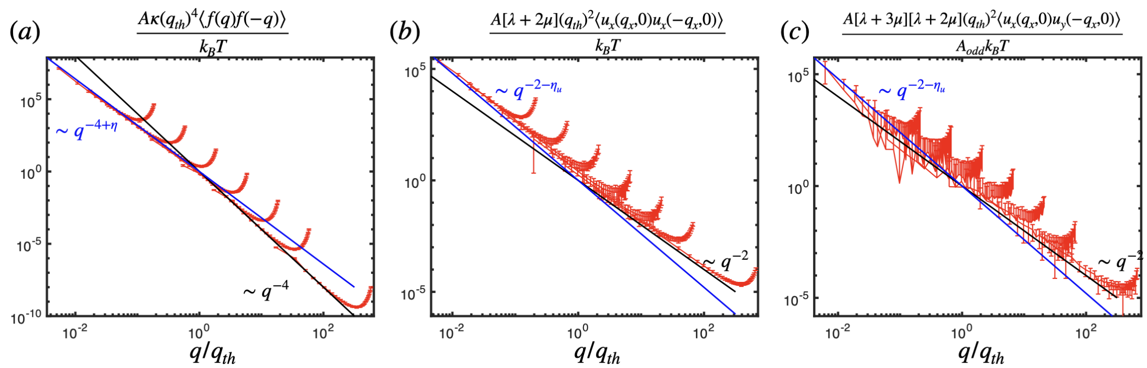

Figure 4.7: Above are plotted the correlation functions, non-dimensionalized by their microscopic parameters, of the (a) flexural modes, (b) longitudinal in-plane modes and (c) transverse-longitudinal in-plane modes. (a) and (b) show that the Aronovitz-Lubensky exponents ($\{\eta, \eta_u\} \approx \{.8, .4\}$) are un-perturbed by A_{odd}. The black curves mark the harmonic approximation and the blue curves mark the slopes with anomalous exponents. The simulations do perhaps suggest that A_{odd} is at most a marginal perturbation of the theory: thus meaning that A^R_{odd} scales with exponent at most η_u. However more simulations are necessary to confirm this.

for the former. We established that whilst for equilibrium statistical mechanics, where a Boltzmann weight is used, a Ward identity guarantees the structure of the strain tensor is preserved, for dynamical equations this is no longer the case. Thus fluctuation-dissipation is an unstable condition, with or without the presence of the odd elastic parameters. If we do enforce fluctuation-dissipation then we obtain that K_{odd} is an irrelevant perturbation whereas A_{odd} is a marginal perturbation. Thus it seems that breaking conservation of energy with the presence of K_{odd} is not a sufficiently strong perturbation to the Aronovitz-Lubensky fixed point. However the fact that A_{odd} is a marginal perturbation means that breaking conservation of angular momentum is a more significant perturbation. More simulations are required to further explore the phase space of different odd elastic parameters. On the theoretical side, more comprehension is necessary as to how to carefully treat the Feynman diagrams we ignored using a $1/d_c$ analysis, which in an ϵ-expansion would typically be summed over. Given that the upper critical dimension of the theory is $D_{uc} = 4$

whereas isotropy is only compatible with chiral odd elasticity for $D = 2$ makes this a future project to undertake.

Chapter 5

Future Directions

5.1 Monoclinic Elastic ϵ-Expansion

As noted during the chapter, the presence of zero eigen-vectors associated with orthorhombic perturbations of the Aronoviz-Lubensky fixed point at the 1-loop order merits further theoretical expansion to 2-loop order. This is in contrast to Toner's results [81]. This would further allude to whether there is indeed a true marginal perturbation of the fixed point or not (and thus perhaps a symmetry that maintains this as a zero eigen-value to all higher loop orders). Simulations with microscopically monoclinic elastic systems show us, however, that they become increasingly isotropic with smaller q/q_{th}. This may indicate that to 2-loop order, these zero eigen-values may become negative; performing the calculation is the only way to know for certain.

In addition, the fact that our renormalization scheme in $D = 2$ presented us with erroneous results requires a deeper theoretical comprehension. Developing such a theory would help us to calculate more pertinent observables for the exact dimension $D = 2$ and help make further phenomenological predictions mentioned in the next paragraph.

In addition, [69] showed that even with a local anisotropy in bending rigidity, one could obtain a flat-to-tubule phase transition. This local anisotropy is much weaker than that presented in [66] and thus it would be of interest to see whether this phase transition could be replicated with a simple in-plane anisotropy. This is also indicative of the fact that even if in the flat phase, the Aronovitz-Lubensky fixed point is stable, at higher temperatures where a tubule phase forms, it certainly is not any longer. A further general criterion for determining along which axis tubulization occurs, given some general anisotropic parameters, would be an interesting endeavor to take. For this, knowing the renormalization group equation in $D = 2$ would be necessary so that one can use those results to calculate the correlation of normals. A $D = 2$ renormalization group scheme thus becomes important to calculate

5.2 Anisotropic Stress

Anisotropic stress applied to a $D = 2$ elastic materials presented us with a scaling theory that turned out to be identical to that of tubules [66]. As previously mentioned, [69] obtained a flat-to-tubule transition with just a local bending anisotropy. Thus more explicit investigation is required to comprehend the behavior of anisotropic systems as a whole.

Other studies applying stress or varying boundary conditions have been done and reveal a rich set of results and phenomena upon which temperature has a non-trivial effect [33, 103, 30, 34, 36]. Further variations would be of interest, such as comprehending the effect of pure shear and simple shear, although one may expect these to induce an instability as well. An interesting scientific inquiry may be to have a disordered but quenched traction-force boundary condition to observe if such variations of the theory are important to consider or not, and thus if it is important to experiments. Further strain-controlled boundary conditions merit investigation.

Despite the opportunity for other rich variations, it is clear that in all cases, a deeper investigation into the simulations and theory regarding the absolute and differential Poisson ratios is necessary. In particular, significant work is necessary on the side of simulations where varying results have been obtained [35, 79].

5.3 Odd Elasticity

Our investigation has revealed that there are many fruitful opportunities from examining the dynamics of elastic membranes as opposed to using a Boltzmann weight associated with a static energy-derived weight. In particular, the form of the strain tensor is no longer protected by a symmetry and thus fluctuation-dissipation need not generally hold. This is true even for non-odd elastic membranes. A comprehension of where a globally stable fixed point is thus becomes necessary.

In addition, we have not considered the feedback effect from a solvent-membrane interaction such as those considered in [83]. Simulations along these lines are also necessary.

Though we found that the odd elastic parameters A_{odd}, K_{odd} act as either marginal or irrelevant perturbations to the Aronovitz-Lubensky fixed point (when fluctuation-dissipation is enforced), what occurs when these parameters are significantly larger than λ, μ remains an open question. Instabilities or a different fixed point may appear. Theory and simulations will both be necessary.

Considering finite-size effects and non-periodic boundary conditions would also be of interest to investigate. A_{odd} can exert a net torque at the boundary and thus may lead to further nuanced phenomenology.

Finally, our non-equilibrium parameters were purely local, what if we have more long ranged non-equilibrium forces? [81] indicates that even just anisotropic long-range forces can take us away from the Aronovitz-Lubensky fixed point. There is no reason to think that this would not be the case for a non-energy-derivable set of forces that are long-ranged.

Appendix A

Mathematica Code For Cubic Elastic Epsilon Expansion

```
In[•]:= Quit


δC1113 = δC1112;
δC1114 = δC1112;
δC1133 = δC1122;
δC1144 = δC1122;
δC1313 = δC1212;
δC1414 = δC1212;
δC3331 = δC2221;
δC4441 = δC2221;
δC2221 = 0;
δC1112 = 0;


In[•]:= C0 = Table[λ * KroneckerDelta[i, j] KroneckerDelta[k, l] +
     μ * KroneckerDelta[i, k] KroneckerDelta[j, l] + μ * KroneckerDelta[i, l]
      KroneckerDelta[j, k] + (δδ) * δC1111 * KroneckerDelta[i, 1] *
      KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
     (δδ) * δC1111 * KroneckerDelta[i, 2] *
      KroneckerDelta[j, 2] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
     (δδ) * δC1111 * KroneckerDelta[i, 3] *
      KroneckerDelta[j, 3] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
     (δδ) * δC1111 * KroneckerDelta[i, 4] *
      KroneckerDelta[j, 4] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
     (δδ) * δC1122 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 1] * KroneckerDelta[k, 2] *
         KroneckerDelta[l, 2] + KroneckerDelta[i, 2] * KroneckerDelta[j, 2] *
         KroneckerDelta[k, 1] * KroneckerDelta[l, 1] + KroneckerDelta[i, 1] *
         KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
        KroneckerDelta[i, 3] * KroneckerDelta[j, 3] * KroneckerDelta[k, 1] *
         KroneckerDelta[l, 1] + KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
         KroneckerDelta[k, 4] * KroneckerDelta[l, 4] + KroneckerDelta[i, 4] *
         KroneckerDelta[j, 4] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
        KroneckerDelta[i, 3] * KroneckerDelta[j, 3] * KroneckerDelta[k, 2] *
         KroneckerDelta[l, 2] + KroneckerDelta[i, 2] * KroneckerDelta[j, 2] *
         KroneckerDelta[k, 3] * KroneckerDelta[l, 3] + KroneckerDelta[i, 4] *
         KroneckerDelta[j, 4] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
        KroneckerDelta[i, 2] * KroneckerDelta[j, 2] * KroneckerDelta[k, 4] *
         KroneckerDelta[l, 4] + KroneckerDelta[i, 3] * KroneckerDelta[j, 3] *
         KroneckerDelta[k, 4] * KroneckerDelta[l, 4] + KroneckerDelta[i, 4] *
         KroneckerDelta[j, 4] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3]) +
     (δδ) * δC1212 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 2] * KroneckerDelta[k, 1] *
         KroneckerDelta[l, 2] + KroneckerDelta[i, 2] * KroneckerDelta[j, 1] *
         KroneckerDelta[k, 2] * KroneckerDelta[l, 1] + KroneckerDelta[i, 2] *
```

```
 KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 2] +
KroneckerDelta[i, 1] *
 KroneckerDelta[j, 2] * KroneckerDelta[k, 2] * KroneckerDelta[l, 1] +

KroneckerDelta[i, 1] * KroneckerDelta[j, 3] *
 KroneckerDelta[k, 1] * KroneckerDelta[l, 3] + KroneckerDelta[i, 3] *
 KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 1] +
KroneckerDelta[i, 3] *
 KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 3] +
KroneckerDelta[i, 1] *
 KroneckerDelta[j, 3] * KroneckerDelta[k, 3] * KroneckerDelta[l, 1] +

KroneckerDelta[i, 1] * KroneckerDelta[j, 4] *
 KroneckerDelta[k, 1] * KroneckerDelta[l, 4] + KroneckerDelta[i, 4] *
 KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 1] +
KroneckerDelta[i, 1] * KroneckerDelta[j, 4] *
 KroneckerDelta[k, 4] * KroneckerDelta[l, 1] + KroneckerDelta[i, 4] *
 KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 4] +


KroneckerDelta[i, 3] * KroneckerDelta[j, 2] *
 KroneckerDelta[k, 3] * KroneckerDelta[l, 2] + KroneckerDelta[i, 2] *
 KroneckerDelta[j, 3] * KroneckerDelta[k, 2] * KroneckerDelta[l, 3] +
KroneckerDelta[i, 3] * KroneckerDelta[j, 2] *
 KroneckerDelta[k, 2] * KroneckerDelta[l, 3] + KroneckerDelta[i, 2] *
 KroneckerDelta[j, 3] * KroneckerDelta[k, 3] * KroneckerDelta[l, 2] +

KroneckerDelta[i, 4] * KroneckerDelta[j, 2] *
 KroneckerDelta[k, 4] * KroneckerDelta[l, 2] + KroneckerDelta[i, 2] *
 KroneckerDelta[j, 4] * KroneckerDelta[k, 2] * KroneckerDelta[l, 4] +
KroneckerDelta[i, 4] * KroneckerDelta[j, 2] *
 KroneckerDelta[k, 2] * KroneckerDelta[l, 4] + KroneckerDelta[i, 2] *
 KroneckerDelta[j, 4] * KroneckerDelta[k, 4] * KroneckerDelta[l, 2] +


KroneckerDelta[i, 3] * KroneckerDelta[j, 4] * KroneckerDelta[k, 3] *
 KroneckerDelta[l, 4] + KroneckerDelta[i, 4] * KroneckerDelta[j, 3] *
 KroneckerDelta[k, 4] * KroneckerDelta[l, 3] + KroneckerDelta[i, 3] *
 KroneckerDelta[j, 4] * KroneckerDelta[k, 4] * KroneckerDelta[l, 3] +
KroneckerDelta[i, 4] * KroneckerDelta[j, 3] * KroneckerDelta[k, 3] *
 KroneckerDelta[l, 4]), {i, 4}, {j, 4}, {k, 4}, {l, 4}];
```

commenting out κ is just a product of the non-dimensionalization of all the elastic constants.

```
δB1133 = δB1122;
δB1144 = δB1122;
δB1113 = δB1112;
δB1114 = δB1112;
δB3331 = δB2221;
δB4441 = δB2221;
δB2221 = 0;
δB1112 = 0;
```

```
In[]:= B0 = Table[(*κ*) (KroneckerDelta[i, 1] *
          KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
         KroneckerDelta[i, 2] *
          KroneckerDelta[j, 2] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
         KroneckerDelta[i, 3] * KroneckerDelta[j, 3] *
          KroneckerDelta[k, 3] * KroneckerDelta[l, 3] + KroneckerDelta[i, 4] *
          KroneckerDelta[j, 4] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
         KroneckerDelta[i, 1] *
          KroneckerDelta[j, 1] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
         KroneckerDelta[i, 2] *
          KroneckerDelta[j, 2] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
         KroneckerDelta[i, 1] *
          KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
         KroneckerDelta[i, 3] *
          KroneckerDelta[j, 3] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
         KroneckerDelta[i, 4] *
          KroneckerDelta[j, 4] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
         KroneckerDelta[i, 1] *
          KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
         KroneckerDelta[i, 3] *
          KroneckerDelta[j, 3] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
         KroneckerDelta[i, 2] *
          KroneckerDelta[j, 2] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
         KroneckerDelta[i, 4] *
          KroneckerDelta[j, 4] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
         KroneckerDelta[i, 2] *
          KroneckerDelta[j, 2] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
         KroneckerDelta[i, 3] *
          KroneckerDelta[j, 3] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
         KroneckerDelta[i, 4] * KroneckerDelta[j, 4] *
          KroneckerDelta[k, 3] * KroneckerDelta[l, 3]) +
       δδ * δB1111 * KroneckerDelta[i, 1] *
        KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
```

```
δδ * δB1111 * KroneckerDelta[i, 2] *
 KroneckerDelta[j, 2] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
δδ * δB1111 * KroneckerDelta[i, 3] *
 KroneckerDelta[j, 3] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
δδ * δB1111 * KroneckerDelta[i, 4] *
 KroneckerDelta[j, 4] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
δδ * δB1122 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
    KroneckerDelta[k, 2] * KroneckerDelta[l, 2] + KroneckerDelta[i, 2] *
    KroneckerDelta[j, 2] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
   KroneckerDelta[i, 3] * KroneckerDelta[j, 3] * KroneckerDelta[k, 1] *
    KroneckerDelta[l, 1] + KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
    KroneckerDelta[k, 3] * KroneckerDelta[l, 3] + KroneckerDelta[i, 4] *
    KroneckerDelta[j, 4] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
   KroneckerDelta[i, 1] * KroneckerDelta[j, 1] * KroneckerDelta[k, 4] *
    KroneckerDelta[l, 4] + KroneckerDelta[i, 2] * KroneckerDelta[j, 2] *
    KroneckerDelta[k, 3] * KroneckerDelta[l, 3] + KroneckerDelta[i, 3] *
    KroneckerDelta[j, 3] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
   KroneckerDelta[i, 2] * KroneckerDelta[j, 2] * KroneckerDelta[k, 4] *
    KroneckerDelta[l, 4] + KroneckerDelta[i, 4] * KroneckerDelta[j, 4] *
    KroneckerDelta[k, 2] * KroneckerDelta[l, 2] + KroneckerDelta[i, 3] *
    KroneckerDelta[j, 3] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
   KroneckerDelta[i, 4] * KroneckerDelta[j, 4] * KroneckerDelta[k, 3] *
    KroneckerDelta[l, 3]), {i, 4}, {j, 4}, {k, 4}, {l, 4}];
```

Calculating In-Plane

```
In[ ]:= P0 = Table[Cos[θ1] * KroneckerDelta[i, 1] + Sin[θ1] Cos[θ2] KroneckerDelta[i, 2] +
     Sin[θ1] Sin[θ2] Cos[θ3] KroneckerDelta[i, 3] +
     Sin[θ1] Sin[θ2] Sin[θ3] KroneckerDelta[i, 4], {i, 4}];

In[ ]:= HHH[i_, j_, k_, l_] := (Sum[C0⟦i, j, x, y⟧ * C0⟦k, l, w, z⟧ * P0⟦x⟧ * P0⟦y⟧ * P0⟦w⟧ * P0⟦z⟧,
     {x, 1, 4}, {y, 1, 4}, {w, 1, 4}, {z, 1, 4}]);
```

Equation For λ via 2233

```
HHHH2233 = Simplify[Normal[
    Series[(HHH[2, 2, 3, 3]) * (1 / (Sum[B0[[d, f, g, h]] × P0[[d]] × P0[[f]] × P0[[g]] × P0[[h]],
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^ 2), {δδ, 0, 1}]]];
GG2233 = HHHH2233;
KK2233 = Simplify[GG2233
    - δB1111 *
     (D[GG2233, δB1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1111 *
     (D[GG2233, δC1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δB1122 *
     (D[GG2233, δB1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1122 *
     (D[GG2233, δC1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1212 * (D[GG2233, δC1212] /.
       {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})];
```

```
In[ ]:= JJ2233 =
   (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) * (Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] * KK2233, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] - (- δB1111 *
         Integrate[Sin[θ1] ^ 2 * Sin[θ2] * (D[GG2233, δB1111] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1122 *
         Integrate[Sin[θ1] ^ 2 * Sin[θ2] * (D[GG2233, δB1122] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1111 *
         Integrate[Sin[θ1] ^ 2 * Sin[θ2] * (D[GG2233, δC1111] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1122 *
         Integrate[Sin[θ1] ^ 2 * Sin[θ2] * (D[GG2233, δC1122] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1212 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] *
           (D[GG2233, δC1212] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
              δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for μ via 2323

```
HHHH2323 = Simplify[Normal[
    Series[(HHH[2, 3, 2, 3]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {δδ, 0, 1}]]];
GG2323 = (HHHH2323);
KK2323 = Simplify[GG2323
    - δB1111 *
     (D[GG2323, δB1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1111 *
     (D[GG2323, δC1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δB1122 *
     (D[GG2323, δB1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1122 *
     (D[GG2323, δC1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1212 * (D[GG2323, δC1212] /.
       {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})];
```

In[ ]:=
```
JJ2323 =
  (1 / 4) (1 / (kBT) ^2) * (2 * dc) * ((kBT / 1 (*A*)) ^2) * (1 (*A*) / (2 π) ^4) * (Integrate[
      Sin[θ1] ^2 * Sin[θ2] * KK2323, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] - (-δB1111 *
        Integrate[Sin[θ1] ^2 * Sin[θ2] * (D[GG2323, δB1111] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1122 *
        Integrate[Sin[θ1] ^2 * Sin[θ2] * (D[GG2323, δB1122] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1111 *
        Integrate[Sin[θ1] ^2 * Sin[θ2] * (D[GG2323, δC1111] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1122 *
        Integrate[Sin[θ1] ^2 * Sin[θ2] * (D[GG2323, δC1122] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1212 * Integrate[Sin[θ1] ^2 * Sin[θ2] *
          (D[GG2323, δC1212] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
              δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for $\lambda+2\mu+\delta C1111$

```
HHHH1111 = Simplify[Normal[
    Series[(HHH[1, 1, 1, 1]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^ 2), {δδ, 0, 1}]]];
GG1111 = (HHHH1111);
KK1111 = Simplify[GG1111
    - δB1111 *
     (D[GG1111, δB1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1111 *
     (D[GG1111, δC1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δB1122 *
     (D[GG1111, δB1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1122 *
     (D[GG1111, δC1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})
    - δC1212 * (D[GG1111, δC1212] /.
       {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})];
```

```
In[ ]:= JJ1111 =
  (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) * (Integrate[
      Sin[θ1] ^ 2 * Sin[θ2] * KK1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] - (-δB1111 *
        Integrate[Sin[θ1] ^ 2 * Sin[θ2] * (D[GG1111, δB1111] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1122 *
        Integrate[Sin[θ1] ^ 2 * Sin[θ2] * (D[GG1111, δB1122] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1111 *
        Integrate[Sin[θ1] ^ 2 * Sin[θ2] * (D[GG1111, δC1111] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1122 *
        Integrate[Sin[θ1] ^ 2 * Sin[θ2] * (D[GG1111, δC1122] /. {δC1111 → 0, δB1111 → 0,
              δB1122 → 0, δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1212 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] *
           (D[GG1111, δC1212] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
              δC1122 → 0, δC1212 → 0}), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Integrating Out

```
U0 = Table[U1 * KroneckerDelta[i, 1] + U2 * KroneckerDelta[i, 2] +
     U3 * KroneckerDelta[i, 3] + U4 * KroneckerDelta[i, 4], {i, 4}];
Q10 = Table[P1 * KroneckerDelta[i, 1] + P2 * KroneckerDelta[i, 2] +
     P3 * KroneckerDelta[i, 3] + P4 * KroneckerDelta[i, 4], {i, 4}];
Q20 = Table[ω * Q1 * KroneckerDelta[i, 1] + ω * Q2 * KroneckerDelta[i, 2] +
     ω * Q3 * KroneckerDelta[i, 3] + ω * Q4 * KroneckerDelta[i, 4], {i, 4}];
Q30 = -Q10;
Q40 = -Q20;
Q0 = Table[(Q10[[1]] + Q20[[1]]) * KroneckerDelta[i, 1] +
     (Q10[[2]] + Q20[[2]]) * KroneckerDelta[i, 2] + (Q10[[3]] + Q20[[3]]) * KroneckerDelta[i, 3] +
     (Q10[[4]] + Q20[[4]]) * KroneckerDelta[i, 4], {i, 4}];

QuadraticU = (Sum[C0[[i, j, k, l]] * Q0[[i]] * U0[[j]] * Q0[[k]] * U0[[l]],
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}]);

MatrixQuadraticU =
   {{Coefficient[QuadraticU, U1^2], Coefficient[QuadraticU, U1 * U2] / 2,
     Coefficient[QuadraticU, U1 * U3] / 2, Coefficient[QuadraticU, U1 * U4] / 2},
    {Coefficient[QuadraticU, U1 * U2] / 2, Coefficient[QuadraticU, U2^2],
     Coefficient[QuadraticU, U2 * U3] / 2, Coefficient[QuadraticU, U2 * U4] / 2},
    {Coefficient[QuadraticU, U1 * U3] / 2, Coefficient[QuadraticU, U2 * U3] / 2,
     Coefficient[QuadraticU, U3^2], Coefficient[QuadraticU, U3 * U4] / 2},
    {Coefficient[QuadraticU, U1 * U4] / 2, Coefficient[QuadraticU, U2 * U4] / 2,
     Coefficient[QuadraticU, U4 * U3] / 2, Coefficient[QuadraticU, U4^2]}};

US = {U1, U2, U3, U4};

Dominant = (D[MatrixQuadraticU] /. {δδ → 0});

In[◦]:= GHK = Map[Reverse, Minors[Dominant], {0, 1}];

GHKTrue = Table[((GHK[[i, j]])) * (-1)^(i + j), {i, 4}, {j, 4}];

epsilon = δδ * (D[MatrixQuadraticU, {δδ, 1}] /. {δδ → 0});

In[◦]:= Denom1 = Simplify[Det[Dominant]];

GFrac = Simplify[Normal[Series[1 / Denom1, {ω, 0, 2}]] (*/.{ω→1}*)];
GFrac2 = Simplify[Normal[Series[(1 / Denom1)^2, {ω, 0, 2}]] (*/.{ω→1}*)];
Denom12 = Simplify[Normal[Series[(Denom1)^2, {ω, 0, 2}]] (*/.{ω→1}*)];

InverseMatIso = GFrac * GHKTrue;
InverseMatPert = (GFrac2) GHKTrue.epsilon.GHKTrue;
```

```
LinearUAlpha = (I / 2) *
    Sum[(δδ) * (D[C0⟦i, j, k, l⟧, {δδ}] /. {δδ → 0}) * Q0⟦i⟧ * U0⟦j⟧ * Q10⟦k⟧ * Q20⟦l⟧,
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
LinearUVectorAlpha = {Coefficient[LinearUAlpha, U1], Coefficient[LinearUAlpha, U2],
    Coefficient[LinearUAlpha, U3], Coefficient[LinearUAlpha, U4]};
LinearIsoUAlpha =
   (I / 2) * Sum[(C0⟦i, j, k, l⟧ /. {δδ → 0}) * Q0⟦i⟧ * U0⟦j⟧ * Q10⟦k⟧ * Q20⟦l⟧,
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
LinearIsoUVectorAlpha =
   {Coefficient[LinearIsoUAlpha, U1], Coefficient[LinearIsoUAlpha, U2],
    Coefficient[LinearIsoUAlpha, U3], Coefficient[LinearIsoUAlpha, U4]};
LinearUBeta = (I / 2) *
    Sum[(δδ) * (D[C0⟦i, j, k, l⟧, {δδ}] /. {δδ → 0}) * (-Q0⟦i⟧) * U0⟦j⟧ * Q30⟦k⟧ * Q40⟦l⟧,
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
LinearUVectorBeta = {Coefficient[LinearUBeta, U1], Coefficient[LinearUBeta, U2],
    Coefficient[LinearUBeta, U3], Coefficient[LinearUBeta, U4]};
LinearIsoUBeta =
   (I / 2) * Sum[(C0⟦i, j, k, l⟧ /. {δδ → 0}) * (-Q0⟦i⟧) * U0⟦j⟧ * Q30⟦k⟧ * Q40⟦l⟧,
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
LinearIsoUVectorBeta =
   {Coefficient[LinearIsoUBeta, U1], Coefficient[LinearIsoUBeta, U2],
    Coefficient[LinearIsoUBeta, U3], Coefficient[LinearIsoUBeta, U4]};

FFourthTerm = - (1 / 8) Sum[C0⟦i, j, k, l⟧ * Q10⟦i⟧ * Q20⟦j⟧ * Q30⟦k⟧ * Q40⟦l⟧,
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];

TotalFFourth = (1 / 2)
    (LinearIsoUVectorAlpha.(InverseMatIso - InverseMatPert).LinearIsoUVectorBeta +
      LinearUVectorAlpha.InverseMatIso.LinearIsoUVectorBeta +
      LinearIsoUVectorAlpha.InverseMatIso.LinearUVectorBeta);

FFourthTerm = - (1 / 8) Sum[C0⟦i, j, k, l⟧ * Q10⟦i⟧ * Q20⟦j⟧ * Q30⟦k⟧ * Q40⟦l⟧,
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];

TotalFFourth = (1 / 2)
    (LinearIsoUVectorAlpha.(InverseMatIso - InverseMatPert).LinearIsoUVectorBeta +
      LinearUVectorAlpha.InverseMatIso.LinearIsoUVectorBeta +
      LinearIsoUVectorAlpha.InverseMatIso.LinearUVectorBeta);
```

```
In[ ]:= DenomFull = Simplify[((Sum[B0⟦d, f, g, h⟧ × Q10⟦d⟧ × Q10⟦f⟧ × Q10⟦g⟧ × Q10⟦h⟧,
        {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]))];

JJJJJP = Normal[Series[(1 / DenomFull) TotalFFourth, {δδ, 0, 1}]];
```

```
In[ ]:= JJJJJ = JJJJJP;
```

Renormalization of κ+δB1111

```
JJJ1111 = (Simplify[Coefficient[(JJJJJ /. {ω → 1, Q2 → 0, Q3 → 0, Q4 → 0}), Q1, 4]]) /.
   {P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
    P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};
```

```
In[∘]:= NJJJ1111 = Simplify[Numerator[JJJ1111]];
DJJJ1111 = Simplify[Denominator[JJJ1111]];
```

```
BBB1111 = (2 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) * Integrate[
   Sin[θ1] ^2 * Sin[θ2] NJJJ1111 / DJJJ1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}];
```

Renormalization of κ

```
JJJ1122 =
  (Simplify[Coefficient[JJJJJ /. {ω → 1, Q4 → 0, Q3 → 0}, Q1^2 Q2^2]]) /. {P1 → Cos[θ1],
   P2 → Sin[θ1] Cos[θ2], P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};
```

```
In[∘]:= NJJJ1122 = Simplify[Numerator[JJJ1122]];
```

```
In[∘]:= DJJJ1122 = Simplify[Denominator[JJJ1122]];
```

```
In[∘]:= NJJJ1122C1111 = δC1111 * (D[NJJJ1122, δC1111] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
NJJJ1122C1122 = δC1122 * (D[NJJJ1122, δC1122] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
NJJJ1122C1212 = δC1212 * (D[NJJJ1122, δC1212] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
NJJJ1122B1122 = δB1122 * (D[NJJJ1122, δB1122] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
NJJJ1122B1111 = δB1111 * (D[NJJJ1122, δB1111] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
NJJJ1122None =
  (NJJJ1122 /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
```

```
In[∘]:= BBB1122C1111 = (1 / DJJJ1122) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
   Integrate[Sin[θ1] ^2 * Sin[θ2] NJJJ1122C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}];
```

```
In[∘]:= BBB1122B1111 = (1 / DJJJ1122) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
   Integrate[Sin[θ1] ^2 * Sin[θ2] NJJJ1122B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}];
```

```
In[∘]:= BBB1122C1122 = (1 / DJJJ1122) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
   Integrate[Sin[θ1] ^2 * Sin[θ2] NJJJ1122C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}];
```

```
In[∘]:= BBB1122C1212 = (1 / DJJJ1122) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
   Integrate[Sin[θ1] ^2 * Sin[θ2] NJJJ1122C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}];
```

```
In[•]:= BBB1122B1122 = (1 / DJJJ1122) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^ 4) *
          Integrate[Sin[θ1] ^ 2 * Sin[θ2] NJJJ1122B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}];

In[•]:= BBB1122None = (1 / DJJJ1122) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^ 4) *
          Integrate[Sin[θ1] ^ 2 * Sin[θ2] NJJJ1122None, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}];

In[•]:= BBB1122 = BBB1122None + BBB1122C1111 +
          BBB1122B1111 + BBB1122C1122 + BBB1122C1212 + BBB1122B1122;
```

Renormalization of κ+δB2222

```
JJJ2222 = (Simplify[Coefficient[JJJJJ /. {ω → 1, Q1 → 0, Q3 → 0, Q4 → 0}, Q2 ^ 4]]) /.
    {P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
     P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

In[•]:= NJJJ2222 = Simplify[Numerator[JJJ2222]];
        DJJJ2222 = Simplify[Denominator[JJJ2222]];

        BBB2222 = (2 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^ 4) * Integrate[
           Sin[θ1] ^ 2 * Sin[θ2] (NJJJ2222 / DJJJ2222), {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}];
```

Full Stability Analysis

```
In[•]:= ξf = (Simplify[(((ϵ + (BBB1122)) / 2))]);
        ξff = (Simplify[(((ϵ + (BBB1122)) / 2))]) /.
           {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0};
```

In[ ]:=
```
δC1212Toner1 = Simplify[(Simplify[Simplify[(4 * ξf - ϵ) * (μ + δδ * δC1212) - (JJ2323)]]) -
    (Simplify[Simplify[(4 * ξf - ϵ) * (μ) - (JJ2323)]] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})];
δC1212Toner =
  (δC1212Toner1 /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δC1111 * (D[δC1212Toner1, δC1111] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δC1122 * (D[δC1212Toner1, δC1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
       δC1122 → 0, δC1212 → 0}) + δC1212 * (D[δC1212Toner1, δC1212] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δB1111 * (D[δC1212Toner1, δB1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
       δC1122 → 0, δC1212 → 0}) + δB1122 * (D[δC1212Toner1, δB1122] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
δC1122Toner1 = Simplify[(Simplify[Simplify[(4 * ξf - ϵ) * (λ + δδ * δC1122) - (JJ2233)]]) -
    (Simplify[Simplify[(4 * ξf - ϵ) * (λ) - (JJ2233)]] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})];
δC1122Toner =
  (δC1122Toner1 /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δC1111 * (D[δC1122Toner1, δC1111] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δC1122 * (D[δC1122Toner1, δC1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
       δC1122 → 0, δC1212 → 0}) + δC1212 * (D[δC1122Toner1, δC1212] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δB1111 * (D[δC1122Toner1, δB1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
       δC1122 → 0, δC1212 → 0}) + δB1122 * (D[δC1122Toner1, δB1122] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
```

In[ ]:=
```
μToner = ((4 * ξf - ϵ) * (μ) - (JJ2323)) /.
   {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0};
```

In[ ]:=
```
δC1111Toner1 = Simplify[Simplify[(((4 * ξf - ϵ) * (λ + 2 μ + δδ * δC1111))) - ((JJ1111))] -
    Simplify[((4 * ξf - ϵ) * (λ + 2 μ) - (JJ1111)) /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}]];
δC1111Toner =
  (δC1111Toner1 /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δC1111 * (D[δC1111Toner1, δC1111] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δC1122 * (D[δC1111Toner1, δC1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
       δC1122 → 0, δC1212 → 0}) + δC1212 * (D[δC1111Toner1, δC1212] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
   δB1111 * (D[δC1111Toner1, δB1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
       δC1122 → 0, δC1212 → 0}) + δB1122 * (D[δC1111Toner1, δB1122] /.
      {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
```

```
In[∘]:= δB1111Toner1 = Simplify[Simplify[(((2 (ξf) - ϵ) * (1 + δδ * δB1111) - (BBB1111)))] -
      (Simplify[(((2 (ξf) - ϵ) * (1) - (BBB1111)))] /.
        {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})];
δB1111Toner =
   (δB1111Toner1 /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
    δC1111 * (D[δB1111Toner1, δC1111] /.
        {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
    δC1122 * (D[δB1111Toner1, δC1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
         δC1122 → 0, δC1212 → 0}) + δC1212 * (D[δB1111Toner1, δC1212] /.
        {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
    δB1111 * (D[δB1111Toner1, δB1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
         δC1122 → 0, δC1212 → 0}) + δB1122 * (D[δB1111Toner1, δB1122] /.
        {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});
δB1122Toner1 = Simplify[Simplify[(((2 (ξf) - ϵ) * (1 + δδ * δB1122) - (BBB1122)))] -
     (Simplify[(((2 (ξf) - ϵ) * (1) - (BBB1122)))] /.
        {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0})];
δB1122Toner =
   (δB1122Toner1 /. {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
    δC1111 * (D[δB1122Toner1, δC1111] /.
        {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
    δC1122 * (D[δB1122Toner1, δC1122] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
         δC1122 → 0, δC1212 → 0}) + δC1212 * (D[δB1122Toner1, δC1212] /.
        {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0}) +
    δB1111 * (D[δB1122Toner1, δB1111] /. {δC1111 → 0, δB1111 → 0, δB1122 → 0,
         δC1122 → 0, δC1212 → 0}) + δB1122 * (D[δB1122Toner1, δB1122] /.
        {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0});

B1111Equation = Simplify[δB1111Toner];
B1122Equation = Simplify[δB1122Toner];
C1111Equation = Simplify[(μ * δC1111Toner - δδ * δC1111 * μToner) / (μ) ^2];
C1122Equation = Simplify[(μ * δC1122Toner - δδ * δC1122 * μToner) / (μ) ^2];
C1212Equation = Simplify[(μ * δC1212Toner - δδ * δC1212 * μToner) / (μ) ^2];
```

```
In[ ]:= MatrixStability =
  Simplify[(1 / (δδ)) {{μ * D[C1111Equation, δC1111], μ * D[C1111Equation, δC1212],
       μ * D[C1111Equation, δC1122], D[C1111Equation, δB1111],
       D[C1111Equation, δB1122]}, {μ * D[C1212Equation, δC1111],
       μ * D[C1212Equation, δC1212], μ * D[C1212Equation, δC1122],
       D[C1212Equation, δB1111], D[C1212Equation, δB1122]},
      {μ * D[C1122Equation, δC1111], μ * D[C1122Equation, δC1212],
       μ * D[C1122Equation, δC1122], D[C1122Equation, δB1111],
       D[C1122Equation, δB1122]}, {μ * D[B1111Equation, δC1111],
       μ * D[B1111Equation, δC1212], μ * D[B1111Equation, δC1122], D[B1111Equation,
        δB1111], D[B1111Equation, δB1122]}, {μ * D[B1122Equation, δC1111],
       μ * D[B1122Equation, δC1212], μ * D[B1122Equation, δC1122],
       D[B1122Equation, δB1111], D[B1122Equation, δB1122]}} /.
    {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0,
     μ → (96 * ϵ * π ^ 2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc)}];
```

```
In[ ]:= MatrixStability =
  Simplify[{{D[δC1111Toner, δC1111], D[δC1111Toner, δC1212], D[δC1111Toner, δC1122],
      D[δC1111Toner, δB1111], D[δC1111Toner, δB1122]},
     {D[δC1212Toner, δC1111], D[δC1212Toner, δC1212], D[δC1212Toner, δC1122],
      D[δC1212Toner, δB1111], D[δC1212Toner, δB1122]},
     {D[δC1122Toner, δC1111], D[δC1122Toner, δC1212], D[δC1122Toner, δC1122],
      D[δC1122Toner, δB1111], D[δC1122Toner, δB1122]},
     {D[δB1111Toner, δC1111], D[δB1111Toner, δC1212], D[δB1111Toner, δC1122],
      D[δB1111Toner, δB1111], D[δB1111Toner, δB1122]},
     {D[δB1122Toner, δC1111], D[δB1122Toner, δC1212], D[δB1122Toner, δC1122],
      D[δB1122Toner, δB1111], D[δB1122Toner, δB1122]}} /.
    {δC1111 → 0, δB1111 → 0, δB1122 → 0, δC1122 → 0, δC1212 → 0,
     μ → (96 * ϵ * π ^ 2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc)}];
```

```
In[ ]:= Eigenvalues[MatrixStability]
```

Appendix B

Mathematica Code For Monoclinic Elastic Epsilon Expansion

In[ ]:= Quit

Defining the Elastic Perturbations

```
δC1113 = δC1112;
δC1114 = δC1112;
δC1133 = δC1122;
δC1144 = δC1122;
δC1313 = δC1212;
δC1414 = δC1212;
δC3331 = δC2221;
δC4441 = δC2221;
```

In[ ]:=
```
C0 = Table[λ * KroneckerDelta[i, j] KroneckerDelta[k, l] +
    μ * KroneckerDelta[i, k] KroneckerDelta[j, l] + μ * KroneckerDelta[i, l]
     KroneckerDelta[j, k] + (δδ) * δC1111 * KroneckerDelta[i, 1] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +

    (δδ) * δC1122 * KroneckerDelta[i, 1] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
    (δδ) * δC1122 * KroneckerDelta[i, 2] *
     KroneckerDelta[j, 2] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
    (δδ) * δC1122 * KroneckerDelta[i, 1] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
    (δδ) * δC1122 * KroneckerDelta[i, 3] *
     KroneckerDelta[j, 3] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
    (δδ) * δC1122 * KroneckerDelta[i, 1] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
    (δδ) * δC1122 * KroneckerDelta[i, 4] *
     KroneckerDelta[j, 4] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
    (δδ) * δC1212 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 2] *
        KroneckerDelta[k, 1] * KroneckerDelta[l, 2] + KroneckerDelta[i, 2] *
        KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 2] +
       KroneckerDelta[i, 2] *
        KroneckerDelta[j, 1] * KroneckerDelta[k, 2] * KroneckerDelta[l, 1] +
       KroneckerDelta[i, 1] * KroneckerDelta[j, 2] *
        KroneckerDelta[k, 2] * KroneckerDelta[l, 1]) +
    (δδ) * δC1212 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 3] *
        KroneckerDelta[k, 1] * KroneckerDelta[l, 3] + KroneckerDelta[i, 3] *
        KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 3] +
       KroneckerDelta[i, 3] *
        KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 1] +
```

```
    KroneckerDelta[i, 1] * KroneckerDelta[j, 3] *
     KroneckerDelta[k, 3] * KroneckerDelta[l, 1]) +
(δδ) * δC1212 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 4] *
     KroneckerDelta[k, 1] * KroneckerDelta[l, 4] + KroneckerDelta[i, 4] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 4] +
    KroneckerDelta[i, 4] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 1] +
    KroneckerDelta[i, 1] * KroneckerDelta[j, 4] *
     KroneckerDelta[k, 4] * KroneckerDelta[l, 1]) +
(δδ) * δC1112 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
     KroneckerDelta[k, 1] * KroneckerDelta[l, 2] + KroneckerDelta[i, 2] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
    KroneckerDelta[i, 1] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 2] * KroneckerDelta[l, 1] +
    KroneckerDelta[i, 1] * KroneckerDelta[j, 2] *
     KroneckerDelta[k, 1] * KroneckerDelta[l, 1]) +
(δδ) * δC1112 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
     KroneckerDelta[k, 1] * KroneckerDelta[l, 3] + KroneckerDelta[i, 3] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
    KroneckerDelta[i, 1] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 1] +
    KroneckerDelta[i, 1] * KroneckerDelta[j, 3] *
     KroneckerDelta[k, 1] * KroneckerDelta[l, 1]) +
(δδ) * δC1112 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
     KroneckerDelta[k, 1] * KroneckerDelta[l, 4] + KroneckerDelta[i, 4] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
    KroneckerDelta[i, 1] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 1] +
    KroneckerDelta[i, 1] * KroneckerDelta[j, 4] *
     KroneckerDelta[k, 1] * KroneckerDelta[l, 1]) +
(δδ) * δC2221 * (KroneckerDelta[i, 2] * KroneckerDelta[j, 2] *
     KroneckerDelta[k, 2] * KroneckerDelta[l, 1] + KroneckerDelta[i, 2] *
     KroneckerDelta[j, 2] * KroneckerDelta[k, 1] * KroneckerDelta[l, 2] +
    KroneckerDelta[i, 1] *
     KroneckerDelta[j, 2] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
    KroneckerDelta[i, 2] * KroneckerDelta[j, 1] *
     KroneckerDelta[k, 2] * KroneckerDelta[l, 2]) +
(δδ) * δC2221 * (KroneckerDelta[i, 3] * KroneckerDelta[j, 3] *
     KroneckerDelta[k, 3] * KroneckerDelta[l, 1] + KroneckerDelta[i, 3] *
     KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
    KroneckerDelta[i, 1] *
     KroneckerDelta[j, 3] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
    KroneckerDelta[i, 3] * KroneckerDelta[j, 3] *
     KroneckerDelta[k, 1] * KroneckerDelta[l, 3]) +
```

```
(δδ) * δC2221 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 4] *
    KroneckerDelta[k, 4] * KroneckerDelta[l, 4] + KroneckerDelta[i, 4] *
    KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
   KroneckerDelta[i, 4] *
    KroneckerDelta[j, 4] * KroneckerDelta[k, 1] * KroneckerDelta[l, 4] +
   KroneckerDelta[i, 4] * KroneckerDelta[j, 4] * KroneckerDelta[k, 4] *
    KroneckerDelta[l, 1]), {i, 4}, {j, 4}, {k, 4}, {l, 4}];
```

commenting out κ is just a product of the non-dimensionalization of all the elastic constants.

```
In[◦]:= δB1133 = δB1122;
δB1144 = δB1122;
δB1113 = δB1112;
δB1114 = δB1112;
δB3331 = δB2221;
δB4441 = δB2221;
```

```
In[◦]:= B0 = Table[(*κ*) (KroneckerDelta[i, 1] *
      KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
     KroneckerDelta[i, 2] *
      KroneckerDelta[j, 2] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
     KroneckerDelta[i, 3] * KroneckerDelta[j, 3] *
      KroneckerDelta[k, 3] * KroneckerDelta[l, 3] + KroneckerDelta[i, 4] *
      KroneckerDelta[j, 4] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
     KroneckerDelta[i, 1] *
      KroneckerDelta[j, 1] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
     KroneckerDelta[i, 2] *
      KroneckerDelta[j, 2] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
     KroneckerDelta[i, 1] *
      KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
     KroneckerDelta[i, 3] *
      KroneckerDelta[j, 3] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
     KroneckerDelta[i, 4] *
      KroneckerDelta[j, 4] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
     KroneckerDelta[i, 1] *
      KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
     KroneckerDelta[i, 3] *
      KroneckerDelta[j, 3] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
     KroneckerDelta[i, 2] *
      KroneckerDelta[j, 2] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
     KroneckerDelta[i, 4] *
      KroneckerDelta[j, 4] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
     KroneckerDelta[i, 2] *
      KroneckerDelta[j, 2] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
     KroneckerDelta[i, 3] *
```

```
      KroneckerDelta[j, 3] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
     KroneckerDelta[i, 4] * KroneckerDelta[j, 4] *
      KroneckerDelta[k, 3] * KroneckerDelta[l, 3]) +
  δδ * δB1111 * KroneckerDelta[i, 1] *
   KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
  δδ * δB1122 * KroneckerDelta[i, 1] *
   KroneckerDelta[j, 1] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
  δδ * δB1122 * KroneckerDelta[i, 2] *
   KroneckerDelta[j, 2] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
  δδ * δB1122 * KroneckerDelta[i, 1] *
   KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
  δδ * δB1122 * KroneckerDelta[i, 3] *
   KroneckerDelta[j, 3] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
  δδ * δB1122 * KroneckerDelta[i, 1] *
   KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
  δδ * δB1122 * KroneckerDelta[i, 4] *
   KroneckerDelta[j, 4] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
  δδ * δB1112 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
       KroneckerDelta[k, 1] * KroneckerDelta[l, 2] + KroneckerDelta[i, 2] *
       KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
      KroneckerDelta[i, 1] *
       KroneckerDelta[j, 1] * KroneckerDelta[k, 2] * KroneckerDelta[l, 1] +
      KroneckerDelta[i, 1] * KroneckerDelta[j, 2] *
       KroneckerDelta[k, 1] * KroneckerDelta[l, 1]) +
  δδ * δB1112 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
       KroneckerDelta[k, 1] * KroneckerDelta[l, 3] + KroneckerDelta[i, 3] *
       KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
      KroneckerDelta[i, 1] *
       KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 1] +
      KroneckerDelta[i, 1] * KroneckerDelta[j, 3] *
       KroneckerDelta[k, 1] * KroneckerDelta[l, 1]) +
  δδ * δB1112 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 1] *
       KroneckerDelta[k, 1] * KroneckerDelta[l, 4] + KroneckerDelta[i, 4] *
       KroneckerDelta[j, 1] * KroneckerDelta[k, 1] * KroneckerDelta[l, 1] +
      KroneckerDelta[i, 1] *
       KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 1] +
      KroneckerDelta[i, 1] * KroneckerDelta[j, 4] *
       KroneckerDelta[k, 1] * KroneckerDelta[l, 1]) +
  δδ * δB2221 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 2] *
       KroneckerDelta[k, 2] * KroneckerDelta[l, 2] + KroneckerDelta[i, 2] *
       KroneckerDelta[j, 1] * KroneckerDelta[k, 2] * KroneckerDelta[l, 2] +
      KroneckerDelta[i, 2] *
       KroneckerDelta[j, 2] * KroneckerDelta[k, 1] * KroneckerDelta[l, 2] +
      KroneckerDelta[i, 2] * KroneckerDelta[j, 2] *
```

```
        KroneckerDelta[k, 2] * KroneckerDelta[l, 1]) +
    δδ * δB2221 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 3] *
        KroneckerDelta[k, 3] * KroneckerDelta[l, 3] + KroneckerDelta[i, 3] *
        KroneckerDelta[j, 1] * KroneckerDelta[k, 3] * KroneckerDelta[l, 3] +
       KroneckerDelta[i, 3] *
        KroneckerDelta[j, 3] * KroneckerDelta[k, 1] * KroneckerDelta[l, 3] +
       KroneckerDelta[i, 3] * KroneckerDelta[j, 3] *
        KroneckerDelta[k, 3] * KroneckerDelta[l, 1]) +
    δδ * δB2221 * (KroneckerDelta[i, 1] * KroneckerDelta[j, 4] *
        KroneckerDelta[k, 4] * KroneckerDelta[l, 4] + KroneckerDelta[i, 4] *
        KroneckerDelta[j, 1] * KroneckerDelta[k, 4] * KroneckerDelta[l, 4] +
       KroneckerDelta[i, 4] *
        KroneckerDelta[j, 4] * KroneckerDelta[k, 1] * KroneckerDelta[l, 4] +
       KroneckerDelta[i, 4] * KroneckerDelta[j, 4] * KroneckerDelta[k, 4] *
        KroneckerDelta[l, 1]), {i, 4}, {j, 4}, {k, 4}, {l, 4}];
```

Integrating Out In-Plane Modes

```
In[•]:= U0 = Table[U1 * KroneckerDelta[i, 1] + U2 * KroneckerDelta[i, 2] +
         U3 * KroneckerDelta[i, 3] + U4 * KroneckerDelta[i, 4], {i, 4}];
       Q10 = Table[P1 * KroneckerDelta[i, 1] + P2 * KroneckerDelta[i, 2] +
         P3 * KroneckerDelta[i, 3] + P4 * KroneckerDelta[i, 4], {i, 4}];
       Q20 = Table[Q1 * KroneckerDelta[i, 1] + Q2 * KroneckerDelta[i, 2] +
         Q3 * KroneckerDelta[i, 3] + Q4 * KroneckerDelta[i, 4], {i, 4}];
       Q30 = Table[Z1 * KroneckerDelta[i, 1] + Z2 * KroneckerDelta[i, 2] +
         Z3 * KroneckerDelta[i, 3] + Z4 * KroneckerDelta[i, 4], {i, 4}];
       Q40 = Table[(-Z1 - Q1 - P1) * KroneckerDelta[i, 1] +
          (-Z2 - Q2 - P2) * KroneckerDelta[i, 2] + (-Z3 - Q3 - P3) * KroneckerDelta[i, 3] +
          (-Z4 - Q4 - P4) * KroneckerDelta[i, 4], {i, 4}];
       Q0 = Table[-(Q10[[1]] + Q20[[1]]) * KroneckerDelta[i, 1] -
          (Q10[[2]] + Q20[[2]]) * KroneckerDelta[i, 2] - (Q10[[3]] + Q20[[3]]) * KroneckerDelta[i, 3] -
          (Q10[[4]] + Q20[[4]]) * KroneckerDelta[i, 4], {i, 4}];

       QuadraticU = (Sum[C0[[i, j, k, l]] * Q0[[i]] * U0[[j]] * Q0[[k]] * U0[[l]],
          {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}]);
```

```
In[ ]:= MatrixQuadraticU =
          {{Coefficient[QuadraticU, U1^2], Coefficient[QuadraticU, U1*U2] / 2,
            Coefficient[QuadraticU, U1*U3] / 2, Coefficient[QuadraticU, U1*U4] / 2},
           {Coefficient[QuadraticU, U1*U2] / 2, Coefficient[QuadraticU, U2^2],
            Coefficient[QuadraticU, U2*U3] / 2, Coefficient[QuadraticU, U2*U4] / 2},
           {Coefficient[QuadraticU, U1*U3] / 2, Coefficient[QuadraticU, U2*U3] / 2,
            Coefficient[QuadraticU, U3^2], Coefficient[QuadraticU, U3*U4] / 2},
           {Coefficient[QuadraticU, U1*U4] / 2, Coefficient[QuadraticU, U2*U4] / 2,
            Coefficient[QuadraticU, U4*U3] / 2, Coefficient[QuadraticU, U4^2]}};

In[ ]:= US = {U1, U2, U3, U4};

In[ ]:= Dominant = (D[MatrixQuadraticU] /. {δδ → 0});

In[ ]:= GHK = Map[Reverse, Minors[Dominant], {0, 1}];

In[ ]:= GHKTrue = Table[((GHK[[i, j]])) * (-1)^(i + j), {i, 4}, {j, 4}];

In[ ]:= epsilon = δδ * (D[MatrixQuadraticU, {δδ, 1}] /. {δδ → 0});

In[ ]:= Denom1 = Simplify[Det[Dominant]];

        GFrac = (*Simplify[Normal[Series[*)1 / Denom1(*,{ω,0,2}]]]*);
        GFrac2 = (*Simplify[Normal[Series[*)(1 / Denom1)^2(*,{ω,0,2}]]]*);

        InverseMatIso = GFrac * GHKTrue;
        InverseMatPert = (GFrac2) GHKTrue.epsilon.GHKTrue;
```

In[ ]:=
```
LinearUAlpha = (I / 2) *
    Sum[(δδ) * (D[C0[[i, j, k, l]], {δδ}] /. {δδ → 0}) * Q0[[i]] * U0[[j]] * Q10[[k]] * Q20[[l]],
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
LinearUVectorAlpha = {Coefficient[LinearUAlpha, U1], Coefficient[LinearUAlpha, U2],
    Coefficient[LinearUAlpha, U3], Coefficient[LinearUAlpha, U4]};
LinearIsoUAlpha =
   (I / 2) * Sum[(C0[[i, j, k, l]] /. {δδ → 0}) * Q0[[i]] * U0[[j]] * Q10[[k]] * Q20[[l]],
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
LinearIsoUVectorAlpha =
   {Coefficient[LinearIsoUAlpha, U1], Coefficient[LinearIsoUAlpha, U2],
    Coefficient[LinearIsoUAlpha, U3], Coefficient[LinearIsoUAlpha, U4]};
LinearUBeta = (I / 2) *
    Sum[(δδ) * (D[C0[[i, j, k, l]], {δδ}] /. {δδ → 0}) * (-Q0[[i]]) * U0[[j]] * Q30[[k]] * Q40[[l]],
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
LinearUVectorBeta = {Coefficient[LinearUBeta, U1], Coefficient[LinearUBeta, U2],
    Coefficient[LinearUBeta, U3], Coefficient[LinearUBeta, U4]};
LinearIsoUBeta =
   (I / 2) * Sum[(C0[[i, j, k, l]] /. {δδ → 0}) * (-Q0[[i]]) * U0[[j]] * Q30[[k]] * Q40[[l]],
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
LinearIsoUVectorBeta =
   {Coefficient[LinearIsoUBeta, U1], Coefficient[LinearIsoUBeta, U2],
    Coefficient[LinearIsoUBeta, U3], Coefficient[LinearIsoUBeta, U4]};

FFourthTerm = -(1 / 8) Sum[C0[[i, j, k, l]] * Q10[[i]] * Q20[[j]] * Q30[[k]] * Q40[[l]],
     {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}];
```

In[ ]:=
```
TotalFFourth = (1 / 2)
    (LinearIsoUVectorAlpha.(InverseMatIso - InverseMatPert).LinearIsoUVectorBeta +
      LinearUVectorAlpha.InverseMatIso.LinearIsoUVectorBeta +
      LinearIsoUVectorAlpha.InverseMatIso.LinearUVectorBeta);
```

In[ ]:=
```
JJJJJP = Normal[Series[TotalFFourth, {δδ, 0, 1}]];
```

In[ ]:=
```
JJJJJ0 = JJJJJP /. {δδ → 0};
JJJJJ1 = δδ * (D[JJJJJP, δδ] /. {δδ → 0});
```

For the one loop f^4 diagram

In[ ]:=
```
DenomFull = Simplify[((Sum[B0[[d, f, g, h]] × Q10[[d]] × Q10[[f]] × Q10[[g]] × Q10[[h]],
       {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]))];
JJJJJ =
   ((Normal[Series[(1 / DenomFull) TotalFFourth, {δδ, 0, 1}]]) /. {Z1 → -Q1, Z2 → -Q2,
       Z3 → -Q3, Z4 → -Q4}) /. {Q1 → ω * Q1, Q2 → ω * Q2, Q3 → ω * Q3, Q4 → ω * Q4};
```

Renormalization of κ+δB1111

```
In[◦]:= JJJ1111P = 1 / ((4 !) ^2) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q2 → 0, Q3 → 0, Q4 → 0}, {Q1, 4}] /.
     {Q1 → 0, P1 → P * Cos[θ1], P2 → P * Sin[θ1] Cos[θ2],
      P3 → P * Sin[θ1] Sin[θ2] Cos[θ3], P4 → P * Sin[θ1] Sin[θ2] Sin[θ3]};

In[◦]:= JJJ1111T = Together[JJJ1111P];

In[◦]:= JJJ1111N = Simplify[Numerator[JJJ1111T]];
        JJJ1111D = Simplify[Denominator[JJJ1111T]];

In[◦]:= JJJ1111 = JJJ1111N / JJJ1111D;

In[◦]:= KKK1111 =
    Simplify[JJJ1111 - δB1111 * (D[JJJ1111, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
           δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
      - δB1112 * (D[JJJ1111, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
           δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
      - δB1122 * (D[JJJ1111, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
           δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
      - δB2221 * (D[JJJ1111, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
           δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
      - δC1111 * (D[JJJ1111, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
           δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
      - δC1122 * (D[JJJ1111, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
           δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
      - δC1212 * (D[JJJ1111, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
           δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
      - δC1112 * (D[JJJ1111, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
           δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
      - δC2221 * (D[JJJ1111, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
           δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
In[ ]:= KKK1111B1111 =
          Simplify[(D[JJJ1111, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1111B1112 =
          Simplify[(D[JJJ1111, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1111B1122 =
          Simplify[(D[JJJ1111, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1111B2221 =
          Simplify[(D[JJJ1111, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1111C1111 =
          Simplify[(D[JJJ1111, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1111C1122 =
          Simplify[(D[JJJ1111, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1111C1212 =
          Simplify[(D[JJJ1111, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1111C1112 =
          Simplify[(D[JJJ1111, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1111C2221 =
          Simplify[(D[JJJ1111, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
              δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
BBB1111 = -2 * (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
   (Integrate[Sin[θ1] ^2 * Sin[θ2] KKK1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1111 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1112 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1122 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB2221 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1111 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1122 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1212 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1112 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC2221 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1111C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Renormalization of κ+δB1122

```
JJJ1122P =
  1 / ((4 !) * 2 ! * 2 !) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q3 → 0, Q4 → 0}, {Q1, 2}, {Q2, 2}] /.
   {Q2 → 0, Q1 → 0, P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
    P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

JJJ1122T = Together[JJJ1122P];

JJJ1122N = Simplify[Numerator[JJJ1122T]];
JJJ1122D = Simplify[Denominator[JJJ1122T]];

JJJ1122 = JJJ1111N / JJJ1122D;
```

```
In[ ]:= KKK1122 =
  Simplify[JJJ1122 - δB1111 * (D[JJJ1122, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1112 * (D[JJJ1122, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1122 * (D[JJJ1122, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB2221 * (D[JJJ1122, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1111 * (D[JJJ1122, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1122 * (D[JJJ1122, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1212 * (D[JJJ1122, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1112 * (D[JJJ1122, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC2221 * (D[JJJ1122, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
        δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
KKK1122B1111 =
  Simplify[(D[JJJ1122, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK1122B1112 =
  Simplify[(D[JJJ1122, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK1122B1122 =
  Simplify[(D[JJJ1122, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK1122B2221 =
  Simplify[(D[JJJ1122, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK1122C1111 =
  Simplify[(D[JJJ1122, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK1122C1122 =
  Simplify[(D[JJJ1122, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK1122C1212 =
  Simplify[(D[JJJ1122, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK1122C1112 =
  Simplify[(D[JJJ1122, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK1122C2221 =
  Simplify[(D[JJJ1122, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

The extra factor of 1/2 is to account for prefactors

```
BBB1122 = -2 * (1 / 2) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
   (Integrate[Sin[θ1] ^2 * Sin[θ2] KKK1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1111 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1122B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1112 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1122B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1122 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1122B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB2221 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1122B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1111 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1122C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1122 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1122C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1212 * Integrate[Sin[θ1] ^2 * Sin[θ2] KKK1122C1212,
       {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] + δC1112 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1122C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC2221 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK1122C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Renormalization of κ+δB2233

```
JJJ2233P =
  1 / ((4 !) * 2 ! * 2 !) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q1 → 0, Q4 → 0}, {Q2, 2}, {Q3, 2}] /.
   {Q2 → 0, Q3 → 0, P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
    P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

JJJ2233T = Together[JJJ2233P];

JJJ2233N = Simplify[Numerator[JJJ2233T]];
JJJ2233D = Simplify[Denominator[JJJ2233T]];

JJJ2233 = JJJ2233N / JJJ2233D;
```

```
In[◦]:= KKK2233 =
  Simplify[JJJ2233 - δB1111 * (D[JJJ2233, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1112 * (D[JJJ2233, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1122 * (D[JJJ2233, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB2221 * (D[JJJ2233, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1111 * (D[JJJ2233, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1122 * (D[JJJ2233, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1212 * (D[JJJ2233, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1112 * (D[JJJ2233, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC2221 * (D[JJJ2233, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
        δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
In[ ]:= KKK2233B1111 =
  Simplify[(D[JJJ2233, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2233B1112 =
  Simplify[(D[JJJ2233, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2233B1122 =
  Simplify[(D[JJJ2233, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2233B2221 =
  Simplify[(D[JJJ2233, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2233C1111 =
  Simplify[(D[JJJ2233, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2233C1122 =
  Simplify[(D[JJJ2233, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2233C1212 =
  Simplify[(D[JJJ2233, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2233C1112 =
  Simplify[(D[JJJ2233, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2233C2221 =
  Simplify[(D[JJJ2233, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
BBB2233 = (1 / (kBT)) * 4 * ((kBT / 1 (*A*))) * (1 (*A*) / (2 π) ^ 4) *
   (Integrate[Sin[θ1] ^ 2 * Sin[θ2] KKK2233, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1111 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2233B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1112 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2233B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1122 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2233B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB2221 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2233B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1122 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2233C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1111 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2233C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1212 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] KKK2233C1212,
       {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] + δC1112 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2233C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC2221 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2233C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Renormalization of κ+δB2244

```
JJJ2244P =
  1 / ((4 !) * 2 ! * 2 !) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q1 → 0, Q3 → 0}, {Q2, 2}, {Q4, 2}] /.
   {Q2 → 0, Q4 → 0, P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
    P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

JJJ2244T = Together[JJJ2244P];

JJJ2244N = Simplify[Numerator[JJJ2244T]];
JJJ2244D = Simplify[Denominator[JJJ2244T]];

JJJ2244 = JJJ2244N / JJJ2244D;
```

In[ ]:=
```
KKK2244 =
  Simplify[JJJ2244 - δB1111 * (D[JJJ2244, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1112 * (D[JJJ2244, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1122 * (D[JJJ2244, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB2221 * (D[JJJ2244, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1111 * (D[JJJ2244, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1122 * (D[JJJ2244, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1212 * (D[JJJ2244, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1112 * (D[JJJ2244, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC2221 * (D[JJJ2244, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
        δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
KKK2244B1111 =
  Simplify[(D[JJJ2244, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2244B1112 =
  Simplify[(D[JJJ2244, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2244B1122 =
  Simplify[(D[JJJ2244, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2244B2221 =
  Simplify[(D[JJJ2244, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2244C1111 =
  Simplify[(D[JJJ2244, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2244C1122 =
  Simplify[(D[JJJ2244, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2244C1212 =
  Simplify[(D[JJJ2244, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2244C1112 =
  Simplify[(D[JJJ2244, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2244C2221 =
  Simplify[(D[JJJ2244, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
BBB2244 = (1 / (kBT)) * 4 * ((kBT / 1 (*A*))) * (1 (*A*) / (2 π) ^ 4) *
    (Integrate[Sin[θ1] ^ 2 * Sin[θ2] KKK2244, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δB1111 * Integrate[
        Sin[θ1] ^ 2 * Sin[θ2] KKK2244B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δB1112 * Integrate[
        Sin[θ1] ^ 2 * Sin[θ2] KKK2244B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δB1122 * Integrate[
        Sin[θ1] ^ 2 * Sin[θ2] KKK2244B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δB2221 * Integrate[
        Sin[θ1] ^ 2 * Sin[θ2] KKK2244B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δC1122 * Integrate[
        Sin[θ1] ^ 2 * Sin[θ2] KKK2244C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δC1111 * Integrate[
        Sin[θ1] ^ 2 * Sin[θ2] KKK2244C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δC1212 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] KKK2244C1212,
        {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] + δC1112 * Integrate[
        Sin[θ1] ^ 2 * Sin[θ2] KKK2244C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δC2221 * Integrate[
        Sin[θ1] ^ 2 * Sin[θ2] KKK2244C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Renormalization of κ+δB2222

```
JJJ2222P = 1 / ((4 !) ^ 2) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q1 → 0, Q3 → 0, Q4 → 0}, {Q2, 4}] /.
    {Q2 → 0, P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
     P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

JJJ2222T = Together[JJJ2222P];

JJJ2222N = Simplify[Numerator[JJJ2222T]];
JJJ2222D = Simplify[Denominator[JJJ2222T]];

JJJ2222 = JJJ2222N / JJJ2222D;
```

In[ ]:=
```
KKK2222 =
  Simplify[JJJ2222 - δB1111 * (D[JJJ2222, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1112 * (D[JJJ2222, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1122 * (D[JJJ2222, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB2221 * (D[JJJ2222, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1111 * (D[JJJ2222, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1122 * (D[JJJ2222, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1212 * (D[JJJ2222, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1112 * (D[JJJ2222, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC2221 * (D[JJJ2222, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
        δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
KKK2222B1111 =
  Simplify[(D[JJJ2222, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2222B1112 =
  Simplify[(D[JJJ2222, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2222B1122 =
  Simplify[(D[JJJ2222, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2222B2221 =
  Simplify[(D[JJJ2222, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2222C1111 =
  Simplify[(D[JJJ2222, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2222C1122 =
  Simplify[(D[JJJ2222, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2222C1212 =
  Simplify[(D[JJJ2222, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2222C1112 =
  Simplify[(D[JJJ2222, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2222C2221 =
  Simplify[(D[JJJ2222, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
BBB2222 = -2 * (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
   (Integrate[Sin[θ1] ^2 * Sin[θ2] KKK2222, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1111 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1112 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1122 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB2221 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1122 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1111 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1212 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1112 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC2221 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2222C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Renormalization of κ+δB3333

```
JJJ3333P = 1 / ((4 !) ^2) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q1 → 0, Q2 → 0, Q4 → 0}, {Q3, 4}] /.
   {Q3 → 0, P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
    P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

JJJ3333T = Together[JJJ3333P];

JJJ3333N = Simplify[Numerator[JJJ3333T]];
JJJ3333D = Simplify[Denominator[JJJ3333T]];

JJJ3333 = JJJ3333N / JJJ3333D;
```

In[ ]:=
```
KKK3333 =
  Simplify[JJJ3333 - δB1111 * (D[JJJ3333, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1112 * (D[JJJ3333, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1122 * (D[JJJ3333, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB2221 * (D[JJJ3333, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1111 * (D[JJJ3333, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1122 * (D[JJJ3333, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1212 * (D[JJJ3333, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1112 * (D[JJJ3333, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC2221 * (D[JJJ3333, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
        δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
KKK3333B1111 =
  Simplify[(D[JJJ3333, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK3333B1112 =
  Simplify[(D[JJJ3333, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK3333B1122 =
  Simplify[(D[JJJ3333, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK3333B2221 =
  Simplify[(D[JJJ3333, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK3333C1111 =
  Simplify[(D[JJJ3333, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK3333C1122 =
  Simplify[(D[JJJ3333, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK3333C1212 =
  Simplify[(D[JJJ3333, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK3333C1112 =
  Simplify[(D[JJJ3333, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK3333C2221 =
  Simplify[(D[JJJ3333, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
BBB3333 = -2 * (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
    (Integrate[Sin[θ1] ^2 * Sin[θ2] KKK3333, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δB1111 * Integrate[
        Sin[θ1] ^2 * Sin[θ2] KKK3333B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δB1112 * Integrate[
        Sin[θ1] ^2 * Sin[θ2] KKK3333B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δB1122 * Integrate[
        Sin[θ1] ^2 * Sin[θ2] KKK3333B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δB2221 * Integrate[
        Sin[θ1] ^2 * Sin[θ2] KKK3333B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δC1122 * Integrate[
        Sin[θ1] ^2 * Sin[θ2] KKK3333C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δC1111 * Integrate[
        Sin[θ1] ^2 * Sin[θ2] KKK3333C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δC1212 * Integrate[Sin[θ1] ^2 * Sin[θ2] KKK3333C1212,
        {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] + δC1112 * Integrate[
        Sin[θ1] ^2 * Sin[θ2] KKK3333C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
      δC2221 * Integrate[
        Sin[θ1] ^2 * Sin[θ2] KKK3333C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Renormalization of κ+δB1112

```
JJJ1112P =
  1 / ((4 !) * 3 ! * 1 !) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q3 → 0, Q4 → 0}, {Q1, 3}, {Q2, 1}] /.
    {Q2 → 0, Q1 → 0, P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
      P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

JJJ1112T = Together[JJJ1112P];

JJJ1112N = Simplify[Numerator[JJJ1112T]];
JJJ1112D = Simplify[Denominator[JJJ1112T]];

JJJ1112 = JJJ1112N / JJJ1112D;
```

In[ ]:=
```
KKK1112 =
  Simplify[JJJ1112 - δB1111 * (D[JJJ1112, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1112 * (D[JJJ1112, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1122 * (D[JJJ1112, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB2221 * (D[JJJ1112, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1111 * (D[JJJ1112, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1122 * (D[JJJ1112, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1212 * (D[JJJ1112, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1112 * (D[JJJ1112, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC2221 * (D[JJJ1112, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
        δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
In[ ]:= KKK1112B1111 =
          Simplify[(D[JJJ1112, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1112B1112 =
          Simplify[(D[JJJ1112, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1112B1122 =
          Simplify[(D[JJJ1112, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1112B2221 =
          Simplify[(D[JJJ1112, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1112C1111 =
          Simplify[(D[JJJ1112, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1112C1122 =
          Simplify[(D[JJJ1112, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1112C1212 =
          Simplify[(D[JJJ1112, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1112C1112 =
          Simplify[(D[JJJ1112, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
        KKK1112C2221 =
          Simplify[(D[JJJ1112, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
               δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
BBB1112 = -2 * (1 / 4) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^ 4) *
   (Integrate[Sin[θ1] ^ 2 * Sin[θ2] KKK1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1111 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK1112B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1112 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK1112B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1122 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK1112B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB2221 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK1112B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1122 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK1112C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1111 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK1112C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1212 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] KKK1112C1212,
       {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] + δC1112 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK1112C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC2221 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK1112C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Renormalization of $\kappa+\delta$B2221

```
JJJ2221P =
  1 / ((4 !) * 3 ! * 1 !) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q3 → 0, Q4 → 0}, {Q2, 3}, {Q1, 1}] /.
   {Q2 → 0, Q1 → 0, P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
    P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

JJJ2221T = Together[JJJ2221P];

JJJ2221N = Simplify[Numerator[JJJ2221T]];
JJJ2221D = Simplify[Denominator[JJJ2221T]];

JJJ2221 = JJJ2221N / JJJ2221D;
```

In[ ]:=
```
KKK2221 =
  Simplify[JJJ2221 - δB1111 * (D[JJJ2221, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1112 * (D[JJJ2221, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB1122 * (D[JJJ2221, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δB2221 * (D[JJJ2221, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1111 * (D[JJJ2221, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1122 * (D[JJJ2221, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1212 * (D[JJJ2221, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC1112 * (D[JJJ2221, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
        δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})
    - δC2221 * (D[JJJ2221, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
        δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
KKK2221B1111 =
  Simplify[(D[JJJ2221, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2221B1112 =
  Simplify[(D[JJJ2221, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2221B1122 =
  Simplify[(D[JJJ2221, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2221B2221 =
  Simplify[(D[JJJ2221, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2221C1111 =
  Simplify[(D[JJJ2221, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2221C1122 =
  Simplify[(D[JJJ2221, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2221C1212 =
  Simplify[(D[JJJ2221, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2221C1112 =
  Simplify[(D[JJJ2221, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
KKK2221C2221 =
  Simplify[(D[JJJ2221, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0, δB2221 → 0,
      δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
BBB2221 = -2 * (1 / 4) (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^ 4) *
   (Integrate[Sin[θ1] ^ 2 * Sin[θ2] KKK2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1111 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2221B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1112 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2221B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1122 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2221B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB2221 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2221B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1122 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2221C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1111 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2221C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1212 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] KKK2221C1212,
       {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] + δC1112 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2221C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC2221 * Integrate[
       Sin[θ1] ^ 2 * Sin[θ2] KKK2221C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Renormalization of κ+δB2223

```
JJJ2223P =
  1 / ((4 !) * 3 ! * 1 !) D[D[(JJJJJ), {ω, 4}] /. {ω → 0, Q1 → 0, Q4 → 0}, {Q2, 3}, {Q3, 1}] /.
   {Q1 → 0, P1 → Cos[θ1], P2 → Sin[θ1] Cos[θ2],
    P3 → Sin[θ1] Sin[θ2] Cos[θ3], P4 → Sin[θ1] Sin[θ2] Sin[θ3]};

JJJ2223T = Together[JJJ2223P];

JJJ2223N = Simplify[Numerator[JJJ2223T]];
JJJ2223D = Simplify[Denominator[JJJ2223T]];

JJJ2223 = JJJ2223N / JJJ2223D;

KKK2223 = Simplify[JJJ2223 - δB1111 * Coefficient[JJJ2223, δB1111]
     - δB1112 * Coefficient[JJJ2223, δB1112]
     - δB1122 * Coefficient[JJJ2223, δB1122]
     - δB2221 * Coefficient[JJJ2223, δB2221]
     - δC1111 * Coefficient[JJJ2223, δC1111]
     - δC1122 * Coefficient[JJJ2223, δC1122]
     - δC1212 * Coefficient[JJJ2223, δC1212]
     - δC1112 * Coefficient[JJJ2223, δC1112]
     - δC2221 * Coefficient[JJJ2223, δC2221]];
```

```
KKK2223B1111 = Simplify[Coefficient[JJJ2223, δB1111]];
KKK2223B1112 = Simplify[Coefficient[JJJ2223, δB1112]];
KKK2223B1122 = Simplify[Coefficient[JJJ2223, δB1122]];
KKK2223B2221 = Simplify[Coefficient[JJJ2223, δB2221]];
KKK2223C1111 = Simplify[Coefficient[JJJ2223, δC1111]];
KKK2223C1122 = Simplify[Coefficient[JJJ2223, δC1122]];
KKK2223C1212 = Simplify[Coefficient[JJJ2223, δC1212]];
KKK2223C1112 = Simplify[Coefficient[JJJ2223, δC1112]];
KKK2223C2221 = Simplify[Coefficient[JJJ2223, δC2221]];

BBB2223 = (1 / (kBT)) * 4 * ((kBT / 1(*A*))) * (1(*A*) / (2 π) ^4) *
   (Integrate[Sin[θ1] ^2 * Sin[θ2] KKK2223, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1111 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2223B1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1112 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2223B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB1122 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2223B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δB2221 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2223B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1122 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2223C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1111 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2223C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC1212 * Integrate[Sin[θ1] ^2 * Sin[θ2] KKK2223C1212,
       {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] + δC1112 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2223C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] +
     δC2221 * Integrate[
       Sin[θ1] ^2 * Sin[θ2] KKK2223C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]);
```

Calculating In-Plane For One Loop

In[ ]:=
```
P0 = Table[Cos[θ1] * KroneckerDelta[i, 1] + Sin[θ1] Cos[θ2] KroneckerDelta[i, 2] +
    Sin[θ1] Sin[θ2] Cos[θ3] KroneckerDelta[i, 3] +
    Sin[θ1] Sin[θ2] Sin[θ3] KroneckerDelta[i, 4], {i, 4}];
```

In[ ]:=
```
HHH[i_, j_, k_, l_] := (Sum[C0⟦i, j, x, y⟧ * C0⟦k, l, w, z⟧ * P0⟦x⟧ * P0⟦y⟧ * P0⟦w⟧ * P0⟦z⟧,
    {x, 1, 4}, {y, 1, 4}, {w, 1, 4}, {z, 1, 4}]);
```

Equation For $\lambda+\delta$C2233

```
HHHH2233 = Simplify[Normal[
    Series[(HHH[2, 2, 3, 3]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG2233 = HHHH2233;
KK2233B1111 = (D[GG2233, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233B1112 = (D[GG2233, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233B1122 = (D[GG2233, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233B2221 = (D[GG2233, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233C1111 = (D[GG2233, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233C1112 = (D[GG2233, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233C1122 = (D[GG2233, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233C1212 = (D[GG2233, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233C2221 = (D[GG2233, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2233 = Simplify[(GG2233 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
In[ ]:= JJ2233 = - (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) *
    (Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK2233, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
      (- δB1111 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK2233B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2233B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2233B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2233B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1111 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2233C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2233C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2233C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1212 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2233C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2233C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for $\mu+\delta$C2323

```
HHHH2323 = Simplify[Normal[
    Series[(HHH[2, 3, 2, 3]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG2323 = HHHH2323;
KK2323B1111 = (D[GG2323, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323B1112 = (D[GG2323, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323B1122 = (D[GG2323, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323B2221 = (D[GG2323, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323C1111 = (D[GG2323, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323C1112 = (D[GG2323, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323C1122 = (D[GG2323, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323C1212 = (D[GG2323, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323C2221 = (D[GG2323, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2323 = Simplify[(GG2323 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
In[ ]:= JJ2323 = - (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) *
    (Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK2323, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
      (-δB1111 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK2323B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2323B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2323B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2323B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1111 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2323C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2323C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2323C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1212 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2323C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2323C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for $\lambda+2\mu+\delta C1111$

```
HHHH1111 = Simplify[Normal[
    Series[(HHH[1, 1, 1, 1]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG1111 = HHHH1111;
KK1111B1111 = (D[GG1111, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111B1112 = (D[GG1111, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111B1122 = (D[GG1111, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111B2221 = (D[GG1111, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111C1111 = (D[GG1111, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111C1112 = (D[GG1111, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111C1122 = (D[GG1111, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111C1212 = (D[GG1111, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111C2221 = (D[GG1111, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1111 = Simplify[(GG1111 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
In[ ]:= JJ1111 = -(1/4) (1/(kBT)^2) * (2 * dc) * ((kBT/1(*A*))^2) * (1(*A*)/(2 π)^4) *
    (Integrate[Sin[θ1]^2 * Sin[θ2] * KK1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
      (-δB1111 * Integrate[Sin[θ1]^2 * Sin[θ2] * KK1111B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1112 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK1111B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1122 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK1111B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB2221 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK1111B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1111 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK1111C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1112 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK1111C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1122 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK1111C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1212 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK1111C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC2221 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK1111C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for λ+δC1122

```
HHHH1122 = Simplify[Normal[
    Series[(HHH[1, 1, 2, 2]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG1122 = HHHH1122;
KK1122B1111 = (D[GG1122, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122B1112 = (D[GG1122, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122B1122 = (D[GG1122, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122B2221 = (D[GG1122, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122C1111 = (D[GG1122, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122C1112 = (D[GG1122, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122C1122 = (D[GG1122, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122C1212 = (D[GG1122, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122C2221 = (D[GG1122, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1122 = Simplify[(GG1122 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
       δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
JJ1122 = - (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) *
   (Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
     (- δB1111 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK1122B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1122B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1122B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1122B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1111 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1122C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1122C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1122C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1212 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1122C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1122C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for $\mu+\delta$C1212

```
HHHH1212 = Simplify[Normal[
   Series[(HHH[1, 2, 1, 2]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
        {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG1212 = HHHH1212;
KK1212B1111 = (D[GG1212, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212B1112 = (D[GG1212, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212B1122 = (D[GG1212, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212B2221 = (D[GG1212, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212C1111 = (D[GG1212, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212C1112 = (D[GG1212, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212C1122 = (D[GG1212, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212C1212 = (D[GG1212, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212C2221 = (D[GG1212, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1212 = Simplify[(GG1212 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
JJ1212 = - (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) *
   (Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
     (- δB1111 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK1212B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1212B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1212B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1212B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1111 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1212C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1212C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1212C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1212 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1212C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1212C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for δC1112

```
HHHH1112 = Simplify[Normal[
    Series[(HHH[1, 1, 1, 2]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG1112 = HHHH1112;
KK1112B1111 = (D[GG1112, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112B1112 = (D[GG1112, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112B1122 = (D[GG1112, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112B2221 = (D[GG1112, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112C1111 = (D[GG1112, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112C1112 = (D[GG1112, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112C1122 = (D[GG1112, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112C1212 = (D[GG1112, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112C2221 = (D[GG1112, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK1112 = Simplify[(GG1112 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
       δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
JJ1112 = - (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) *
   (Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
     (- δB1111 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK1112B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1112B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1112B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1112B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1111 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1112C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1112C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1112C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1212 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1112C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK1112C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for δC2223

```
HHHH2223 = Simplify[Normal[
    Series[(HHH[2, 2, 2, 3]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG2223 = HHHH2223;
KK2223B1111 = (D[GG2223, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223B1112 = (D[GG2223, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223B1122 = (D[GG2223, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223B2221 = (D[GG2223, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223C1111 = (D[GG2223, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223C1112 = (D[GG2223, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223C1122 = (D[GG2223, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223C1212 = (D[GG2223, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223C2221 = (D[GG2223, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2223 = Simplify[(GG2223 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
In[ ]:= JJ2223 = -(1/4) (1/(kBT)^2) * (2 * dc) * ((kBT/1(*A*))^2) * (1(*A*)/(2 π)^4) *
    (Integrate[Sin[θ1]^2 * Sin[θ2] * KK2223, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
      (-δB1111 * Integrate[Sin[θ1]^2 * Sin[θ2] * KK2223B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1112 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK2223B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1122 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK2223B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB2221 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK2223B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1111 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK2223C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1112 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK2223C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1122 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK2223C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1212 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK2223C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC2221 * Integrate[
          Sin[θ1]^2 * Sin[θ2] * KK2223C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for δC2221

```
HHHH2221 = Simplify[Normal[
    Series[(HHH[2, 2, 2, 1]) * (1 / (Sum[B0[[d, f, g, h]] × P0[[d]] × P0[[f]] × P0[[g]] × P0[[h]],
          {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG2221 = HHHH2221;
KK2221B1111 = (D[GG2221, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221B1112 = (D[GG2221, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221B1122 = (D[GG2221, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221B2221 = (D[GG2221, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221C1111 = (D[GG2221, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221C1112 = (D[GG2221, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221C1122 = (D[GG2221, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221C1212 = (D[GG2221, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221C2221 = (D[GG2221, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2221 = Simplify[(GG2221 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

In[ ]:=
```
JJ2221 = - (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) *
   (Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
     (- δB1111 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK2221B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2221B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2221B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δB2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2221B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1111 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2221C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2221C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2221C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC1212 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2221C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
       - δC2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2221C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Equation for $\lambda+2\mu+\delta C2222$

```
HHHH2222 = Simplify[Normal[
    Series[(HHH[2, 2, 2, 2]) * (1 / (Sum[B0⟦d, f, g, h⟧ × P0⟦d⟧ × P0⟦f⟧ × P0⟦g⟧ × P0⟦h⟧,
            {d, 1, 4}, {f, 1, 4}, {g, 1, 4}, {h, 1, 4}]) ^2), {ϵ, 0, 3}]]];
GG2222 = HHHH2222;
KK2222B1111 = (D[GG2222, δB1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222B1112 = (D[GG2222, δB1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222B1122 = (D[GG2222, δB1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222B2221 = (D[GG2222, δB2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222C1111 = (D[GG2222, δC1111] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222C1112 = (D[GG2222, δC1112] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222C1122 = (D[GG2222, δC1122] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222C1212 = (D[GG2222, δC1212] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222C2221 = (D[GG2222, δC2221] /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
     δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0});
KK2222 = Simplify[(GG2222 /. {δB1111 → 0, δB1112 → 0, δB1122 → 0,
      δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0})];
```

```
In[ ]:= JJ2222 = - (1 / 4) (1 / (kBT) ^ 2) * (2 * dc) * ((kBT / 1 (*A*)) ^ 2) * (1 (*A*) / (2 π) ^ 4) *
    (Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK2222, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}] -
      (-δB1111 * Integrate[Sin[θ1] ^ 2 * Sin[θ2] * KK2222B1111,
          {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2222B1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2222B1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δB2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2222B2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1111 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2222C1111, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1112 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2222C1112, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1122 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2222C1122, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC1212 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2222C1212, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]
        - δC2221 * Integrate[
          Sin[θ1] ^ 2 * Sin[θ2] * KK2222C2221, {θ1, 0, π}, {θ2, 0, π}, {θ3, 0, 2 π}]));
```

Obtaining contributions to Linear Stability Matrix

```
ξf = ((ϵ - (BBB2222)) / 2);

μToner = Simplify[Simplify[(4 * ξf - ϵ) * μ + (JJ2323)] /.
    {μ → (96 * ϵ * π ^ 2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc)}];

λToner = Simplify[Simplify[(4 * ξf - ϵ) * λ + (JJ2233)] /.
    {μ → (96 * ϵ * π ^ 2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc)}];
```

```
In[ ]:= δC1111Toner = Simplify[((4 * ξf - ϵ) * δδ * δC1111 + ((JJ1111 - (JJ2233) - 2 * (JJ2323)))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        δC1122Toner = Simplify[((4 * ξf - ϵ) * δδ * δC1122 + (((JJ1122 - JJ2233)))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        δC1212Toner = Simplify[((4 * ξf - ϵ) * δδ * δC1212 + (((JJ1212 - JJ2323)))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        δC1112Toner = Simplify[((4 * ξf - ϵ) * δδ * δC1112 + (((JJ1112)))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        δC2221Toner = Simplify[((4 * ξf - ϵ) * δδ * δC2221 + (((JJ2221)))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        FB1111κ = Simplify[(((2 ξf - ϵ) * (1 + δδ * δB1111) + (BBB1111))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        FB1112κ = Simplify[(((2 ξf - ϵ) * (δδ * δB1112) + (BBB1112))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        FB1122κ = Simplify[(((2 ξf - ϵ) * (1 + δδ * δB1122) + (BBB1122))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        FB2221κ = Simplify[(((2 ξf - ϵ) * (δδ * δB2221) + (BBB2221))) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];

In[ ]:= FC1111μ = Simplify[((δC1111Toner * μ - μToner * δδ * δC1111) / (μ^2)) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        FC1122μ = Simplify[((δC1122Toner * μ - μToner * δδ * δC1122) / (μ^2)) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        FC1212μ = Simplify[((δC1212Toner * μ - μToner * δδ * δC1212) / (μ^2)) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        FC2221μ = Simplify[((δC2221Toner * μ - μToner * δδ * δC2221) / (μ^2)) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];
        FC1112μ = Simplify[((δC1112Toner * μ - μToner * δδ * δC1112) / (μ^2)) /.
          {μ → (96 * ϵ * π^2) / (24 + dc), λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc)}];

In[ ]:= C1111C1111 = (μ * D[FC1111μ, δC1111]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
          λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
          δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
        C1111C1112 = (μ * D[FC1111μ, δC1112]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
          λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
          δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
        C1111C1122 = (μ * D[FC1111μ, δC1122]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
          λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
          δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
        C1111C1212 = (μ * D[FC1111μ, δC1212]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
          λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
          δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
        C1111C2221 = (μ * D[FC1111μ, δC2221]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
          λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
```

```
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1111B1111 = (D[FC1111μ, δB1111]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1111B1122 = (D[FC1111μ, δB1122]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1111B1112 = (D[FC1111μ, δB1112]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1111B2221 = (D[FC1111μ, δB2221]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};


C1112C1111 = (μ * D[FC1112μ, δC1111]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1112C1112 = (μ * D[FC1112μ, δC1112]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1112C1122 = (μ * D[FC1112μ, δC1122]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1112C1212 = (μ * D[FC1112μ, δC1212]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1112C2221 = (μ * D[FC1112μ, δC2221]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1112B1111 = (D[FC1112μ, δB1111]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1112B1122 = (D[FC1112μ, δB1122]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1112B1112 = (D[FC1112μ, δB1112]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1112B2221 = (D[FC1112μ, δB2221]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};

C1122C1111 = (μ * D[FC1122μ, δC1111]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
```

```
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1122C1112 = (μ * D[FC1122μ, δC1112]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1122C1122 = (μ * D[FC1122μ, δC1122]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1122C1212 = (μ * D[FC1122μ, δC1212]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1122C2221 = (μ * D[FC1122μ, δC2221]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1122B1111 = (D[FC1122μ, δB1111]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1122B1122 = (D[FC1122μ, δB1122]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1122B1112 = (D[FC1122μ, δB1112]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1122B2221 = (D[FC1122μ, δB2221]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};

C1212C1111 = (μ * D[FC1212μ, δC1111]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1212C1112 = (μ * D[FC1212μ, δC1112]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1212C1122 = (μ * D[FC1212μ, δC1122]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1212C1212 = (μ * D[FC1212μ, δC1212]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1212C2221 = (μ * D[FC1212μ, δC2221]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1212B1111 = (D[FC1212μ, δB1111]) /. {μ → (96 * ∊ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∊ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
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    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1212B1122 = (D[FC1212μ, δB1122]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1212B1112 = (D[FC1212μ, δB1112]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C1212B2221 = (D[FC1212μ, δB2221]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};

C2221C1111 = (μ * D[FC2221μ, δC1111]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C2221C1112 = (μ * D[FC2221μ, δC1112]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C2221C1122 = (μ * D[FC2221μ, δC1122]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C2221C1212 = (μ * D[FC2221μ, δC1212]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C2221C2221 = (μ * D[FC2221μ, δC2221]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C2221B1111 = (D[FC2221μ, δB1111]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C2221B1122 = (D[FC2221μ, δB1122]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C2221B1112 = (D[FC2221μ, δB1112]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
C2221B2221 = (D[FC2221μ, δB2221]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};

B1111C1111 = (μ * D[FB1111κ, δC1111]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1111C1112 = (μ * D[FB1111κ, δC1112]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
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    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1111C1122 = (μ * D[FB1111κ, δC1122]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1111C1212 = (μ * D[FB1111κ, δC1212]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1111C2221 = (μ * D[FB1111κ, δC2221]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1111B1111 = (D[FB1111κ, δB1111]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1111B1122 = (D[FB1111κ, δB1122]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1111B1112 = (D[FB1111κ, δB1112]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1111B2221 = (D[FB1111κ, δB2221]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};

B1112C1111 = (μ * D[FB1112κ, δC1111]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1112C1112 = (μ * D[FB1112κ, δC1112]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1112C1122 = (μ * D[FB1112κ, δC1122]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1112C1212 = (μ * D[FB1112κ, δC1212]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1112C2221 = (μ * D[FB1112κ, δC2221]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1112B1111 = (D[FB1112κ, δB1111]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1112B1122 = (D[FB1112κ, δB1122]) /. {μ → (96 * ϵ * π ^ 2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π ^ 2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
```

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B1112B1112 = (D[FB1112κ, δB1112]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1112B2221 = (D[FB1112κ, δB2221]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};

B1122C1111 = (μ * D[FB1122κ, δC1111]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1122C1112 = (μ * D[FB1122κ, δC1112]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1122C1122 = (μ * D[FB1122κ, δC1122]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1122C1212 = (μ * D[FB1122κ, δC1212]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1122C2221 = (μ * D[FB1122κ, δC2221]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1122B1111 = (D[FB1122κ, δB1111]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1122B1122 = (D[FB1122κ, δB1122]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1122B1112 = (D[FB1122κ, δB1112]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B1122B2221 = (D[FB1122κ, δB2221]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};

B2221C1111 = (μ * D[FB2221κ, δC1111]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B2221C1112 = (μ * D[FB2221κ, δC1112]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B2221C1122 = (μ * D[FB2221κ, δC1122]) /. {μ → (96 * ∈ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ∈ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
```

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B2221C1212 = (μ * D[FB2221κ, δC1212]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B2221C2221 = (μ * D[FB2221κ, δC2221]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B2221B1111 = (D[FB2221κ, δB1111]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B2221B1122 = (D[FB2221κ, δB1122]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B2221B1112 = (D[FB2221κ, δB1112]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
B2221B2221 = (D[FB2221κ, δB2221]) /. {μ → (96 * ϵ * π^2) / (24 + dc),
    λ → (-1 / 3) (96 * ϵ * π^2) / (24 + dc), δB1111 → 0, δB1112 → 0, δB1122 → 0,
    δB2221 → 0, δC1111 → 0, δC1122 → 0, δC1212 → 0, δC1112 → 0, δC2221 → 0};
```

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In[ ]:= MatrixStability =
          Simplify[{{Simplify[C1111C1111], Simplify[C1111C1112], Simplify[C1111C1122],
              Simplify[C1111C1212], Simplify[C1111C2221], Simplify[C1111B1111],
              Simplify[C1111B1112], Simplify[C1111B1122], Simplify[C1111B2221]},
            {Simplify[C1112C1111], Simplify[C1112C1112], Simplify[C1112C1122],
              Simplify[C1112C1212], Simplify[C1112C2221], Simplify[C1112B1111],
              Simplify[C1112B1112], Simplify[C1112B1122], Simplify[C1112B2221]},
            {Simplify[C1122C1111], Simplify[C1122C1112], Simplify[C1122C1122],
              Simplify[C1122C1212], Simplify[C1122C2221], Simplify[C1122B1111],
              Simplify[C1122B1112], Simplify[C1122B1122], Simplify[C1122B2221]},
            {Simplify[C1212C1111], Simplify[C1212C1112], Simplify[C1212C1122],
              Simplify[C1212C1212], Simplify[C1212C2221], Simplify[C1212B1111],
              Simplify[C1212B1112], Simplify[C1212B1122], Simplify[C1212B2221]},
            {Simplify[C2221C1111], Simplify[C2221C1112], Simplify[C2221C1122],
              Simplify[C2221C1212], Simplify[C2221C2221], Simplify[C2221B1111],
              Simplify[C2221B1112], Simplify[C2221B1122], Simplify[C2221B2221]},
            {Simplify[B1111C1111], Simplify[B1111C1112], Simplify[B1111C1122],
              Simplify[B1111C1212], Simplify[B1111C2221], Simplify[B1111B1111],
              Simplify[B1111B1112], Simplify[B1111B1122], Simplify[B1111B2221]},
            {Simplify[B1112C1111], Simplify[B1112C1112], Simplify[B1112C1122],
              Simplify[B1112C1212], Simplify[B1112C2221], Simplify[B1112B1111],
              Simplify[B1112B1112], Simplify[B1112B1122], Simplify[B1112B2221]},
            {Simplify[B1122C1111], Simplify[B1122C1112], Simplify[B1122C1122],
              Simplify[B1122C1212], Simplify[B1122C2221], Simplify[B1122B1111],
              Simplify[B1122B1112], Simplify[B1122B1122], Simplify[B1122B2221]},
            {Simplify[B2221C1111], Simplify[B2221C1112], Simplify[B2221C1122],
              Simplify[B2221C1212], Simplify[B2221C2221], Simplify[B2221B1111],
              Simplify[B2221B1112], Simplify[B2221B1122], Simplify[B2221B2221]}}];

In[ ]:= Eigenvalues[MatrixStability]

In[ ]:= Eigenvectors[MatrixStability]

In[ ]:= MatrixStabilityOrthorhombic =
          Simplify[{{Simplify[C1111C1111], Simplify[C1111C1122],
              Simplify[C1111C1212], Simplify[C1111B1111], Simplify[C1111B1122]},
            {Simplify[C1122C1111], Simplify[C1122C1122], Simplify[C1122C1212],
              Simplify[C1122B1111], Simplify[C1122B1122]},
            {Simplify[C1212C1111], Simplify[C1212C1122], Simplify[C1212C1212],
              Simplify[C1212B1111], Simplify[C1212B1122]},
            {Simplify[B1111C1111], Simplify[B1111C1122], Simplify[B1111C1212],
              Simplify[B1111B1111], Simplify[B1111B1122]},
            {Simplify[B1122C1111], Simplify[B1122C1122], Simplify[B1122C1212],
              Simplify[B1122B1111], Simplify[B1122B1122]}}];
```

In[ ]:=
```
Eigenvalues[MatrixStabilityOrthorhombic]
```

In[ ]:=
```
MatrixStabilityOrthorhombic
```

Appendix C

Mathematica Code For Odd Elastic Sheets

Integrating Out In-Plane Phonons

In[ ]:=
```
Quit
```

```
(*All analytic calculation parts of the code that have been commented
  out need not be commented in. They have been left there for further
  investigation into the 1-loop analysis of odd elastic membranes.*)
```

In[ ]:=
```
(*dc=1;*)
κ = 1;
CCDfϕuf = Normal[SymmetrizedArray[{i_, j_, k_, l_} ⧴ CDfϕuf[i, j, k, l],
    {2, 2, 2, 2}, {(*(*stress (strain) is symmetric*){Cycles[{{1,2}}],1},*)
     (*strain (stress) is symmetric*){Cycles[{{3, 4}}], 1}(*,
     (*energy is quadratic*){Cycles[{{1,3},{2,4}}],1}*)}]];
CCDfϕf3 = Normal[SymmetrizedArray[{i_, j_, k_, l_} ⧴ CDfϕf3[i, j, k, l],
    {2, 2, 2, 2}, {(*(*stress (strain) is symmetric*){Cycles[{{1,2}}],1},*)
     (*strain (stress) is symmetric*){Cycles[{{3, 4}}], 1}(*,
     (*energy is quadratic*){Cycles[{{1,3},{2,4}}],1}*)}]];
CCDDYu = Normal[SymmetrizedArray[{i_, j_, k_, l_} ⧴ CDDYu[i, j, k, l],
    {2, 2, 2, 2}, {(*(*stress (strain) is symmetric*){Cycles[{{1,2}}],1},*)
     (*strain (stress) is symmetric*){Cycles[{{3, 4}}], 1}(*,
     (*energy is quadratic*){Cycles[{{1,3},{2,4}}],1}*)}]];
CCDDYf2 = Normal[SymmetrizedArray[{i_, j_, k_, l_} ⧴ CDDYf2[i, j, k, l],
    {2, 2, 2, 2}, {(*(*stress (strain) is symmetric*){Cycles[{{1,2}}],1},*)
     (*strain (stress) is symmetric*){Cycles[{{3, 4}}], 1}(*,
     (*energy is quadratic*){Cycles[{{1,3},{2,4}}],1}*)}]];
BBB = Normal[SymmetrizedArray[{i_, j_, k_, l_} ⧴ B[i, j, k, l],
    {2, 2, 2, 2}, {(*stress (strain) is symmetric*){Cycles[{{1, 2}}], 1},
     (*strain (stress) is symmetric*){Cycles[{{3, 4}}], 1},
     (*energy is quadratic*){Cycles[{{1, 3}, {2, 4}}], 1}}]];
yy[q_, ω_] = {y1[q, ω], y2[q, ω]};
uu[q_, ω_] = {u1[q, ω], u2[q, ω]};
qq = {q[1], q[2]};
FF[q_, ω_] = F[q, ω];
ϕϕ[q_, ω_] = ϕ[q, ω];

CCDfϕuf[[1, 1, 1, 1]] = (DfϕufΛ + 2 Dfϕufμ) (*/dc*);
CCDfϕuf[[1, 2, 1, 2]] = Dfϕufμ (*/dc*);
CCDfϕuf[[2, 1, 1, 2]] = Dfϕufμ (*/dc*);
CCDfϕuf[[1, 2, 2, 1]] = Dfϕufμ (*/dc*);
CCDfϕuf[[2, 1, 2, 1]] = Dfϕufμ (*/dc*);
CCDfϕuf[[1, 1, 2, 2]] = Dfϕufλ (*/dc*);
CCDfϕuf[[2, 2, 1, 1]] = Dfϕufλ (*/dc*);
```

```
CCDfϕuf⟦2, 2, 2, 2⟧ = (Dfϕufλ + 2 Dfϕufμ) (*/dc*);
CCDfϕuf⟦1, 1, 1, 2⟧ = DfϕufK(*/dc*);
CCDfϕuf⟦1, 1, 2, 1⟧ = DfϕufK(*/dc*);
CCDfϕuf⟦2, 2, 2, 1⟧ = -DfϕufK(*/dc*);
CCDfϕuf⟦2, 2, 1, 2⟧ = -DfϕufK(*/dc*);
CCDfϕuf⟦1, 2, 1, 1⟧ = -DfϕufK(*/dc*) - DfϕufA(*/dc*);
CCDfϕuf⟦2, 1, 1, 1⟧ = -DfϕufK(*/dc*) + DfϕufA(*/dc*);
CCDfϕuf⟦2, 1, 2, 2⟧ = DfϕufK(*/dc*) + DfϕufA(*/dc*);
CCDfϕuf⟦1, 2, 2, 2⟧ = DfϕufK(*/dc*) - DfϕufA(*/dc*);

CCDfϕf3⟦1, 1, 1, 1⟧ = (Dfϕf3λ + 2 Dfϕf3μ) (*/dc*);
CCDfϕf3⟦1, 2, 1, 2⟧ = Dfϕf3μ(*/dc*);
CCDfϕf3⟦2, 1, 1, 2⟧ = Dfϕf3μ(*/dc*);
CCDfϕf3⟦1, 2, 2, 1⟧ = Dfϕf3μ(*/dc*);
CCDfϕf3⟦2, 1, 2, 1⟧ = Dfϕf3μ(*/dc*);
CCDfϕf3⟦1, 1, 2, 2⟧ = Dfϕf3λ(*/dc*);
CCDfϕf3⟦2, 2, 1, 1⟧ = Dfϕf3λ(*/dc*);
CCDfϕf3⟦2, 2, 2, 2⟧ = (Dfϕf3λ + 2 Dfϕf3μ) (*/dc*);
CCDfϕf3⟦1, 1, 1, 2⟧ = Dfϕf3K(*/dc*);
CCDfϕf3⟦1, 1, 2, 1⟧ = Dfϕf3K(*/dc*);
CCDfϕf3⟦2, 2, 2, 1⟧ = -Dfϕf3K(*/dc*);
CCDfϕf3⟦2, 2, 1, 2⟧ = -Dfϕf3K(*/dc*);
CCDfϕf3⟦1, 2, 1, 1⟧ = -Dfϕf3K(*/dc*) - Dfϕf3A(*/dc*);
CCDfϕf3⟦2, 1, 1, 1⟧ = -Dfϕf3K(*/dc*) + Dfϕf3A(*/dc*);
CCDfϕf3⟦2, 1, 2, 2⟧ = Dfϕf3K(*/dc*) + Dfϕf3A(*/dc*);
CCDfϕf3⟦1, 2, 2, 2⟧ = Dfϕf3K(*/dc*) - Dfϕf3A(*/dc*);

BBB⟦1, 1, 1, 1⟧ = κ;
BBB⟦1, 1, 1, 2⟧ = 0;
BBB⟦1, 1, 2, 1⟧ = 0;
BBB⟦1, 2, 1, 1⟧ = 0;
BBB⟦2, 1, 1, 1⟧ = 0;
BBB⟦1, 1, 2, 2⟧ = κ / 2;
BBB⟦2, 2, 1, 1⟧ = κ / 2;
BBB⟦1, 2, 1, 2⟧ = κ / 4;
BBB⟦1, 2, 2, 1⟧ = κ / 4;
BBB⟦2, 1, 2, 1⟧ = κ / 4;
BBB⟦2, 1, 1, 2⟧ = κ / 4;
BBB⟦1, 2, 2, 2⟧ = 0;
BBB⟦2, 1, 2, 2⟧ = 0;
BBB⟦2, 2, 1, 2⟧ = 0;
BBB⟦2, 2, 2, 1⟧ = 0;
BBB⟦2, 2, 2, 2⟧ = κ;
```

```
CCDDYu⟦1, 1, 1, 1⟧ = DDYuλ(*/dc*) + 2 DDYuμ(*/dc*);
CCDDYu⟦1, 2, 1, 2⟧ = DDYuμ(*/dc*);
CCDDYu⟦2, 1, 1, 2⟧ = DDYuμ(*/dc*);
CCDDYu⟦1, 2, 2, 1⟧ = DDYuμ(*/dc*);
CCDDYu⟦2, 1, 2, 1⟧ = DDYuμ(*/dc*);
CCDDYu⟦1, 1, 2, 2⟧ = DDYuλ(*/dc*);
CCDDYu⟦2, 2, 1, 1⟧ = DDYuλ(*/dc*);
CCDDYu⟦2, 2, 2, 2⟧ = DDYuλ(*/dc*) + 2 DDYuμ(*/dc*);
CCDDYu⟦1, 1, 1, 2⟧ = DDYuK(*/dc*);
CCDDYu⟦1, 1, 2, 1⟧ = DDYuK(*/dc*);
CCDDYu⟦2, 2, 2, 1⟧ = -DDYuK(*/dc*);
CCDDYu⟦2, 2, 1, 2⟧ = -DDYuK(*/dc*);
CCDDYu⟦1, 2, 1, 1⟧ = -DDYuK(*/dc*) - DDYuA(*/dc*);
CCDDYu⟦2, 1, 1, 1⟧ = -DDYuK(*/dc*) + DDYuA(*/dc*);
CCDDYu⟦2, 1, 2, 2⟧ = DDYuK(*/dc*) + DDYuA(*/dc*);
CCDDYu⟦1, 2, 2, 2⟧ = DDYuK(*/dc*) - DDYuA(*/dc*);

CCDDYf2⟦1, 1, 1, 1⟧ = DDYf2λ(*/dc*) + 2 DDYf2μ(*/dc*);
CCDDYf2⟦1, 2, 1, 2⟧ = DDYf2μ(*/dc*);
CCDDYf2⟦2, 1, 1, 2⟧ = DDYf2μ(*/dc*);
CCDDYf2⟦1, 2, 2, 1⟧ = DDYf2μ(*/dc*);
CCDDYf2⟦2, 1, 2, 1⟧ = DDYf2μ(*/dc*);
CCDDYf2⟦1, 1, 2, 2⟧ = DDYf2λ(*/dc*);
CCDDYf2⟦2, 2, 1, 1⟧ = DDYf2λ(*/dc*);
CCDDYf2⟦2, 2, 2, 2⟧ = DDYf2λ(*/dc*) + 2 DDYf2μ(*/dc*);
CCDDYf2⟦1, 1, 1, 2⟧ = DDYf2K(*/dc*);
CCDDYf2⟦1, 1, 2, 1⟧ = DDYf2K(*/dc*);
CCDDYf2⟦2, 2, 2, 1⟧ = -DDYf2K(*/dc*);
CCDDYf2⟦2, 2, 1, 2⟧ = -DDYf2K(*/dc*);
CCDDYf2⟦1, 2, 1, 1⟧ = -DDYf2K(*/dc*) - DDYf2A(*/dc*);
CCDDYf2⟦2, 1, 1, 1⟧ = -DDYf2K(*/dc*) + DDYf2A(*/dc*);
CCDDYf2⟦2, 1, 2, 2⟧ = DDYf2K(*/dc*) + DDYf2A(*/dc*);
CCDDYf2⟦1, 2, 2, 2⟧ = DDYf2K(*/dc*) - DDYf2A(*/dc*);
pp1 = Table[{yy[q, ω]⟦j⟧, uu[q, ω]⟦j⟧}, {j, 2}];
pp2 = Table[{yy[-q, -ω]⟦j⟧, uu[-q, -ω]⟦j⟧}, {j, 2}];
LL = {L1, L2};

M = Simplify[Table[{{-2 LL⟦j⟧ * KroneckerDelta[j, l], I * ω * KroneckerDelta[j, l] +
        Sum[CCDDYu⟦i, j, k, l⟧ * q[i] * q[k], {i, 2}, {k, 2}]},
      {-I * ω * KroneckerDelta[j, l] + Sum[CCDDYu⟦i, l, k, j⟧ * q[i] * q[k],
         {i, 2}, {k, 2}], 0}}, {j, 2}, {l, 2}]];

HarmonicExp = Sum[pp1⟦j⟧.M⟦j, l⟧.pp2⟦l⟧, {j, 2}, {l, 2}];
```

```
kk1 = {yy[q, ω]⟦1⟧, yy[q, ω]⟦2⟧, uu[q, ω]⟦1⟧, uu[q, ω]⟦2⟧};
kk2 = {yy[-q, -ω]⟦1⟧, yy[-q, -ω]⟦2⟧, uu[-q, -ω]⟦1⟧, uu[-q, -ω]⟦2⟧};
MM = {{Coefficient[HarmonicExp, yy[q, ω]⟦1⟧ × yy[-q, -ω]⟦1⟧],
     Coefficient[HarmonicExp, yy[q, ω]⟦1⟧ × yy[-q, -ω]⟦2⟧],
     Coefficient[HarmonicExp, yy[q, ω]⟦1⟧ × uu[-q, -ω]⟦1⟧],
     Coefficient[HarmonicExp, yy[q, ω]⟦1⟧ × uu[-q, -ω]⟦2⟧]},
    {Coefficient[HarmonicExp, yy[q, ω]⟦2⟧ × yy[-q, -ω]⟦1⟧],
     Coefficient[HarmonicExp, yy[q, ω]⟦2⟧ × yy[-q, -ω]⟦2⟧],
     Coefficient[HarmonicExp, yy[q, ω]⟦2⟧ × uu[-q, -ω]⟦1⟧],
     Coefficient[HarmonicExp, yy[q, ω]⟦2⟧ × uu[-q, -ω]⟦2⟧]},
    {Coefficient[HarmonicExp, uu[q, ω]⟦1⟧ × yy[-q, -ω]⟦1⟧],
     Coefficient[HarmonicExp, uu[q, ω]⟦1⟧ × yy[-q, -ω]⟦2⟧],
     Coefficient[HarmonicExp, uu[q, ω]⟦1⟧ × uu[-q, -ω]⟦1⟧],
     Coefficient[HarmonicExp, uu[q, ω]⟦1⟧ × uu[-q, -ω]⟦2⟧]},
    {Coefficient[HarmonicExp, uu[q, ω]⟦2⟧ × yy[-q, -ω]⟦1⟧],
     Coefficient[HarmonicExp, uu[q, ω]⟦2⟧ × yy[-q, -ω]⟦2⟧],
     Coefficient[HarmonicExp, uu[q, ω]⟦2⟧ × uu[-q, -ω]⟦1⟧],
     Coefficient[HarmonicExp, uu[q, ω]⟦2⟧ × uu[-q, -ω]⟦2⟧]}};
NonLinear1 = -Sum[I * yy[q, ω]⟦j⟧ * CCDDYf2⟦i, j, k, l⟧ ×
       q[i] × p1[k] (q[l] + p1[l]) FF[p1, Ω1] × FF[-q - p1, -ω - Ω1] (1 / 2) +
      (I) ϕϕ[p1, Ω1] * CCDfϕuf⟦i, j, k, l⟧ × p1[i] (p1[j] + q[j]) * q[k] *
       uu[q, ω]⟦l⟧ * FF[-p1 - q, -ω - Ω1], {i, 2}, {j, 2}, {k, 2}, {l, 2}];
NonLinear2 = -Sum[I * yy[-q, -ω]⟦j⟧ * CCDDYf2⟦i, j, k, l⟧
       (-q[i]) p2[k] (-q[l] + p2[l]) FF[p2, Ω2] × FF[q - p2, ω - Ω2] (1 / 2) +
      (I) ϕϕ[p2, Ω2] * CCDfϕuf⟦i, j, k, l⟧ × p2[i] (p2[j] - q[j]) * (-q[k]) *
       uu[-q, -ω]⟦l⟧ * FF[-p2 + q, ω - Ω2], {i, 2}, {j, 2}, {k, 2}, {l, 2}];
mm1 = {Coefficient[NonLinear1, yy[q, ω]⟦1⟧], Coefficient[NonLinear1, yy[q, ω]⟦2⟧],
    Coefficient[NonLinear1, uu[q, ω]⟦1⟧], Coefficient[NonLinear1, uu[q, ω]⟦2⟧]};
mm2 = {Coefficient[NonLinear2, yy[-q, -ω]⟦1⟧],
    Coefficient[NonLinear2, yy[-q, -ω]⟦2⟧], Coefficient[NonLinear2, uu[-q, -ω]⟦1⟧],
    Coefficient[NonLinear2, uu[-q, -ω]⟦2⟧]};

F4A = Coefficient[(mm1.Inverse[MM].mm2) / 2,
    FF[p1, Ω1] × FF[-q - p1, -ω - Ω1] × ϕϕ[p2, Ω2] × FF[-p2 + q, ω - Ω2]];
F4B = Coefficient[(mm1.Inverse[MM].mm2) / 2,
    ϕϕ[p1, Ω1] × FF[-p1 - q, -ω - Ω1] × FF[p2, Ω2] × FF[q - p2, ω - Ω2]];
F4C = Coefficient[(mm1.Inverse[MM].mm2) / 2,
    ϕϕ[p1, Ω1] × FF[-p1 - q, -ω - Ω1] × ϕϕ[p2, Ω2] × FF[-p2 + q, ω - Ω2]];
```

F4A + Original F4 term

Contraction of two F’s, not allowed to make q=0

F matrix

```
uhom = {{u11, u12}, {u21, u22}};
sigmahom = {{σ11, σ12}, {σ21, σ22}};
MF[q_, ω_] =
  {{-2 * Df, I * ω + Df * (Sum[Sum[BBB⟦i, j, k, l⟧ × q[i] × q[j] × q[k] × q[l], {k, 2},
          {l, 2}] + sigmahom⟦i, j⟧ × q[i] × q[j], {i, 2}, {j, 2}])},
   {-I * ω + Df * (Sum[Sum[BBB⟦i, j, k, l⟧ × q[i] × q[j] × q[k] × q[l] (*+CC⟦i,j,k,l⟧uhom⟦k,
           l⟧*), {k, 2}, {l, 2}] + sigmahom⟦i, j⟧ × q[i] × q[j], {i, 2}, {j, 2}]), 0}};
F4N = (*WAVE VECTORS HERE NOT SYMMETRIZED*)
  Together[((Inverse[MF[-p2 + q, -Ω2 + ω]]⟦2, 2⟧ /.
        {(-p2 + q)[1] → (-p2[1] + q[1]), (-p2 + q)[2] → (-p2[2] + q[2])})
      ((1 / 2) Sum[2 * (-1) * CCDfϕf3⟦i, j, k, l⟧ (p1[k] (-q[l] - p1[l]))
           (*2 ways to contract this diagram*)p2[i] (-p2[j] + q[j]),
          {i, 2}, {j, 2}, {k, 2}, {l, 2}] + 4 * F4A)) /.
    {p1[1] → (-p2[1]), p1[2] → (-p2[2]), Ω1 → -Ω2, Ω2 → 0, Ω1 → 0}];

F4NIso1 = Simplify[
   F4N /. {(*A→0,K→0,*)(*Lf→L,*)L1 → L, L2 → L, (*Df→DD,*)b1 → 0, b2 → 0,
     bf → 0, a1 → 0, a2 → 0, af → 0, p2[1] → p2(**Cos[θ]*), p2[2] → 0(*p2*Sin[θ]*),
     q[1] → q * Cos[θ(*+ϕ*)], q[2] → q * Sin[θ(*+ϕ*)], σ12 → 0, σ21 → 0,
     σ22 → σ, σ11 → σ, u22 → u, u12 → 0, u21 → 0, u11 → u}];
F4NIsoS = FullSimplify[D[F4NIso1, {p2, 2}] /. {p2 → 0}];
F4NIsoB = Simplify[D[F4NIso1, {p2, 4}] /. {p2 → 0}];
NumF4NIsoS = Simplify[Integrate[Numerator[F4NIsoS], {θ, 0, 2 π}]];
FullF4NIsoS = NumF4NIsoS / Denominator[F4NIsoS];
F4NIsoS1 = FullSimplify[2 * π * I (Residue[FullF4NIsoS, {ω, ⅈ Df q^2 (q^2 κ + σ)}])];
NumF4NIsoB = Simplify[Integrate[Numerator[F4NIsoB], {θ, 0, 2 π}]];
FullF4NIsoB = NumF4NIsoB / Denominator[F4NIsoB];
F4NIsoB1 = 2 * π * I (Residue[FullF4NIsoB, {ω, ⅈ Df q^2 (q^2 κ + σ)}]);
```

F4C

```
In[ ]:= F4C11 = (F4C /. {Ω1 → Ω2 - ω, p1[1] → (p2[1] - q[1]), p1[2] → (p2[2] - q[2])});
F4C1 = Together[2 (Inverse[MF[p2 - q, Ω2 - ω]][[1, 2]] /.
      {(p2 - q)[1] → (p2[1] - q[1]), (p2 - q)[2] → (p2[2] - q[2])}) * (F4C11)];
F4CIso1 = (*Full*)
  Simplify[F4C1 /. {Ω2 → 0, L1 → L, L2 → L, b1 → 0, b2 → 0, bf → 0, a1 → 0, a2 → 0,
     af → 0, p2[1] → p2, p2[2] → 0, q[1] → q * Cos[θ(*+ϕ*)], q[2] → q * Sin[θ],
     u22 → u, u12 → 0, u11 → u, σ11 → σ, σ21 → 0, σ12 → 0, σ22 → σ}];
F4CIsoS = Simplify[D[F4CIso1, {p2, 2}] /. {p2 → 0}];
F4CIsoB = Simplify[D[F4CIso1, {p2, 4}] /. {p2 → 0}];
NumF4CIsoS = Simplify[Integrate[Numerator[F4CIsoS], {θ, 0, 2 π}]];
FullF4CIsoS = NumF4CIsoS / Denominator[F4CIsoS];
F4CIsoS1 =
  FullSimplify[2 * π * I (Residue[FullF4CIsoS, {ω, (ⅈ Df * q^2 (q^2 κ + σ))/1}] + Residue[
       FullF4CIsoS, {ω, 1/(2(**dc *)) (ⅈ q^2 (DDYuλ(**λ*) + 3 DDYuμ(**μ*)) - I * q^2
            √(-4 DDYuA * DDYuK(**A K*) - 4 (DDYuK^2) (**K^2*) + (DDYuλ^2)
               (*λ^2*) + 2 (DDYuλ * DDYuμ) (* λ μ*) + (DDYuμ^2) (*μ^2*)))}] +
      Residue[FullF4CIsoS, {ω, 1/(2(**dc*)) (ⅈ q^2 (DDYuλ(**λ*) + 3 DDYuμ(**μ*)) +
            I * q^2 √(-4 DDYuA * DDYuK(**A K*) - 4 (DDYuK^2) (**K^2*) + (DDYuλ^2)
               (*λ^2*) + 2 (DDYuλ * DDYuμ) (*λ μ*) + (DDYuμ^2) (*μ^2*)))}])];
F4SFinal = (1 / 2) (1 / (2 * Pi) ^3) Simplify[(F4CIsoS1 + F4NIsoS1)];
NumF4CIsoB = Simplify[Integrate[Numerator[F4CIsoB], {θ, 0, 2 π}]];
FullF4CIsoB = NumF4CIsoB / Denominator[F4CIsoB];
F4CIsoB1 =
  Simplify[2 * π * I (Residue[FullF4CIsoB, {ω, ⅈ Df q^2 (q^2 κ + σ) }] +
      Residue[FullF4CIsoB, {ω, 1/(2(**dc *)) (ⅈ q^2 (DDYuλ(**λ*) + 3 DDYuμ(**μ*)) -
            I * q^2 √(-4 DDYuA * DDYuK(**A K*) - 4 (DDYuK^2) (**K^2*) + (DDYuλ^2)
               (*λ^2*) + 2 (DDYuλ * DDYuμ) (* λ μ*) + (DDYuμ^2) (*μ^2*)))}] +
      Residue[FullF4CIsoB, {ω, 1/(2(**dc *)) (ⅈ q^2 (DDYuλ(**λ*) + 3 DDYuμ(**μ*)) +
            I * q^2 √(-4 DDYuA * DDYuK(**A K*) - 4 (DDYuK^2) (**K^2*) + (DDYuλ^2)
               (*λ^2*) + 2 (DDYuλ * DDYuμ) (*λ μ*) + (DDYuμ^2) (*μ^2*)))}])];
F4BFinal = (1 / 4!) (1 / (2 * Pi) ^3) Simplify[F4CIsoB1 + F4NIsoB1];
```

Terms to renormalize Cijkl terms

Now contract diagrams together, and locate where they contribute

```
(*Y1U1 Vertex renormalization:*)
```

Renormalization of mu

```
Y1U1Totμ = dc * (1 / 2) (1 / (2 * Pi) ^3) * 2 * (1 / 2) 2 *
    FullSimplify[((((Inverse[MF[p2, Ω2]]〚1, 2〛 × Inverse[MF[-p2 + q, -Ω2 + ω]]〚2, 2〛) /.
            {(-p2 + q) [1] → (-p2[1] + q[1]), (-p2 + q) [2] → (-p2[2] + q[2])})
          (FullSimplify[((FullSimplify[FullSimplify[Coefficient[NonLinear1,
                        yy[q, ω]〚1〛 × F[p1, Ω1] × F[-p1 - q, -ω - Ω1] ] /.
                      {b1 → 0, DD1 → DD}] × FullSimplify[Coefficient[NonLinear2,
                        uu[-q, -ω]〚1〛 × F[-p2 + q, ω - Ω2] × ϕ[p2, Ω2] ] /. {b1 → 0,
                        DD1 → DD, bf → 0(*,Df→DD*)}]]) /. {p1 → -p2, Ω1 → -Ω2}) /.
              {(-p2) [1] → (-p2[1]), (-p2) [2] → (-p2[2])}])) /.
        {σ21 → 0, σ12 → 0, σ11 → σ, σ22 → σ, u12 → 0, u11 → u, u22 → u, q → p2,
         p2 → q, ω → Ω2, Ω2 → ω, L1 → L, L2 → L, af → 0, bf → 0}) /.
      {p2[1] → 0, p2[2] → p2, q[1] → q * Cos[θ], q[2] → q * Sin[θ],
       Lf → Df}];
DDμrenorm = FullSimplify[D[Y1U1Totμ, {p2, 2}] /. {p2 → 0, Ω2 → 0}];
Contμ = Integrate[2 * π * I * (Residue[DDμrenorm, {ω, ⅈ Df q^2 (q^2 κ + σ)}]), {θ, 0, 2 π}];
```

Renormalization of (lambda+2mu)

```
In[∘]:= Y1U1Totλμ = dc * (1 / 2) (1 / (2 * Pi) ^3) 2 * (1 / 2) 2 *
    FullSimplify[((((Inverse[MF[p2, Ω2]]〚1, 2〛 × Inverse[MF[-p2 + q, -Ω2 + ω]]〚2, 2〛) /.
            {(-p2 + q) [1] → (-p2[1] + q[1]), (-p2 + q) [2] → (-p2[2] + q[2])})
          (FullSimplify[((FullSimplify[FullSimplify[Coefficient[NonLinear1,
                        yy[q, ω]〚1〛 × F[p1, Ω1] × F[-p1 - q, -ω - Ω1] ] /.
                      {b1 → 0, DD1 → DD}] × FullSimplify[Coefficient[NonLinear2,
                        uu[-q, -ω]〚1〛 × F[-p2 + q, ω - Ω2] × ϕ[p2, Ω2] ] /. {b1 → 0,
                        DD1 → DD, bf → 0(*,Df→DD*)}]]) /. {p1 → -p2, Ω1 → -Ω2}) /.
              {(-p2) [1] → (-p2[1]), (-p2) [2] → (-p2[2])}])) /.
        {σ21 → 0, σ12 → 0, σ11 → σ, σ22 → σ, u12 → 0, u11 → u, u22 → u, q → p2,
         p2 → q, ω → Ω2, Ω2 → ω, L1 → L, L2 → L, af → 0, bf → 0}) /.
      {p2[1] → p2, p2[2] → 0, q[1] → q * Cos[θ], q[2] → q * Sin[θ],
       Lf → Df}];
DDλμrenorm = FullSimplify[D[Y1U1Totλμ, {p2, 2}] /. {p2 → 0, Ω2 → 0}];
Contλμ =
  Integrate[2 * π * I * (Residue[DDλμrenorm, {ω, ⅈ q^2 Df (q^2 κ + σ)}]), {θ, 0, 2 π}];
```

Renormalization of A

```
In[∘]:= Y1U1TotA = dc * (1 / 2) (1 / (2 * Pi) ^ 3) * 2 * (1 / 2) 2 *
     FullSimplify[((((Inverse[MF[p2, Ω2]]⟦1, 2⟧ × Inverse[MF[-p2 + q, -Ω2 + ω]]⟦2, 2⟧) /.
            {(-p2 + q)[1] → (-p2[1] + q[1]), (-p2 + q)[2] → (-p2[2] + q[2])})
          (FullSimplify[((FullSimplify[FullSimplify[Coefficient[NonLinear1,
                   yy[q, ω]⟦1⟧ × F[p1, Ω1] × F[-p1 - q, -ω - Ω1]] /.
                  {b1 → 0, DD1 → DD}] × FullSimplify[Coefficient[NonLinear2,
                   uu[-q, -ω]⟦1⟧ × F[-p2 + q, ω - Ω2] × ϕ[p2, Ω2]] /. {b1 → 0,
                   DD1 → DD, bf → 0(*,Df→DD*)}]]) /. {p1 → -p2, Ω1 → -Ω2}) /.
             {(-p2)[1] → (-p2[1]), (-p2)[2] → (-p2[2])}])) /.
         {σ21 → 0, σ12 → 0, σ11 → σ, σ22 → σ, u12 → 0, u11 → u, u22 → u, q → p2,
          p2 → q, ω → Ω2, Ω2 → ω, L1 → L, L2 → L, af → 0, bf → 0}) /.
       {p2[1] → p1, p2[2] → p2, q[1] → q * Cos[θ], q[2] → q * Sin[θ],
        Lf → Df}];
DDArenorm = 2 * FullSimplify[D[Y1U1TotA, {p1, 1}, {p2, 1}] /. {p1 → 0, p2 → 0, Ω2 → 0}];
ContA = Integrate[2 * π * I * (Residue[DDArenorm, {ω, ⅈ Df q^2 (q^2 κ + σ) }]), {θ, 0, 2 π}];
```

Renormalization of K

```
In[∘]:= Y1U2TotK = dc * (1 / 2) (1 / (2 * Pi) ^ 3) * 2 * (1 / 2) 2 *
     FullSimplify[((((Inverse[MF[p2, Ω2]]⟦1, 2⟧ × Inverse[MF[-p2 + q, -Ω2 + ω]]⟦2, 2⟧) /.
            {(-p2 + q)[1] → (-p2[1] + q[1]), (-p2 + q)[2] → (-p2[2] + q[2])})
          FullSimplify[((FullSimplify[FullSimplify[Coefficient[NonLinear1,
                 yy[q, ω]⟦1⟧ × F[p1, Ω1] × F[-p1 - q, -ω - Ω1]] /. {b1 → 0, DD1 → DD}] ×
               FullSimplify[Coefficient[NonLinear2, uu[-q, -ω]⟦2⟧ × F[-p2 + q, ω -
                    Ω2] × ϕ[p2, Ω2]] /. {b1 → 0, DD1 → DD, bf → 0(*,Df→DD*)}]]) /.
              {p1 → -p2, Ω1 → -Ω2}) /. {(-p2)[1] → (-p2[1]), (-p2)[2] → (-p2[2])}]) /.
         {σ21 → 0, σ12 → 0, σ11 → σ, σ22 → σ, u12 → 0, u11 → u, u22 → u, q → p2,
          p2 → q, ω → Ω2, Ω2 → ω, L1 → L, L2 → L, af → 0, bf → 0}) /.
       {p2[1] → p2, p2[2] → 0, q[1] → q * Cos[θ], q[2] → q * Sin[θ],
        Lf → Df}];
DDKrenorm = FullSimplify[D[Y1U2TotK, {p2, 2}] /. {p2 → 0, Ω2 → 0}];
ContK = Integrate[2 * π * I * (Residue[DDKrenorm, {ω, ⅈ * Df q^2 (q^2 κ + σ)}]), {θ, 0, 2 π}];
```

Renormalize Cijkl ufphi Via Effective F4 Diagram

```
(*Dont include effective slim fish diagrams*)

(*The diagrams in this entry have been ignored due to the
  fact that they are not 1-PI or that they are lower order in d_c*)
(*F4UFFContributionAB2=
 Together[((*wide fish*)-2(*factor of two for switching f's around*)*
     (1/2)(*factor from second order Taylor expansion*)*
     Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of phi(p1)f(-p1)*)*
     ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
```

```
      {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
     (FullSimplify[Coefficient[NonLinear1,uu[q,ω]⟦1⟧F[-p1-q,-ω-Ω1]ϕ[p1,Ω1]]]
      (*Coefficient of u1(q) f(-p1-q) phi(p1) vertex*))*
     (2*((((F4A/.{p1[1]→p1[1]+ρ[1],p1[2]→p1[2]+ρ[2],Ω1→Ω1+P})/.
            {q[1]→p1[1]+p2[1],q[2]→p1[2]+p2[2],ω→Ω1+Ω2})/.{p2[1]→-p2[1],
           p2[2]→-p2[2],Ω2→-Ω2,p1[1]→-p1[1],p1[2]→-p1[2],Ω1→-Ω1})/.
         {ρ[1]→2*p1[1]+q[1],ρ[2]→2*p1[2]+q[2],P→2Ω1+ω})
        (*+(F4B/.{p1[1]→p2[1],p1[2]→p2[2],Ω1→Ω2,q[1]→-q[1],q[2]→-q[2],ω→-ω})*)))/
  {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
F4UFFContributionAB3=
 Together[((*wide fish*)-2(*factor of two for switching f's around*)*
     (1/2)(*factor from second order Taylor expansion*)*
     Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of phi(p1)f(-p1)*)*
     ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1)*))/.
      {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
     (FullSimplify[Coefficient[NonLinear1,uu[q,ω]⟦1⟧F[-p1-q,-ω-Ω1]ϕ[p1,Ω1]]]
      (*Coefficient of u1(q) f(-p1-q) phi(p1) vertex*))*
     (2*((((F4A/.{p1[1]→p1[1]+ρ[1],p1[2]→p1[2]+ρ[2],Ω1→Ω1+P})/.
            {q[1]→p1[1]+p2[1]+q[1],q[2]→p1[2]+p2[2]+q[2],ω→Ω1+Ω2+ω})/.{p2[1]→
            -p2[1],p2[2]→-p2[2],Ω2→-Ω2})/.{ρ[1]→-2*p1[1],ρ[2]→-2*p1[2],P→-2Ω1})
        (*+(F4B/.{p1[1]→p2[1],p1[2]→p2[2],Ω1→Ω2,q[1]→-q[1],q[2]→-q[2],ω→-ω})*)))/
  {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];*)
(*Dont include effective slim fish diagrams*)
(*F4UFFContributionC1=Together[
   2*((*slim fish*)-dc*2(*factor of two for switching different ends around*)*
       (1/2)(*coefficient of phi^2 f^2 vertex extracted*)*
       (1/2)(*factor from second order Taylor expansion*)*
       (Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of phi(p1)f(-p1)*)*
         ((Inverse[MF[p1+q,Ω1+ω]]⟦1,2⟧(*propagator of phi(p1+q)f(-p1)*))/.
           {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])}))
       (-ⅈ Df((λ+2 μ) p1[1] q[1] (p1[1]+q[1])+(A-K) p1[2] q[1] (p1[1]+q[1])+
           K p1[1] (p1[1]+q[1]) q[2]+μ p1[2] (p1[1]+q[1]) q[2]-
           (A+K) p1[1] q[1] (p1[2]+q[2])+λ p1[2] q[1] (p1[2]+q[2])+
           μ p1[1] q[2] (p1[2]+q[2])-K p1[2] q[2] (p1[2]+q[2]))
        (*Coefficient of u1(q) f(-p1-q) phi(p1) vertex*))*
       ((F4C/.{q[1]→-q[1],q[2]→-q[2],ω→-ω,p2[1]→-p2[1],p2[2]→-p2[2],Ω2→-Ω2})/.
         {p1[1]→p1[1]+q[1],p1[2]→p1[2]+q[2],Ω1→Ω1+ω}))/.
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];*)
(*F4UFFContributionC2=
  Together[(((*wide fish*)-2(*factor of two for switching different ends around*)*
       (1/2)(*factor from second order Taylor expansion*)*
       Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of phi(p1)f(-p1)*)*
       ((Inverse[MF[p1+q,Ω1+ω]]⟦1,2⟧(*propagator of phi(p1+q)f(-p1)*))/.
         {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
```

```
        (FullSimplify[Coefficient[NonLinear1,uu[q,ω]⟦1⟧F[-p1-q,-ω-Ω1]ϕ[p1,Ω1]]]
         (*Coefficient of u1(q) f(-p1-q) phi(p1) vertex*))*
        (((((F4C/.{p1[1]→p1[1]+ρ[1],p1[2]→p1[2]+ρ[2],Ω1→Ω1+P,p2[1]→p2[1]+ξ[1],
                 p2[2]→p2[2]+ξ[2],Ω2→Ω2+Z})/.{q[1]→(-(p1[1]+q[1]-p2[1])),
               q[2]→(-(p1[2]+q[2]-p2[2])),ω→(-(Ω1+ω-Ω2))}))/.
           {ρ[1]→q[1],ρ[2]→q[2],P→ω,ξ[1]→-q[1],ξ[2]→-q[2],Z→-ω})(*+(F4B/.
            {p1[1]→p2[1],p1[2]→p2[2],Ω1→Ω2,q[1]→-q[1],q[2]→-q[2],ω→-ω})*))))/.
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];*)
(*F4UFFContributionC2=
  Together[(((*wide fish*)-2(*factor of two for switching different ends around*)*
        (1/2)(*factor from second order Taylor expansion*)*
        Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of phi(p1)f(-p1)*)*
        ((Inverse[MF[p1+q,Ω1+ω]]⟦1,2⟧(*propagator of phi(p1+q)f(-p1)*))/.
          {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
        (FullSimplify[Coefficient[NonLinear1,uu[q,ω]⟦1⟧F[-p1-q,-ω-Ω1]ϕ[p1,Ω1]]]
         (*Coefficient of u1(q) f(-p1-q) phi(p1) vertex*))*
        ((((F4C/.{p1[1]→p1[1]+ρ[1],p1[2]→p1[2]+ρ[2],Ω1→Ω1+P})/.
              {q[1]→p1[1]+p2[1],q[2]→p1[2]+p2[2],ω→Ω1+Ω2})/.{p2[1]→-p2[1],
             p2[2]→-p2[2],Ω2→-Ω2,p1[1]→-p1[1],p1[2]→-p1[2],Ω1→-Ω1})/.
          {ρ[1]→2*p1[1]+q[1],ρ[2]→2*p1[2]+q[2],P→2Ω1+ω}))))/.
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];*)
```

In[ ]:=
```
(*Diagrams in this entry with a zero in front
 have been ignored because they are lower order in d_c*)
F4UFFContributionOG =
  dc * (1 / dc) ((*slim fish*) - dc * (2) (*factor due to the fact this is a cross
        term in second order Taylor expansion*) * 2 (*factor of two for switching
        f's around*) * (1 / 2) (*coefficient of phi f^3 vertex extracted*) *
       (1 / 2) (*factor from second order Taylor expansion*) *
       Inverse[MF[p1, Ω1]]⟦1, 2⟧ (*propagator of phi(p1) f(-p1)*) *
       ((Inverse[MF[p1 + q, Ω1 + ω]]⟦2, 2⟧ (*propagator of f(p1+q) f(-p1)*)) /.
         {(p1 + q) [1] → (p1[1] + q[1]), (p1 + q) [2] → (p1[2] + q[2])})
       (FullSimplify[Coefficient[NonLinear1, uu[q, ω]⟦1⟧ × F[-p1 - q, -ω - Ω1] ×
            ϕ[p1, Ω1]]] (*Coefficient of u1(q) f(-p1-q) phi(p1) vertex*)
         (*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
         ((Sum[(-1) CCDfϕf3⟦i, j, k, l⟧ * p2[i] * (p1[k] + q[k]) *
              (-p1[l]) * (p2[j] - q[j]), {i, 2}, {j, 2}, {k, 2}, {l, 2}]))) -
      (*wide fish*) 0 * (2) * 2 * (1 / 2) * (1 / 2) * Inverse[MF[p1, Ω1]]⟦1, 2⟧ *
       ((Inverse[MF[p1 + q, Ω1 + ω]]⟦2, 2⟧) /.
         {(p1 + q) [1] → (p1[1] + q[1]), (p1 + q) [2] → (p1[2] + q[2])}) (FullSimplify[
          Coefficient[NonLinear1, uu[q, ω]⟦1⟧ × F[-p1 - q, -ω - Ω1] × ϕ[p1, Ω1]]]
         (*Coefficient of phi(-p2) f(-p1) --- f(p1+q) f(p2-q)) vertex*)
         ((Sum[(-1) CCDfϕf3⟦i, j, k, l⟧ * (p2[l] - q[l]) * (p1[k] + q[k]) *
              (-p1[j]) * (p2[i]), {i, 2}, {j, 2}, {k, 2}, {l, 2}]))) -
      (*wide fish*) 0 * (2) * 2 * (1 / 2) * (1 / 2) * Inverse[MF[p1, Ω1]]⟦1, 2⟧ *
       ((Inverse[MF[p1 + q, Ω1 + ω]]⟦2, 2⟧) /.
         {(p1 + q) [1] → (p1[1] + q[1]), (p1 + q) [2] → (p1[2] + q[2])}) (FullSimplify[
          Coefficient[NonLinear1, uu[q, ω]⟦1⟧ × F[-p1 - q, -ω - Ω1] × ϕ[p1, Ω1]]]
          (*Coefficient of phi(-p2) f(p1+q)--- f(-p1) f(p2-q)) vertex*)
         ((Sum[(-1) CCDfϕf3⟦i, j, k, l⟧ * (p2[i]) * (p1[j] + q[j]) *
              (-p1[k]) * (p2[l] - q[l]), {i, 2}, {j, 2}, {k, 2}, {l, 2}])))) /.
   {σ12 → 0, σ21 → 0, σ11 → σ, σ22 → σ, L1 → L, L2 → L};
```

In[ ]:=
```
(*Renormalization of λ+2μ*)
```

In[ ]:=
```
F4UFFContributionC2qp2squared =
  Together[((1 / 2) D[(F4UFFContributionC2 /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
          Ω2 → 0, ω → 0, p2[1] → p2, p2[2] → 0, q[1] → q, q[2] → 0}),
       {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})];
F4UFFContributionOGqp2squared =
  Simplify[Together[((1 / 2) D[(F4UFFContributionOG /. {Ω2 → 0, ω → 0,
             p2[1] → p2, p2[2] → 0, q[1] → q, q[2] → 0}), {q, 1}, {p2, 2}] /.
       {q → 0, p2 → 0}) /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ]}]];
F4UFFContributionAB2qp2squared =
  Simplify[Together[((1 / 2) D[(F4UFFContributionAB2 /.
           {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
```

```
          p2[2] → 0, q[1] → q, q[2] → 0}), {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})]];
F4UFFContributionAB3qp2squared =
  Simplify[Together[((1 / 2) D[(F4UFFContributionAB3 /.
         {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
          p2[2] → 0, q[1] → q, q[2] → 0}), {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})]];
NumF4UFFContributionOGqp2squared =
  Simplify[Integrate[Numerator[F4UFFContributionOGqp2squared], {θ, 0, 2 π}]];
NumF4UFFContributionAB2qp2squared =
  Simplify[Integrate[Numerator[F4UFFContributionAB2qp2squared], {θ, 0, 2 π}]];
NumF4UFFContributionAB3qp2squared =
  Simplify[Integrate[Numerator[F4UFFContributionAB3qp2squared], {θ, 0, 2 π}]];
NumF4UFFContributionC2qp2squared =
  Simplify[Integrate[Numerator[F4UFFContributionC2qp2squared], {θ, 0, 2 π}]];
FullF4UFFContributionOGqp2squared = NumF4UFFContributionOGqp2squared /
   Simplify[Denominator[F4UFFContributionOGqp2squared]];
FullF4UFFContributionAB2qp2squared = NumF4UFFContributionAB2qp2squared /
   Simplify[Denominator[F4UFFContributionAB2qp2squared]];
FullF4UFFContributionAB3qp2squared = NumF4UFFContributionAB3qp2squared /
   Simplify[Denominator[F4UFFContributionAB3qp2squared]];
FullF4UFFContributionC2qp2squared = NumF4UFFContributionC2qp2squared /
   Simplify[Denominator[F4UFFContributionC2qp2squared]];
FullF4UFFContributionOGqp2squaredRes =
  2 * π * I * (Residue[FullF4UFFContributionOGqp2squared, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
FullF4UFFContributionAB2qp2squaredRes =
  2 * π * I * (Residue[FullF4UFFContributionAB2qp2squared, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
      Residue[FullF4UFFContributionAB2qp2squared, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
            p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
      Residue[FullF4UFFContributionAB2qp2squared, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
            p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4UFFContributionAB3qp2squaredRes =
  2 * π * I * (Residue[FullF4UFFContributionAB3qp2squared, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
      Residue[FullF4UFFContributionAB3qp2squared, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
            p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
      Residue[FullF4UFFContributionAB3qp2squared, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
```

```
            p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2  p1^2]}]]);
FullF4UFFContributionC2qp2squaredRes =
  2 * π * I * (Residue[FullF4UFFContributionC2qp2squared, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4UFFContributionC2qp2squared, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
            p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2  p1^2])}] +
     Residue[FullF4UFFContributionC2qp2squared, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
            p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2  p1^2])}]);
ContDfλμ = (1 / (2 π) ^3) (-FullF4UFFContributionOGqp2squaredRes +
     2 * ((FullF4UFFContributionAB2qp2squaredRes +
          FullF4UFFContributionAB3qp2squaredRes) +
        FullF4UFFContributionC2qp2squaredRes));
```

```
In[∘]:= ContDfλμ = (1 / (2 π) ^3) (-FullF4UFFContributionOGqp2squaredRes);
```

```
In[∘]:= (*Renormalization of μ*)
```

```
In[∘]:= F4UFFContributionC2qsquaredp2 =
  Together[((1 / 2) D[(F4UFFContributionC2 /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
           Ω2 → 0, ω → 0, p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}),
        {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})];
F4UFFContributionOGqsquaredp2 =
  Simplify[Together[((1 / 2) D[(F4UFFContributionOG /. {Ω2 → 0, ω → 0,
             p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}), {q, 2}, {p2, 1}] /.
        {q → 0, p2 → 0}) /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ]}]];
F4UFFContributionAB2qsquaredp2 =
  Simplify[Together[((1 / 2) D[(F4UFFContributionAB2 /.
           {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
            p2[2] → 0, q[1] → 0, q[2] → q}), {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})]];
F4UFFContributionAB3qsquaredp2 =
  Simplify[Together[((1 / 2) D[(F4UFFContributionAB3 /.
           {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
            p2[2] → 0, q[1] → 0, q[2] → q}), {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})]];
NumF4UFFContributionOGqsquaredp2 =
  Simplify[Integrate[Numerator[F4UFFContributionOGqsquaredp2], {θ, 0, 2 π}]];
NumF4UFFContributionAB2qsquaredp2 =
  Simplify[Integrate[Numerator[F4UFFContributionAB2qsquaredp2], {θ, 0, 2 π}]];
NumF4UFFContributionAB3qsquaredp2 =
  Simplify[Integrate[Numerator[F4UFFContributionAB3qsquaredp2], {θ, 0, 2 π}]];
NumF4UFFContributionC2qsquaredp2 =
```

```
  Simplify[Integrate[Numerator[F4UFFContributionC2qsquaredp2], {θ, 0, 2 π}]];
FullF4UFFContributionOGqsquaredp2 = NumF4UFFContributionOGqsquaredp2 /
   Simplify[Denominator[F4UFFContributionOGqsquaredp2]];
FullF4UFFContributionAB2qsquaredp2 = NumF4UFFContributionAB2qsquaredp2 /
   Simplify[Denominator[F4UFFContributionAB2qsquaredp2]];
FullF4UFFContributionAB3qsquaredp2 = NumF4UFFContributionAB3qsquaredp2 /
   Simplify[Denominator[F4UFFContributionAB3qsquaredp2]];
FullF4UFFContributionC2qsquaredp2 = NumF4UFFContributionC2qsquaredp2 /
   Simplify[Denominator[F4UFFContributionC2qsquaredp2]];
FullF4UFFContributionOGqsquaredp2Res =
  2 * π * I * (Residue[FullF4UFFContributionOGqsquaredp2, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
FullF4UFFContributionAB2qsquaredp2Res =
  2 * π * I * (Residue[FullF4UFFContributionAB2qsquaredp2, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
      Residue[FullF4UFFContributionAB2qsquaredp2, {Ω1, 1/(2 (**dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
            p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
      Residue[FullF4UFFContributionAB2qsquaredp2, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
            p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4UFFContributionAB3qsquaredp2Res =
  2 * π * I * (Residue[FullF4UFFContributionAB3qsquaredp2, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
      Residue[FullF4UFFContributionAB3qsquaredp2, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
            p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
      Residue[FullF4UFFContributionAB3qsquaredp2, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
            p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4UFFContributionC2qsquaredp2Res =
  2 * π * I * (Residue[FullF4UFFContributionC2qsquaredp2, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
      Residue[FullF4UFFContributionC2qsquaredp2, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
            p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
      Residue[FullF4UFFContributionC2qsquaredp2, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
```

```
p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]]);
(*ContDfμ=(1/(2π)^3)(-FullF4UFFContributionOGqsquaredp2Res+
   2*((FullF4UFFContributionAB2qsquaredp2Res+
      FullF4UFFContributionAB3qsquaredp2Res)+
     FullF4UFFContributionC2qsquaredp2Res));*)
ContDfμ = (1 / (2 π) ^3) (-FullF4UFFContributionOGqsquaredp2Res);
```

In[◦]:=

```
(*Renormalization of K*)
```

In[◦]:=

```
F4UFFContributionC2qsquaredp2K =
  Together[((1 / 2) D[(F4UFFContributionC2 /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
        Ω2 → 0, ω → 0, p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}),
      {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})];
F4UFFContributionOGqsquaredp2K =
  Simplify[Together[((1 / 2) D[(F4UFFContributionOG /. {Ω2 → 0, ω → 0,
          p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}), {q, 1}, {p2, 2}] /.
      {q → 0, p2 → 0}) /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ]}]];
F4UFFContributionAB2qsquaredp2K =
  Simplify[Together[((1 / 2) D[(F4UFFContributionAB2 /.
        {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
         p2[2] → 0, q[1] → 0, q[2] → q}), {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})]];
F4UFFContributionAB3qsquaredp2K =
  Simplify[Together[((1 / 2) D[(F4UFFContributionAB3 /.
        {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
         p2[2] → 0, q[1] → 0, q[2] → q}), {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})]];
NumF4UFFContributionOGqsquaredp2K =
  Simplify[Integrate[Numerator[F4UFFContributionOGqsquaredp2K], {θ, 0, 2 π}]];
NumF4UFFContributionAB2qsquaredp2K =
  Simplify[Integrate[Numerator[F4UFFContributionAB2qsquaredp2K], {θ, 0, 2 π}]];
NumF4UFFContributionAB3qsquaredp2K =
  Simplify[Integrate[Numerator[F4UFFContributionAB3qsquaredp2K], {θ, 0, 2 π}]];
NumF4UFFContributionC2qsquaredp2K =
  Simplify[Integrate[Numerator[F4UFFContributionC2qsquaredp2K], {θ, 0, 2 π}]];
FullF4UFFContributionOGqsquaredp2K = NumF4UFFContributionOGqsquaredp2K /
   Simplify[Denominator[F4UFFContributionOGqsquaredp2K]];
FullF4UFFContributionAB2qsquaredp2K = NumF4UFFContributionAB2qsquaredp2K /
   Simplify[Denominator[F4UFFContributionAB2qsquaredp2K]];
FullF4UFFContributionAB3qsquaredp2K = NumF4UFFContributionAB3qsquaredp2K /
   Simplify[Denominator[F4UFFContributionAB3qsquaredp2K]];
FullF4UFFContributionC2qsquaredp2K = NumF4UFFContributionC2qsquaredp2K /
   Simplify[Denominator[F4UFFContributionC2qsquaredp2K]];
FullF4UFFContributionOGqsquaredp2ResK =
  2 * π * I * (Residue[FullF4UFFContributionOGqsquaredp2K, {Ω1, i Df * p1^2 (p1^2 κ + σ)}]);
```

```
FullF4UFFContributionAB2qsquaredp2ResK =
  2 * π * I * (Residue[FullF4UFFContributionAB2qsquaredp2K, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4UFFContributionAB2qsquaredp2K,
      {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ p1^2 +
          ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] + Residue[
      FullF4UFFContributionAB2qsquaredp2K, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ p1^2 +
          Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4UFFContributionAB3qsquaredp2ResK =
  2 * π * I * (Residue[FullF4UFFContributionAB3qsquaredp2K, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4UFFContributionAB3qsquaredp2K,
      {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ p1^2 +
          ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] + Residue[
      FullF4UFFContributionAB3qsquaredp2K, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ p1^2 +
          Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4UFFContributionC2qsquaredp2ResK =
  2 * π * I * (Residue[FullF4UFFContributionC2qsquaredp2K, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4UFFContributionC2qsquaredp2K, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
           p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
     Residue[FullF4UFFContributionC2qsquaredp2K, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
           p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
(*ContDfK=(1/(2π)^3)(-FullF4UFFContributionOGqsquaredp2ResK+
    2*((FullF4UFFContributionAB2qsquaredp2ResK+
         FullF4UFFContributionAB3qsquaredp2ResK)+
       FullF4UFFContributionC2qsquaredp2ResK));*)
ContDfK = (1 / (2 π) ^3) (-FullF4UFFContributionOGqsquaredp2ResK);
```

```
(*Renormalization of A-K Changed the differentiation frequency of q and p2 here and Chang
```

```
In[ ]:= F4UFFContributionC2qsquaredp2AK =
  Together[((1 / 2) D[(F4UFFContributionC2 /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
```

```
            Ω2 → 0, ω → 0, p2[1] → 0, p2[2] → p2, q[1] → q, q[2] → 0}),
        {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})];
F4UFFContributionOGqsquaredp2AK =
  Simplify[Together[((1 / 2) D[(F4UFFContributionOG /. {Ω2 → 0, ω → 0,
             p2[1] → 0, p2[2] → p2, q[1] → q, q[2] → 0}), {q, 2}, {p2, 1}] /.
        {q → 0, p2 → 0}) /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ]}]];
F4UFFContributionAB2qsquaredp2AK =
  Simplify[Together[((1 / 2) D[(F4UFFContributionAB2 /.
           {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → 0,
            p2[2] → p2, q[1] → q, q[2] → 0}), {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})]];
F4UFFContributionAB3qsquaredp2AK =
  Simplify[Together[((1 / 2) D[(F4UFFContributionAB3 /.
           {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → 0,
            p2[2] → p2, q[1] → q, q[2] → 0}), {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})]];
NumF4UFFContributionOGqsquaredp2AK =
  Simplify[Integrate[Numerator[F4UFFContributionOGqsquaredp2AK], {θ, 0, 2 π}]];
NumF4UFFContributionAB2qsquaredp2AK =
  Simplify[Integrate[Numerator[F4UFFContributionAB2qsquaredp2AK], {θ, 0, 2 π}]];
NumF4UFFContributionAB3qsquaredp2AK =
  Simplify[Integrate[Numerator[F4UFFContributionAB3qsquaredp2AK], {θ, 0, 2 π}]];
NumF4UFFContributionC2qsquaredp2AK =
  Simplify[Integrate[Numerator[F4UFFContributionC2qsquaredp2AK], {θ, 0, 2 π}]];
FullF4UFFContributionOGqsquaredp2AK = NumF4UFFContributionOGqsquaredp2AK /
   Simplify[Denominator[F4UFFContributionOGqsquaredp2AK]];
FullF4UFFContributionAB2qsquaredp2AK = NumF4UFFContributionAB2qsquaredp2AK /
   Simplify[Denominator[F4UFFContributionAB2qsquaredp2AK]];
FullF4UFFContributionAB3qsquaredp2AK = NumF4UFFContributionAB3qsquaredp2AK /
   Simplify[Denominator[F4UFFContributionAB3qsquaredp2AK]];
FullF4UFFContributionC2qsquaredp2AK = NumF4UFFContributionC2qsquaredp2AK /
   Simplify[Denominator[F4UFFContributionC2qsquaredp2AK]];
FullF4UFFContributionOGqsquaredp2ResAK = 2 * π * I *
   (Residue[FullF4UFFContributionOGqsquaredp2AK, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
FullF4UFFContributionAB2qsquaredp2ResAK =
  2 * π * I * (Residue[FullF4UFFContributionAB2qsquaredp2AK, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4UFFContributionAB2qsquaredp2AK,
      {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ p1^2 +
          ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] + Residue[
      FullF4UFFContributionAB2qsquaredp2AK, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ p1^2 +
```

```
         Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4UFFContributionAB3qsquaredp2ResAK =
  2 * π * I * (Residue[FullF4UFFContributionAB3qsquaredp2AK, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4UFFContributionAB3qsquaredp2AK,
      {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ p1^2 +
          ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] + Residue[
      FullF4UFFContributionAB3qsquaredp2AK, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ p1^2 +
         Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4UFFContributionC2qsquaredp2ResAK =
  2 * π * I * (Residue[FullF4UFFContributionC2qsquaredp2AK, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4UFFContributionC2qsquaredp2AK,
      {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ p1^2 +
          ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] + Residue[
      FullF4UFFContributionC2qsquaredp2AK, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ p1^2 +
         Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
ContDfAK = (1 / (2 π) ^3) (-FullF4UFFContributionOGqsquaredp2ResAK +
     2 * ((FullF4UFFContributionAB2qsquaredp2ResAK +
          FullF4UFFContributionAB3qsquaredp2ResAK) +
        FullF4UFFContributionC2qsquaredp2ResK));
ContDfAK = (1 / (2 π) ^3) (-FullF4UFFContributionOGqsquaredp2ResAK);
```

Ward Identity Check YF^2

```
(*Diagrams in this entry have been ignored either because they are not 1-
 PI or they are lower order in d_c*)
(*F4YFFContributionAB2=
 Together[((*wide fish*)-2(*factor of two for switching ends around*)*2
      (*factor of two for switching f's around*)*(1/2)(*factor from second order
       Taylor expansion*)*Inverse[MF[p1,Ω1]]⟦2,1⟧(*propagator of f(p1)phi(-p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      (FullSimplify[Coefficient[NonLinear1,yy[q,ω]⟦1⟧F[p1,Ω1] F[-p1-q,-ω-Ω1] ]]
       (*Coefficient of y1(q) f(-p1-q) f(p1) vertex*))*
      (2*((((F4A/.{p1[1]→p1[1]+ρ[1],p1[2]→p1[2]+ρ[2],Ω1→Ω1+P,p2[1]→p2[1]+z[1],
                p2[2]→p2[2]+z[2],Ω2→Ω2+Z})/.{q[1]→(-p1[1]+p2[1]-q[1]),
              q[2]→(-p1[2]+p2[2]-q[2]),ω→(-Ω1+Ω2-ω)}))/.{ρ[1]→q[1],ρ[2]→q[2],
           P→ω,z[1]→(-p1[1]-p2[1]),z[2]→(-p1[2]-p2[2]),Z→(-Ω1-Ω2)})
         (*+(F4B/.{p1[1]→p2[1],p1[2]→p2[2],Ω1→Ω2,q[1]→-q[1],q[2]→-q[2],ω→-ω})*)))/
   {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
(*Dont include effective slim fish diagrams*)
F4YFFContributionC2=
 Together[(((*wide fish*)-2(*factor of two for switching different ends around*)*
       (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦2,1⟧(*propagator of phi(-p1)f(p1)*)*
       ((Inverse[MF[p1+q,Ω1+ω]]⟦1,2⟧(*propagator of phi(p1+q)f(-p1)*))/.
         {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
       (FullSimplify[Coefficient[NonLinear1,yy[q,ω]⟦1⟧F[p1,Ω1] F[-p1-q,-ω-Ω1] ]]
        (*Coefficient of y1(q) f(-p1-q) f(p1) vertex*))*
       (((((F4C/.{p1[1]→p1[1]+ρ[1],p1[2]→p1[2]+ρ[2],Ω1→Ω1+P,p2[1]→p2[1]+z[1],
                p2[2]→p2[2]+z[2],Ω2→Ω2+Z})/.{q[1]→(-(p1[1]+q[1]-p2[1])),
              q[2]→(-(p1[2]+q[2]-p2[2])),ω→(-(Ω1+ω-Ω2))}))/.{ρ[1]→q[1],ρ[2]→q[2],
           P→ω,z[1]→(-p1[1]-p2[1]),z[2]→(-p1[2]-p2[2]),Z→(-Ω1-Ω2)})(*+(F4B/.
           {p1[1]→p2[1],p1[2]→p2[2],Ω1→Ω2,q[1]→-q[1],q[2]→-q[2],ω→-ω})*))))/.
   {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];*)
```

In[∘]:=
```
(*Diagrams in this entry with a zero in front
 have been ignored because they are lower order in d_c*)
F4YFFContributionOG = dc * (1 / dc) ((*slim fish*) - dc * (2) (*factor due to the
        fact this is a cross term in second order Taylor expansion*) * 2
       (*factor of two for switching f's around*) * (1 / 2) (*coefficient of
        phi f^3 vertex extracted*) * (1 / 2) (*factor from second order Taylor
        expansion*) * Inverse[MF[p1, Ω1]]⟦2, 1⟧ (*propagator of phi(-p1) f(p1)*) *
       ((Inverse[MF[p1 + q, Ω1 + ω]]⟦2, 2⟧ (*propagator of f(p1+q) f(-p1-q)*)) /.
         {(p1 + q) [1] → (p1[1] + q[1]), (p1 + q) [2] → (p1[2] + q[2])})
       ((FullSimplify[Coefficient[NonLinear1, yy[q, ω]⟦1⟧ × F[p1, Ω1] ×
             F[-p1 - q, -ω - Ω1] ]] (*Coefficient of y1(q) f(-p1-q) f(p1) vertex*))
         (*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
        ((Sum[CCDfϕf3⟦i, j, k, l⟧ * (-p2[k]) * (p1[j] + q[j]) * (-p1[i]) *
             (p2[l] - q[l]), {i, 2}, {j, 2}, {k, 2}, {l, 2}]))) -
      (*wide fish*) 0 * 2 * (2) * 2 * (1 / 2) * (1 / 2) * Inverse[MF[p1, Ω1]]⟦2, 1⟧ *
       ((Inverse[MF[p1 + q, Ω1 + ω]]⟦2, 2⟧) /.
         {(p1 + q) [1] → (p1[1] + q[1]), (p1 + q) [2] → (p1[2] + q[2])})
       ((FullSimplify[Coefficient[NonLinear1, yy[q, ω]⟦1⟧ × F[p1, Ω1] ×
             F[-p1 - q, -ω - Ω1] ]] (*Coefficient of y1(q) f(-p1-q) f(p1) vertex*))
         (*Coefficient of phi(-p2) f(-p1) --- f(p1+q) f(p2-q)) vertex*)
        ((Sum[(-1) CCDfϕf3⟦i, j, k, l⟧ * (p2[j] - q[j]) * (p1[k] + q[k]) *
             (-p1[i]) * (p2[l]), {i, 2}, {j, 2}, {k, 2}, {l, 2}])))) /.
   {σ12 → 0, σ21 → 0, σ11 → σ, σ22 → σ, L1 → L, L2 → L};
```

In[∘]:=
```
(*Renormalization of λ+2μ*)
```

In[∘]:=
```
F4YFFContributionC2qp2squared =
  Together[((1 / 2) D[(F4YFFContributionC2 /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
          Ω2 → 0, ω → 0, p2[1] → p2, p2[2] → 0, q[1] → q, q[2] → 0}),
       {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})];
F4YFFContributionOGqp2squared =
  Simplify[Together[((1 / 2) D[(F4YFFContributionOG /. {Ω2 → 0, ω → 0,
            p2[1] → p2, p2[2] → 0, q[1] → q, q[2] → 0}), {q, 1}, {p2, 2}] /.
       {q → 0, p2 → 0}) /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ]}]];
F4YFFContributionAB2qp2squared =
  Simplify[Together[((1 / 2) D[(F4YFFContributionAB2 /.
          {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
           p2[2] → 0, q[1] → q, q[2] → 0}), {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})]];
NumF4YFFContributionOGqp2squared =
  Simplify[Integrate[Numerator[F4YFFContributionOGqp2squared], {θ, 0, 2 π}]];
NumF4YFFContributionAB2qp2squared =
  Simplify[Integrate[Numerator[F4YFFContributionAB2qp2squared], {θ, 0, 2 π}]];
NumF4YFFContributionC2qp2squared =
  Simplify[Integrate[Numerator[F4YFFContributionC2qp2squared], {θ, 0, 2 π}]];
```

```
FullF4YFFContributionOGqp2squared = NumF4YFFContributionOGqp2squared /
   Simplify[Denominator[F4YFFContributionOGqp2squared]];
FullF4YFFContributionAB2qp2squared = NumF4YFFContributionAB2qp2squared /
   Simplify[Denominator[F4YFFContributionAB2qp2squared]];
FullF4YFFContributionC2qp2squared = NumF4YFFContributionC2qp2squared /
   Simplify[Denominator[F4YFFContributionC2qp2squared]];
FullF4YFFContributionOGqp2squaredRes =
  2 * π * I * (Residue[FullF4YFFContributionOGqp2squared, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
FullF4YFFContributionAB2qp2squaredRes =
  2 * π * I * (Residue[FullF4YFFContributionAB2qp2squared, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4YFFContributionAB2qp2squared, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
          p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
     Residue[FullF4YFFContributionAB2qp2squared, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
          p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4YFFContributionC2qp2squaredRes =
  2 * π * I * (Residue[FullF4YFFContributionC2qp2squared, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4YFFContributionC2qp2squared, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
          p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
     Residue[FullF4YFFContributionC2qp2squared, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
          p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
(*ContDλμYFF=2*(1/(2π)^3)(-FullF4YFFContributionOGqp2squaredRes+
    2*((FullF4YFFContributionAB2qp2squaredRes)+
       FullF4YFFContributionC2qp2squaredRes));*)
ContDλμYFF = 2 * (1 / (2 π) ^ 3) (-FullF4YFFContributionOGqp2squaredRes);
```

In[ ]:=
```
(*Renormalization of μ*)
```

In[ ]:=
```
F4YFFContributionC2qsquaredp2 =
  Together[((1 / 2) D[(F4YFFContributionC2 /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
          Ω2 → 0, ω → 0, p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}),
       {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})];
F4YFFContributionOGqsquaredp2 =
  Simplify[Together[((1 / 2) D[(F4YFFContributionOG /. {Ω2 → 0, ω → 0,
            p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}), {q, 2}, {p2, 1}] /.
```

```
      {q → 0, p2 → 0}) /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ]}]];
F4YFFContributionAB2qsquaredp2 =
  Simplify[Together[((1 / 2) D[(F4YFFContributionAB2 /.
          {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
           p2[2] → 0, q[1] → 0, q[2] → q}), {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})]];

NumF4YFFContributionOGqsquaredp2 =
  Simplify[Integrate[Numerator[F4YFFContributionOGqsquaredp2], {θ, 0, 2 π}]];
NumF4YFFContributionAB2qsquaredp2 =
  Simplify[Integrate[Numerator[F4YFFContributionAB2qsquaredp2], {θ, 0, 2 π}]];
NumF4YFFContributionC2qsquaredp2 =
  Simplify[Integrate[Numerator[F4YFFContributionC2qsquaredp2], {θ, 0, 2 π}]];
FullF4YFFContributionOGqsquaredp2 = NumF4YFFContributionOGqsquaredp2 /
   Simplify[Denominator[F4YFFContributionOGqsquaredp2]];
FullF4YFFContributionAB2qsquaredp2 = NumF4YFFContributionAB2qsquaredp2 /
   Simplify[Denominator[F4YFFContributionAB2qsquaredp2]];
FullF4YFFContributionC2qsquaredp2 = NumF4YFFContributionC2qsquaredp2 /
   Simplify[Denominator[F4YFFContributionC2qsquaredp2]];
FullF4YFFContributionOGqsquaredp2Res =
  2 * π * I * (Residue[FullF4YFFContributionOGqsquaredp2, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
FullF4YFFContributionAB2qsquaredp2Res =
  2 * π * I * (Residue[FullF4YFFContributionAB2qsquaredp2, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4YFFContributionAB2qsquaredp2, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
           p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
     Residue[FullF4YFFContributionAB2qsquaredp2, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
           p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4YFFContributionC2qsquaredp2Res =
  2 * π * I * (Residue[FullF4YFFContributionC2qsquaredp2, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4YFFContributionC2qsquaredp2, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
           p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
     Residue[FullF4YFFContributionC2qsquaredp2, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
           p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
(*ContDμYFF=2*(1/(2π)^3)(-FullF4YFFContributionOGqsquaredp2Res+
```

```
    2*((FullF4YFFContributionAB2qsquaredp2Res)+
       FullF4YFFContributionC2qsquaredp2Res));*)
ContDμYFF = 2 * (1 / (2 π) ^3) (-FullF4YFFContributionOGqsquaredp2Res);
```

In[ ]:=
```
(*Renormalization of K*)
```

In[ ]:=
```
F4YFFContributionC2qsquaredp2K =
  Together[((1 / 2) D[(F4YFFContributionC2 /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
          Ω2 → 0, ω → 0, p2[1] → 0, p2[2] → p2, q[1] → q, q[2] → 0}),
      {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})];
F4YFFContributionOGqsquaredp2K =
  Simplify[Together[((1 / 2) D[(F4YFFContributionOG /. {Ω2 → 0, ω → 0,
            p2[1] → 0, p2[2] → p2, q[1] → q, q[2] → 0}), {q, 2}, {p2, 1}] /.
      {q → 0, p2 → 0}) /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ]}]];
F4YFFContributionAB2qsquaredp2K =
  Simplify[Together[((1 / 2) D[(F4YFFContributionAB2 /.
          {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → 0,
           p2[2] → p2, q[1] → q, q[2] → 0}), {q, 2}, {p2, 1}] /. {q → 0, p2 → 0})]];
NumF4YFFContributionOGqsquaredp2K =
  Simplify[Integrate[Numerator[F4YFFContributionOGqsquaredp2K], {θ, 0, 2 π}]];
NumF4YFFContributionAB2qsquaredp2K =
  Simplify[Integrate[Numerator[F4YFFContributionAB2qsquaredp2K], {θ, 0, 2 π}]];
NumF4YFFContributionC2qsquaredp2K =
  Simplify[Integrate[Numerator[F4YFFContributionC2qsquaredp2K], {θ, 0, 2 π}]];
FullF4YFFContributionOGqsquaredp2K = NumF4YFFContributionOGqsquaredp2K /
   Simplify[Denominator[F4YFFContributionOGqsquaredp2K]];
FullF4YFFContributionAB2qsquaredp2K = NumF4YFFContributionAB2qsquaredp2K /
   Simplify[Denominator[F4YFFContributionAB2qsquaredp2K]];
FullF4YFFContributionC2qsquaredp2K = NumF4YFFContributionC2qsquaredp2K /
   Simplify[Denominator[F4YFFContributionC2qsquaredp2K]];
FullF4YFFContributionOGqsquaredp2ResK =
  2 * π * I * (Residue[FullF4YFFContributionOGqsquaredp2K, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
FullF4YFFContributionAB2qsquaredp2ResK =
  2 * π * I * (Residue[FullF4YFFContributionAB2qsquaredp2K, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
     Residue[FullF4YFFContributionAB2qsquaredp2K,
      {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ p1^2 +
          ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] + Residue[
      FullF4YFFContributionAB2qsquaredp2K, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ p1^2 +
          Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
```

```
FullF4YFFContributionC2qsquaredp2ResK =
  2 * π * I * (Residue[FullF4YFFContributionC2qsquaredp2K, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
      Residue[FullF4YFFContributionC2qsquaredp2K, {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ
             p1^2 + ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] +
      Residue[FullF4YFFContributionC2qsquaredp2K, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ
             p1^2 + Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
(*ContDKYFF=2*(1/(2π)^3)(-FullF4YFFContributionOGqsquaredp2ResK+
     2*((FullF4YFFContributionAB2qsquaredp2ResK
         (*+FullF4YFFContributionAB3qsquaredp2ResK*))+
       FullF4YFFContributionC2qsquaredp2ResK));*)
ContDKYFF = 2 * (1 / (2 π) ^3) (-FullF4YFFContributionOGqsquaredp2ResK);
```

In[ ]:=
```
(*Renormalization of A-K*)
```

In[ ]:=
```
F4YFFContributionC2qsquaredp2AK =
  Together[((1 / 2) D[(F4YFFContributionC2 /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
          Ω2 → 0, ω → 0, p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}),
       {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})];
F4YFFContributionOGqsquaredp2AK =
  Simplify[Together[((1 / 2) D[(F4YFFContributionOG /. {Ω2 → 0, ω → 0,
            p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}), {q, 1}, {p2, 2}] /.
       {q → 0, p2 → 0}) /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ]}]];
F4YFFContributionAB2qsquaredp2AK =
  Simplify[Together[((1 / 2) D[(F4YFFContributionAB2 /.
          {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω2 → 0, ω → 0, p2[1] → p2,
           p2[2] → 0, q[1] → 0, q[2] → q}), {q, 1}, {p2, 2}] /. {q → 0, p2 → 0})]];
NumF4YFFContributionOGqsquaredp2AK =
  Simplify[Integrate[Numerator[F4YFFContributionOGqsquaredp2AK], {θ, 0, 2 π}]];
NumF4YFFContributionAB2qsquaredp2AK =
  Simplify[Integrate[Numerator[F4YFFContributionAB2qsquaredp2AK], {θ, 0, 2 π}]];
NumF4YFFContributionC2qsquaredp2AK =
  Simplify[Integrate[Numerator[F4YFFContributionC2qsquaredp2AK], {θ, 0, 2 π}]];
FullF4YFFContributionOGqsquaredp2AK = NumF4YFFContributionOGqsquaredp2AK /
   Simplify[Denominator[F4YFFContributionOGqsquaredp2AK]];
FullF4YFFContributionAB2qsquaredp2AK = NumF4YFFContributionAB2qsquaredp2AK /
   Simplify[Denominator[F4YFFContributionAB2qsquaredp2AK]];
FullF4YFFContributionC2qsquaredp2AK = NumF4YFFContributionC2qsquaredp2AK /
   Simplify[Denominator[F4YFFContributionC2qsquaredp2AK]];
FullF4YFFContributionOGqsquaredp2ResAK = 2 * π * I *
```

```
  (Residue[FullF4YFFContribution0Gqsquaredp2AK, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
FullF4YFFContributionAB2qsquaredp2ResAK =
 2 * π * I * (Residue[FullF4YFFContributionAB2qsquaredp2AK, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
    Residue[FullF4YFFContributionAB2qsquaredp2AK,
     {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ p1^2 +
          ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] + Residue[
     FullF4YFFContributionAB2qsquaredp2AK, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ p1^2 +
          Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
FullF4YFFContributionC2qsquaredp2ResAK =
 2 * π * I * (Residue[FullF4YFFContributionC2qsquaredp2AK, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}] +
    Residue[FullF4YFFContributionC2qsquaredp2AK,
     {Ω1, 1/(2 (*dc*)) ⅈ (DDYuλ p1^2 + 3 DDYuμ p1^2 +
          ⅈ Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}] + Residue[
     FullF4YFFContributionC2qsquaredp2AK, {Ω1, 1/(2 (*dc*)) (ⅈ DDYuλ p1^2 + 3 ⅈ DDYuμ p1^2 +
          Sqrt[4 DDYuA DDYuK + 4 DDYuK^2 - DDYuλ^2 - 2 DDYuλ DDYuμ - DDYuμ^2] p1^2)}]);
(*ContDAKYFF=2*(1/(2π)^3)(-FullF4YFFContribution0Gqsquaredp2ResAK+
    2*((FullF4YFFContributionAB2qsquaredp2ResAK)+
       FullF4YFFContributionC2qsquaredp2ResAK));*)
ContDAKYFF = 2 * (1 / (2 π) ^3) (-FullF4YFFContribution0Gqsquaredp2ResAK);
```

Renormalize Cijkl phi f^3 term.

```
(*Non-effective diagrams are sometimes
  abbreviated by NE whereas E stands for effective*)
```

```
(*Non Effective ^2 Combinations*)
```

In[ ]:=
```
(*Diagrams in this entry that have been commented
 out have been ignored because they are lower order in d_c*)
F4F4Contribution0GSlimFish =
  Together[dc * (1 / dc) ((*slim fish*) - dc * (1) (*factor due to the fact
        this is a cross term in second order Taylor expansion*) * 4
       (*factor of two for switching f's around and factor of two for switching
```

```
        diagrams around*) * (1 / 2)(*factor from second order Taylor expansion*) *
       Inverse[MF[p1, Ω1]]⟦1, 2⟧(*propagator of phi(p1)f(-p1)*) *
       ((Inverse[MF[p1 + q, Ω1 + ω]]⟦2, 2⟧(*propagator of f(p1+q)f(-p1-q)*)) /.
          {(p1 + q)[1] → (p1[1] + q[1]), (p1 + q)[2] → (p1[2] + q[2])})
       ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
        Sum[(-1 / 2)(*coefficient of (phi f^3)^2 vertex extracted*)
          CCDfϕf3⟦i, j, k, l⟧ * (p1[i]) * (-p1[j] - q[j]) * (p2[k]) *
          (-p2[l] + q[l]), {i, 2}, {j, 2}, {k, 2}, {l, 2}])
       ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
        Sum[(-1 / 2) CCDfϕf3⟦i, j, k, l⟧ * (p3[i] - q[i]) * (-p3[j]) *
          (-p1[k]) * (p1[l] + q[l]), {i, 2}, {j, 2}, {k, 2}, {l, 2}])) /.
    {σ12 → 0, σ21 → 0, σ11 → σ, σ22 → σ, L1 → L, L2 → L}];
(*F4F4ContributionOGWideFishNϕ=Together[
  (1/dc)((*slim fish*)-dc*(1)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*8(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of phi(p1)f(-p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)(*coefficient of (phi f^3)^2 vertex extracted*)CCDfϕf3⟦i,j,k,l⟧*
         (p1[i])*(p2[j])*(q[k]-p2[k])*(-p1[l]-q[l]),{i,2},{j,2},{k,2},{l,2}])
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p3[i]-q[i])*(-p3[j])*(-p1[k])*(p1[l]+q[l]),
        {i,2},{j,2},{k,2},{l,2}]))/.{σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
F4F4ContributionOGWideFishWϕ1=Together[
  (1/dc)((*slim fish*)-dc*(1)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*4(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦2,2⟧(*propagator of f(p1)f(-p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]]⟦1,2⟧(*propagator of ϕ(p1+q)f(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)(*coefficient of (phi f^3)^2 vertex extracted*)CCDfϕf3⟦i,j,k,l⟧*
         (p1[i]+q[i])*(-p1[j])*(p3[k]-q[k])*(-p3[l]),{i,2},{j,2},{k,2},{l,2}])
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p2[i])*(p1[j])*(-p1[k]-q[k])*(q[l]-p2[l]),
        {i,2},{j,2},{k,2},{l,2}]))/.{σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
F4F4ContributionOGWideFishWϕ2=Together[
  (1/dc)((*slim fish*)-dc*(1)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*4(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
```

```
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦2,1⟧(*propagator of ϕ(-p1)f(p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)(*coefficient of (phi f^3)^2 vertex extracted*)CCDfϕf3⟦i,j,k,l⟧*
         (-p1[i])*(p1[j]+q[j])*(p3[k]-q[k])*(-p3[l]),{i,2},{j,2},{k,2},{l,2}])
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p2[i])*(p1[j])*(-p1[k]-q[k])*(q[l]-p2[l]),
        {i,2},{j,2},{k,2},{l,2}]))/.{σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
F4F4ContributionOGBunkBedSame=Together[
  (1/dc)((*slim fish*)-dc*(1)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*8(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦2,1⟧(*propagator of ϕ(-p1)f(p1)*)*
      ((Inverse[MF[p1-q,-ω+Ω1]]⟦2,2⟧(*propagator of f(p2-q+p1+p3)f
          (-(p2-q+p1+p3))*))/.{(p1-q)[1]→(-q[1]+p1[1]),(p1-q)[2]→(-q[2]+p1[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)(*coefficient of (phi f^3)^2 vertex extracted*)CCDfϕf3⟦i,j,k,l⟧*
         (p3[i]-q[i])*(p1[j])*(-p1[k]+q[k])*(-p3[l]),{i,2},{j,2},{k,2},{l,2}])
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(-p1[i])*(q[j]-p2[j])*(p2[k])*(p1[l]-q[l]),
        {i,2},{j,2},{k,2},{l,2}]))/.{σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
F4F4ContributionOGBunkBedOpposite=Together[
  (1/dc)((*slim fish*)-dc*(1)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*8(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦2,2⟧(*propagator of f(-p1)f(p1)*)*
      ((Inverse[MF[p1-q,Ω1-ω]]⟦1,2⟧(*propagator of ϕ(p2-q+p1+p3)f
          (-(p2-q+p1+p3))*))/.{(p1-q)[1]→(p1[1]-q[1]),(p1-q)[2]→(p1[2]-q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)(*coefficient of (phi f^3)^2 vertex extracted*)CCDfϕf3⟦i,j,k,l⟧*
         (p3[i]-q[i])*(p1[j])*(-p1[k]+q[k])*(-p3[l]),{i,2},{j,2},{k,2},{l,2}])
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p1[i]-q[i])*(p2[j])*(-p1[k])*(q[l]-p2[l]),
        {i,2},{j,2},{k,2},{l,2}]))/.{σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
(*F4F4ContributionOGBunkBedSame=
 Together[(1/dc)((*slim fish*)-dc*(*factor due to the fact this is a cross term
       in second order Taylor expansion*)*8(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*Inverse[MF[p1,Ω1]]⟦2,1⟧
      (*propagator of ϕ(-p1)f(p1)*)*((Inverse[MF[p2-q+p1+p3,Ω2-ω+Ω1+Ω3]]⟦2,2⟧
```

```
(*propagator of f(p2-q+p1+p3)f(-(p2-q+p1+p3))*))/.{(p2-q+p1+p3)[1]→
 (p2[1]-q[1]+p1[1]+p3[1]),(p2-q+p1+p3)[2]→(p2[2]-q[2]+p1[2]+p3[2])})
((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
 Sum[(-1/2)(*coefficient of (phi f^3)^2 vertex extracted*)
  CCDfϕf3⟦i,j,k,l⟧*(p3[i]-q[i])*(p1[j])*(p2[k])*
  (-p2[l]+q[l]-p1[l]-p3[l]),{i,2},{j,2},{k,2},{l,2}])
((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
 Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(-p1[i])*(-p3[j])*(q[k]-p2[k])*
  (p2[l]-q[l]+p1[l]+p3[l]),{i,2},{j,2},{k,2},{l,2}]))/.
{σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
F4F4ContributionOGBunkBedOpposite=
Together[(1/dc)((*slim fish*)-dc*(*factor due to the fact this is a cross term
  in second order Taylor expansion*)*8(*factor of two for switching f's
  around on each diagram and factor of two for switching diagrams around*)*
 (1/2)(*factor from second order Taylor expansion*)*Inverse[MF[p1,Ω1]]⟦2,2⟧
 (*propagator of f(-p1)f(p1)*)*((Inverse[MF[p2-q+p1+p3,Ω2-ω+Ω1+Ω3]]⟦1,2⟧
  (*propagator of ϕ(p2-q+p1+p3)f(-(p2-q+p1+p3))*))/.{(p2-q+p1+p3)[1]→
   (p2[1]-q[1]+p1[1]+p3[1]),(p2-q+p1+p3)[2]→(p2[2]-q[2]+p1[2]+p3[2])})
 ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
  Sum[(-1/2)(*coefficient of (phi f^3)^2 vertex extracted*)
   CCDfϕf3⟦i,j,k,l⟧*(p3[i]-q[i])*(p1[j])*(p2[k])*
   (-p2[l]+q[l]-p1[l]-p3[l]),{i,2},{j,2},{k,2},{l,2}])
 ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
  Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p2[i]-q[i]+p1[i]+p3[i])*
   (q[j]-p2[j])*(-p1[l])*(-p3[k]),{i,2},{j,2},{k,2},{l,2}]))/.
{σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];*)*)
```

In[∘]:=
```
(*Non Effective Times Effective Combinations*)
```

In[∘]:=
```
(*(ϕf^3)_NE (ϕf^3)_E*)
```

```
(*Diagrams in this entry that have been commented
 out have been ignored because they are lower order in d_c*)
```

```
(*F4ϕF3ContributionNEEWideFishNϕ=Together[
 (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
  second order Taylor expansion*)*4(*factor of two for switching f's around
  on each diagram *)*(1/2)(*factor from second order Taylor expansion*)*
 Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of phi(p1)f(-p1)*)*
 ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
  {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
 ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
  Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p3[i]-q[i])*(-p3[j])*
   (-p1[k])*(p1[l]+q[l]),{i,2},{j,2},{k,2},{l,2}])
 (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
```

```
                p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
            {q[1]→(p1[1]+p2[1]),q[2]→(p1[2]+p2[2]),ω→(Ω1+Ω2)})/.
          {ρ[1]→(-2p1[1]-q[1]),ρ[2]→(-2p1[2]-q[2]),P→(-2Ω1-ω),
           ξ[1]→(p1[1]-p2[1]),ξ[2]→(p1[2]-p2[2]),Z→(Ω1-Ω2)}))/.
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

F4ϕF3ContributionNEEWideFishWϕ1=Together[
  (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*2(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦2,2⟧(*propagator of f(-p1)f(p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]]⟦2,1⟧(*propagator of f(p1+q)ϕ(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(-p1[i]-q[i])*(p1[j])*
         (p2[k])*(-p2[l]+q[l]),{i,2},{j,2},{k,2},{l,2}])
      (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
                p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
            {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
          {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
           ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)}))/.
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

F4ϕF3ContributionNEEWideFishWϕ2=Together[
  (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*2(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of f(-p1)ϕ(p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p1[i])*(-p1[j]-q[j])*
         (p2[k])*(-p2[l]+q[l]),{i,2},{j,2},{k,2},{l,2}])
      (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
                p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
            {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
          {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
           ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)}))/.
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

F4ϕF3ContributionNEEBunkBedNEϕSame=Together[
  (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
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```
        second order Taylor expansion*)*4(*factor of two for switching f's
        around on each diagram and factor of two for switching diagrams around*)*
       (1/2)(*factor from second order Taylor expansion*)*
       Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of f(-p1)ϕ(p1)*)*
       ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
         {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
       ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
        Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p3[i]-q[i])*(-p1[j])*
          (-p3[k])*(p1[l]+q[l]),{i,2},{j,2},{k,2},{l,2}])
       (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
               p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
           {q[1]→(p1[1]+p2[1]),q[2]→(p1[2]+p2[2]),ω→(Ω1+Ω2)})/.
         {ρ[1]→(-2p1[1]-q[1]),ρ[2]→(-2p1[2]-q[2]),P→(-2Ω1-ω),
          ξ[1]→(p1[1]-p2[1]),ξ[2]→(p1[2]-p2[2]),Z→(Ω1-Ω2)}))/.
   {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

F4ϕF3ContributionNEEBunkBedNEϕOpposite=Together[
  (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
        second order Taylor expansion*)*4(*factor of two for switching f's
        around on each diagram and factor of two for switching diagrams around*)*
       (1/2)(*factor from second order Taylor expansion*)*
       Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of f(-p1)ϕ(p1)*)*
       ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
         {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
       ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
        Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p3[i]-q[i])*(p1[j]+q[j])*(-p3[k])*(-p1[l]),
         {i,2},{j,2},{k,2},{l,2}])
       (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
               p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
           {q[1]→(p1[1]+p2[1]),q[2]→(p1[2]+p2[2]),ω→(Ω1+Ω2)})/.
         {ρ[1]→(-2p1[1]-q[1]),ρ[2]→(-2p1[2]-q[2]),P→(-2Ω1-ω),
          ξ[1]→(p1[1]-p2[1]),ξ[2]→(p1[2]-p2[2]),Z→(Ω1-Ω2)}))/.
   {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];
*)
```

```
(*F4ϕF3ContributionNEEBunkBedEϕSame=Together[
  (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*4(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of f(-p1)ϕ(p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]]⟦2,2⟧(*propagator of f(p1+q)f(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(p1[i])*(p2[j])*(-p1[k]-q[k])*(-p2[l]+q[l]),
        {i,2},{j,2},{k,2},{l,2}])
      (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
             p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
          {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
        {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
         ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)}))/.
  {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

F4ϕF3ContributionNEEBunkBedEϕOpposite=Together[
  (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*4(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]]⟦2,2⟧(*propagator of f(-p1)f(p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]]⟦2,1⟧(*propagator of f(p1+q)ϕ(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3⟦i,j,k,l⟧*(-p1[i]-q[i])*(-p2[j]+q[j])*(p1[k])*(p2[l]),
        {i,2},{j,2},{k,2},{l,2}])
      (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
             p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
          {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
        {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
         ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)}))/.
  {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];*)
```

```
In[◦]:= (*(ϕf^3)_NE (ϕ^2f^2)_E*)
```

```
(*Diagrams in this entry that have been commented
 out have been ignored because they are lower order in d_c*)

(*F4ϕ2F2ContributionNEEWideFishNϕ=Together[
  (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*2(*factor of two for switching f's around
```

```
      on each diagram *)*(1/2)(*factor from second order Taylor expansion*)*
     Inverse[MF[p1,Ω1]]〚1,2〛(*propagator of phi(p1)f(-p1)*)*
     ((Inverse[MF[p1+q,Ω1+ω]]〚2,1〛(*propagator of f(p1+q)ϕ(-p1-q)*))/.
       {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
     ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
      Sum[(-1/2)CCDfϕf3〚i,j,k,l〛*(p3[i]-q[i])*(-p3[j])*(-p1[k])*(p1[l]+q[l]),
       {i,2},{j,2},{k,2},{l,2}])(((F4C/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),
            Ω1→(Ω1+P),p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
         {q[1]→(p1[1]+p2[1]),q[2]→(p1[2]+p2[2]),ω→(Ω1+Ω2)})/.
       {ρ[1]→(-2p1[1]-q[1]),ρ[2]→(-2p1[2]-q[2]),P→(-2Ω1-ω),
        ξ[1]→(p1[1]-p2[1]),ξ[2]→(p1[2]-p2[2]),Z→(Ω1-Ω2)}))/.
   {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

F4ϕ2F2ContributionNEEWideFishWϕ=
 Together[(1/dc)((*slim fish*)-dc*(2)(*factor due to the fact
      this is a cross term in second order Taylor expansion*)*2
     (*factor of two for switching f's and factor of two for switching
      ends around*)*(1/2)(*factor from second order Taylor expansion*)*
     Inverse[MF[p1,Ω1]]〚1,2〛(*propagator of f(-p1)ϕ(p1)*)*
     ((Inverse[MF[p1+q,Ω1+ω]]〚1,2〛(*propagator of ϕ(p1+q)f(-p1-q)*))/.
       {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
     ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
      Sum[(-1/2)CCDfϕf3〚i,j,k,l〛*(p1[i])*(-p1[j]-q[j])*(p2[k])*(-p2[l]+q[l]),
       {i,2},{j,2},{k,2},{l,2}])(((F4C/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),
            Ω1→(Ω1+P),p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
         {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
       {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
        ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)}))/.
   {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

F4ϕ2F2ContributionNEEBunkBedNEϕ=Together[
  (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
      second order Taylor expansion*)*4(*factor of two for switching f's
      around on each diagram and factor of two for switching diagrams around*)*
     (1/2)(*factor from second order Taylor expansion*)*
     Inverse[MF[p1,Ω1]]〚1,2〛(*propagator of f(-p1)ϕ(p1)*)*
     ((Inverse[MF[p1+q,Ω1+ω]]〚2,1〛(*propagator of f(p1+q)ϕ(-p1-q)*))/.
       {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
     ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
      Sum[(-1/2)CCDfϕf3〚i,j,k,l〛*(p3[i]-q[i])*(-p1[j])*(-p3[k])*(p1[l]+q[l]),
       {i,2},{j,2},{k,2},{l,2}])(((F4C/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),
            Ω1→(Ω1+P),p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
         {q[1]→(p1[1]+p2[1]),q[2]→(p1[2]+p2[2]),ω→(Ω1+Ω2)})/.
       {ρ[1]→(-2p1[1]-q[1]),ρ[2]→(-2p1[2]-q[2]),P→(-2Ω1-ω),
```

```
        ξ[1]→(p1[1]-p2[1]),ξ[2]→(p1[2]-p2[2]),Z→(Ω1-Ω2)}))/.
  {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

F4ϕ2F2ContributionNEEBunkBedEϕ=
 Together[(1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross
       term in second order Taylor expansion*)*4(*for which end of ϕ's around
       on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]][[1,2]](*propagator of f(-p1)ϕ(p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]][[1,2]](*propagator of ϕ(p1+q)f(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      ((*Coefficient of phi(-p2) f(p2-q) --- f(p1+q) f(-p1)) vertex*)
       Sum[(-1/2)CCDfϕf3[[i,j,k,l]]*(p1[i])*(p2[j])*(-p1[k]-q[k])*(-p2[l]+q[l]),
        {i,2},{j,2},{k,2},{l,2}])(((F4C/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),
             Ω1→(Ω1+P),p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
          {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
        {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
         ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)}))/.
  {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

*)
```

In[∘]:=

```
(*Effective^2 Combinations*)
```

```
(*Diagrams in this entry that have been commented
 out have been ignored because they are lower order in d_c*)

(*ϕF3ϕF3ContributionBunkBedSame=Together[
   (1/dc)((*slim fish*)-dc*(1)(*factor due to the fact this is a cross term in
       second order Taylor expansion*)*8(*factor of two for switching f's
       around on each diagram and factor of two for switching diagrams around*)*
      (1/2)(*factor from second order Taylor expansion*)*
      Inverse[MF[p1,Ω1]][[1,2]](*propagator of f(-p1)ϕ(p1)*)*
      ((Inverse[MF[p1+q,Ω1+ω]][[2,2]](*propagator of f(p1+q)f(-p1-q)*))/.
        {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
      (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
             p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
          {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
        {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
         ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)})
      (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
             p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
          {q[1]→(p1[1]+p2[1]),q[2]→(p1[2]+p2[2]),ω→(Ω1+Ω2)})/.
        {ρ[1]→(-2p1[1]-q[1]),ρ[2]→(-2p1[2]-q[2]),P→(-2Ω1-ω),
         ξ[1]→(p1[1]-p2[1]),ξ[2]→(p1[2]-p2[2]),Z→(Ω1-Ω2)}))/.
```

```
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

ϕF3ϕF3ContributionBunkBedOpposite=Together[
   (1/dc)((*slim fish*)-dc*(1)(*factor due to the fact this is a cross term in
         second order Taylor expansion*)*8(*factor of two for switching f's
         around on each diagram and factor of two for switching diagrams around*)*
        (1/2)(*factor from second order Taylor expansion*)*
        Inverse[MF[p1,Ω1]]⟦2,2⟧(*propagator of f(-p1)f(p1)*)*
        ((Inverse[MF[p1+q,Ω1+ω]]⟦2,1⟧(*propagator of f(p1+q)ϕ(-p1-q)*))/.
          {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
        (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
                p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
            {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
          {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
           ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)})
        (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
                p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
            {q[1]→(-p1[1]-q[1]+p2[1]),q[2]→(-p1[2]-q[2]+p2[2]),ω→(-Ω1-ω+Ω2)})/.
          {ρ[1]→(0),ρ[2]→(0),P→(0),ξ[1]→(-p1[1]-p2[1]-q[1]),
           ξ[2]→(-p1[2]-p2[2]-q[2]),Z→(-ω-Ω1-Ω2)}))/.
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

ϕF3Extϕ2F2ContributionBunkBed=Together[
   (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
         second order Taylor expansion*)*4(*factor of two for switching f's
         around on each diagram and factor of two for switching diagrams around*)*
        (1/2)(*factor from second order Taylor expansion*)*
        Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of f(-p1)ϕ(p1)*)*
        ((Inverse[MF[p1+q,Ω1+ω]]⟦2,1⟧(*propagator of f(p1+q)ϕ(-p1-q)*))/.
          {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
        (((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
                p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
            {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
          {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
           ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)})
        (((F4C/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
                p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
            {q[1]→(p1[1]+p2[1]),q[2]→(p1[2]+p2[2]),ω→(Ω1+Ω2)})/.
          {ρ[1]→(-2p1[1]-q[1]),ρ[2]→(-2p1[2]-q[2]),P→(-2Ω1-ω),
           ξ[1]→(p1[1]-p2[1]),ξ[2]→(p1[2]-p2[2]),Z→(Ω1-Ω2)}))/.
    {σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];

ϕF3ϕ2F2ExtContributionBunkBed=Together[
   (1/dc)((*slim fish*)-dc*(2)(*factor due to the fact this is a cross term in
```

```
second order Taylor expansion*)*4(*factor of two for switching f's
around on each diagram and factor of two for switching diagrams around*)*
(1/2)(*factor from second order Taylor expansion*)*
Inverse[MF[p1,Ω1]]⟦1,2⟧(*propagator of f(-p1)ϕ(p1)*)*
((Inverse[MF[p1+q,Ω1+ω]]⟦1,2⟧(*propagator of ϕ(p1+q)f(-p1-q)*))/.
  {(p1+q)[1]→(p1[1]+q[1]),(p1+q)[2]→(p1[2]+q[2])})
(((F4C/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
      p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
    {q[1]→(p3[1]-p1[1]-q[1]),q[2]→(p3[2]-p1[2]-q[2]),ω→(Ω3-Ω1-ω)})/.
  {ρ[1]→(q[1]),ρ[2]→(q[2]),P→(ω),ξ[1]→(p3[1]-p2[1]-q[1]),
   ξ[2]→(p3[2]-p2[2]-q[2]),Z→(Ω3-Ω2-ω)})
(((2*F4A/.{p1[1]→(p1[1]+ρ[1]),p1[2]→(p1[2]+ρ[2]),Ω1→(Ω1+P),
      p2[1]→(p2[1]+ξ[1]),p2[2]→(p2[2]+ξ[2]),Ω2→(Ω2+Z)})/.
    {q[1]→(p1[1]+p2[1]),q[2]→(p1[2]+p2[2]),ω→(Ω1+Ω2)})/.
  {ρ[1]→(-2p1[1]-q[1]),ρ[2]→(-2p1[2]-q[2]),P→(-2Ω1-ω),
   ξ[1]→(p1[1]-p2[1]),ξ[2]→(p1[2]-p2[2]),Z→(Ω1-Ω2)}))/.
{σ12→0,σ21→0,σ11→σ,σ22→σ,L1→L,L2→L}];*)
```

Extract contribution of $\lambda+2\mu$

In[ ]:=
```
(*F4F4ContributionOGSlimFishλμ=
   Simplify[Together[((1/4)D[(F4F4ContributionOGSlimFish/.{p1[1]→p1*Cos[θ],
         p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
         p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGWideFishNϕλμ=
   Simplify[Together[((1/4)D[(F4F4ContributionOGWideFishNϕ/.{p1[1]→p1*Cos[θ],
         p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
         p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGWideFishWϕ1λμ=
   Simplify[Together[((1/4)D[(F4F4ContributionOGWideFishWϕ1/.{p1[1]→p1*Cos[θ],
         p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
         p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];

F4F4ContributionOGWideFishWϕ2λμ=
   Simplify[Together[((1/4)D[(F4F4ContributionOGWideFishWϕ2/.{p1[1]→p1*Cos[θ],
         p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
         p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];

F4F4ContributionOGBunkBedSameλμ=
   Simplify[Together[((1/4)D[(F4F4ContributionOGBunkBedSame/.{p1[1]→p1*Cos[θ],
         p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
         p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGBunkBedOppositeλμ=Simplify[
    Together[((1/4)D[(F4F4ContributionOGBunkBedOpposite/.{p1[1]→p1*Cos[θ],
         p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
```

```
            p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];
F4ϕF3ContributionNEEWideFishNϕλμ=
    Simplify[Together[((1/4)D[(F4ϕF3ContributionNEEWideFishNϕ/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
            p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEWideFishWϕ1λμ=
    Simplify[Together[((1/4)D[(F4ϕF3ContributionNEEWideFishWϕ1/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
            p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEWideFishWϕ2λμ=
    Simplify[Together[((1/4)D[(F4ϕF3ContributionNEEWideFishWϕ2/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
            p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];
F4ϕF3ContributionNEEBunkBedNEϕSameλμ=Simplify[
     Together[((1/4)D[(F4ϕF3ContributionNEEBunkBedNEϕSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
            p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedNEϕOppositeλμ=Simplify[
     Together[((1/4)D[(F4ϕF3ContributionNEEBunkBedNEϕOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
            p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedEϕSameλμ=Simplify[
     Together[((1/4)D[(F4ϕF3ContributionNEEBunkBedEϕSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
            p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedEϕOppositeλμ=Simplify[
     Together[((1/4)D[(F4ϕF3ContributionNEEBunkBedEϕOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
            p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})]];
F4ϕ2F2ContributionNEEWideFishNϕλμ=
    Together[((1/4)D[(F4ϕ2F2ContributionNEEWideFishNϕ/.
          {p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,
           p2[1]→p2,p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEWideFishWϕλμ=
    Together[((1/4)D[(F4ϕ2F2ContributionNEEWideFishWϕ/.
          {p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,
           p2[1]→p2,p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEBunkBedNEϕλμ=
    Together[((1/4)D[(F4ϕ2F2ContributionNEEBunkBedNEϕ/.
```

```
            {p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,
             p2[1]→p2,p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEBunkBedEϕλμ=
    Together[((1/4)D[(F4ϕ2F2ContributionNEEBunkBedEϕ/.
            {p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,
             p2[1]→p2,p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})];
ϕF3ϕF3ContributionBunkBedSameλμ=
    Together[((1/4)D[(ϕF3ϕF3ContributionBunkBedSame/.{p1[1]→p1*Cos[θ],
             p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
             p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})];
ϕF3ϕF3ContributionBunkBedOppositeλμ=
    Together[((1/4)D[(ϕF3ϕF3ContributionBunkBedOpposite/.
            {p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,
             p2[1]→p2,p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})];
ϕF3Extϕ2F2ContributionBunkBedλμ=
    Together[((1/4)D[(ϕF3Extϕ2F2ContributionBunkBed/.{p1[1]→p1*Cos[θ],
             p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
             p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})];
ϕF3ϕ2F2ExtContributionBunkBedλμ=
    Together[((1/4)D[(ϕF3ϕ2F2ExtContributionBunkBed/.{p1[1]→p1*Cos[θ],
             p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,
             p2[2]→0,q[1]→q,q[2]→0}),{p3,2},{p2,2}]/.{p3→0,q→0,p2→0})];
NumF4F4ContributionOGSlimFishλμ=
    Simplify[Integrate[Numerator[F4F4ContributionOGSlimFishλμ],{θ,0,2π}]];
  NumF4F4ContributionOGWideFishNϕλμ=
    Simplify[Integrate[Numerator[F4F4ContributionOGWideFishNϕλμ],{θ,0,2π}]];
  NumF4F4ContributionOGWideFishWϕ1λμ=
    Simplify[Integrate[Numerator[F4F4ContributionOGWideFishWϕ1λμ],{θ,0,2π}]];
  NumF4F4ContributionOGWideFishWϕ2λμ=
    Simplify[Integrate[Numerator[F4F4ContributionOGWideFishWϕ2λμ],{θ,0,2π}]];
  NumF4F4ContributionOGBunkBedSameλμ=
    Simplify[Integrate[Numerator[F4F4ContributionOGBunkBedSameλμ],{θ,0,2π}]];
  NumF4F4ContributionOGBunkBedOppositeλμ=
    Simplify[Integrate[Numerator[F4F4ContributionOGBunkBedOppositeλμ],{θ,0,2π}]];
  NumF4ϕF3ContributionNEEWideFishNϕλμ=
    Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishNϕλμ],{θ,0,2π}]];
  NumF4ϕF3ContributionNEEWideFishWϕ1λμ=
    Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishWϕ1λμ],{θ,0,2π}]];
NumF4ϕF3ContributionNEEWideFishWϕ2λμ=
    Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishWϕ2λμ],{θ,0,2π}]];
NumF4ϕF3ContributionNEEBunkBedNEϕSameλμ=Simplify[
     Integrate[Numerator[F4ϕF3ContributionNEEBunkBedNEϕSameλμ],{θ,0,2π}]];
  NumF4ϕF3ContributionNEEBunkBedNEϕOppositeλμ=Simplify[
     Integrate[Numerator[F4ϕF3ContributionNEEBunkBedNEϕOppositeλμ],{θ,0,2π}]];
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NumF4ϕF3ContributionNEEBunkBedEϕSameλμ=
  Simplify[Integrate[Numerator[F4ϕF3ContributionNEEBunkBedEϕSameλμ],{θ,0,2π}]];
NumF4ϕF3ContributionNEEBunkBedEϕOppositeλμ=Simplify[
   Integrate[Numerator[F4ϕF3ContributionNEEBunkBedEϕOppositeλμ],{θ,0,2π}]];
NumF4ϕ2F2ContributionNEEWideFishNϕλμ=
  Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEWideFishNϕλμ],{θ,0,2π}]];
NumF4ϕ2F2ContributionNEEWideFishWϕλμ=
  Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEWideFishWϕλμ],{θ,0,2π}]];

NumF4ϕ2F2ContributionNEEBunkBedNEϕλμ=
  Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEBunkBedNEϕλμ],{θ,0,2π}]];
NumF4ϕ2F2ContributionNEEBunkBedEϕλμ=
  Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEBunkBedEϕλμ],{θ,0,2π}]];
NumϕF3ϕF3ContributionBunkBedSameλμ=
  Simplify[Integrate[Numerator[ϕF3ϕF3ContributionBunkBedSameλμ],{θ,0,2π}]];
NumϕF3ϕF3ContributionBunkBedOppositeλμ=
  Simplify[Integrate[Numerator[ϕF3ϕF3ContributionBunkBedOppositeλμ],{θ,0,2π}]];
NumϕF3Extϕ2F2ContributionBunkBedλμ=
  Integrate[Numerator[ϕF3Extϕ2F2ContributionBunkBedλμ],{θ,0,2π}];
NumϕF3ϕ2F2ExtContributionBunkBedλμ=
  Integrate[Numerator[ϕF3ϕ2F2ExtContributionBunkBedλμ],{θ,0,2π}];
FullF4F4ContributionOGSlimFishλμ=NumF4F4ContributionOGSlimFishλμ/
   Simplify[Denominator[F4F4ContributionOGSlimFishλμ]];
FullF4F4ContributionOGWideFishNϕλμ=NumF4F4ContributionOGWideFishNϕλμ/
   Simplify[Denominator[F4F4ContributionOGWideFishNϕλμ]];
FullF4F4ContributionOGWideFishWϕ1λμ=NumF4F4ContributionOGWideFishWϕ1λμ/
   Simplify[Denominator[F4F4ContributionOGWideFishWϕ1λμ]];
FullF4F4ContributionOGWideFishWϕ2λμ=NumF4F4ContributionOGWideFishWϕ2λμ/
   Simplify[Denominator[F4F4ContributionOGWideFishWϕ2λμ]];
FullF4F4ContributionOGBunkBedSameλμ=NumF4F4ContributionOGBunkBedSameλμ/
   Simplify[Denominator[F4F4ContributionOGBunkBedSameλμ]];
FullF4F4ContributionOGBunkBedOppositeλμ=NumF4F4ContributionOGBunkBedOppositeλμ/
   Simplify[Denominator[F4F4ContributionOGBunkBedOppositeλμ]];
FullF4ϕF3ContributionNEEWideFishNϕλμ=NumF4ϕF3ContributionNEEWideFishNϕλμ/
   Simplify[Denominator[F4ϕF3ContributionNEEWideFishNϕλμ]];
FullF4ϕF3ContributionNEEWideFishWϕ1λμ=NumF4ϕF3ContributionNEEWideFishWϕ1λμ/
   Simplify[Denominator[F4ϕF3ContributionNEEWideFishWϕ1λμ]];
FullF4ϕF3ContributionNEEWideFishWϕ2λμ=NumF4ϕF3ContributionNEEWideFishWϕ2λμ/
   Simplify[Denominator[F4ϕF3ContributionNEEWideFishWϕ2λμ]];
FullF4ϕF3ContributionNEEBunkBedNEϕSameλμ=
  NumF4ϕF3ContributionNEEBunkBedNEϕSameλμ/
   Simplify[Denominator[F4ϕF3ContributionNEEBunkBedNEϕSameλμ]];
FullF4ϕF3ContributionNEEBunkBedNEϕOppositeλμ=
  NumF4ϕF3ContributionNEEBunkBedNEϕOppositeλμ/
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    Simplify[Denominator[F4ϕF3ContributionNEEBunkBedNEϕOppositeλμ]];
FullF4ϕF3ContributionNEEBunkBedEϕSameλμ=NumF4ϕF3ContributionNEEBunkBedEϕSameλμ/
    Simplify[Denominator[F4ϕF3ContributionNEEBunkBedEϕSameλμ]];
FullF4ϕF3ContributionNEEBunkBedEϕOppositeλμ=
  NumF4ϕF3ContributionNEEBunkBedEϕOppositeλμ/
    Simplify[Denominator[F4ϕF3ContributionNEEBunkBedEϕOppositeλμ]];
FullF4ϕ2F2ContributionNEEWideFishNϕλμ=NumF4ϕ2F2ContributionNEEWideFishNϕλμ/
    Simplify[Denominator[F4ϕ2F2ContributionNEEWideFishNϕλμ]];
FullF4ϕ2F2ContributionNEEWideFishWϕλμ=NumF4ϕ2F2ContributionNEEWideFishWϕλμ/
    Simplify[Denominator[F4ϕ2F2ContributionNEEWideFishWϕλμ]];

FullF4ϕ2F2ContributionNEEBunkBedNEϕλμForIso=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕλμ/
    Simplify[Denominator[F4ϕ2F2ContributionNEEBunkBedNEϕλμ]];
FullF4ϕ2F2ContributionNEEBunkBedEϕλμForIso=NumF4ϕ2F2ContributionNEEBunkBedEϕλμ/
    Simplify[Denominator[F4ϕ2F2ContributionNEEBunkBedEϕλμ]];
FullϕF3ϕF3ContributionBunkBedSameλμForIso=NumϕF3ϕF3ContributionBunkBedSameλμ/
    Simplify[Denominator[ϕF3ϕF3ContributionBunkBedSameλμ]];
FullϕF3ϕF3ContributionBunkBedOppositeλμForIso=
  NumϕF3ϕF3ContributionBunkBedOppositeλμ/
    Simplify[Denominator[ϕF3ϕF3ContributionBunkBedOppositeλμ]];
FullϕF3Extϕ2F2ContributionBunkBedλμForIso=NumϕF3Extϕ2F2ContributionBunkBedλμ/
    Simplify[Denominator[ϕF3Extϕ2F2ContributionBunkBedλμ]];
FullϕF3ϕ2F2ExtContributionBunkBedλμForIso=NumϕF3ϕ2F2ExtContributionBunkBedλμ/
    Simplify[Denominator[ϕF3ϕ2F2ExtContributionBunkBedλμ]];


D11F4ϕ2F2ContributionNEEBunkBedNEϕλμ=Simplify[Distribute[(*dc*)
    I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
D22F4ϕ2F2ContributionNEEBunkBedNEϕλμ=Simplify[Distribute[(*dc*)
    (Df p1^2 (p1^2+σ)+ⅈ Ω1)*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^
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θ((*dc*)Ω1-(1/2 (i DDYuλ p1^2+3 i DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
    (Df p1^2 (p1^2+σ)-i Ω1) ((*dc*)Ω1+(1/2 i (DDYuλ p1^2+3 DDYuμ p1^2+
        i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
    ((*dc*)Ω1+(1/2 (i DDYuλ p1^2+3 i DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
D32F4ϕ2F2ContributionNEEBunkBedNEϕλμ=Simplify[
  Distribute[(*dc*) (Df p1^2 (p1^2+σ)+i Ω1) ((*dc*)Ω1-(1/2 i (DDYuλ p1^2+3 DDYuμ p1^2+
        i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))*
    (*dc*) ((*dc*)Ω1-(1/2 (i DDYuλ p1^2+3 i DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0*
    (Df p1^2 (p1^2+σ)-i Ω1) ((*dc*)Ω1+(1/2 i (DDYuλ p1^2+3 DDYuμ p1^2+
        i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
    ((*dc*)Ω1+(1/2 (i DDYuλ p1^2+3 i DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
Full11F4ϕ2F2ContributionNEEBunkBedNEϕλμ=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕλμ/D11F4ϕ2F2ContributionNEEBunkBedNEϕλμ;
 Full22F4ϕ2F2ContributionNEEBunkBedNEϕλμ=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕλμ/D22F4ϕ2F2ContributionNEEBunkBedNEϕλμ;
 Full32F4ϕ2F2ContributionNEEBunkBedNEϕλμ=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕλμ/D32F4ϕ2F2ContributionNEEBunkBedNEϕλμ;


 Full11F4ϕ2F2ContributionNEEBunkBedNEϕλμRes=
  2*π*I*Full11F4ϕ2F2ContributionNEEBunkBedNEϕλμ/.{Ω1→i Df p1^2 (p1^2+σ)};
 Full22F4ϕ2F2ContributionNEEBunkBedNEϕλμRes=
 2*π*I*Full22F4ϕ2F2ContributionNEEBunkBedNEϕλμ/.
 {Ω1→1/(2(*dc*)) i (DDYuλ p1^2+3 DDYuμ p1^2+
      i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full32F4ϕ2F2ContributionNEEBunkBedNEϕλμRes=
 2*π*I*Full32F4ϕ2F2ContributionNEEBunkBedNEϕλμ/.
 {Ω1→1/(2(*dc*)) (i DDYuλ p1^2+3 i DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullF4ϕ2F2ContributionNEEBunkBedNEϕλμRes=
 Full11F4ϕ2F2ContributionNEEBunkBedNEϕλμRes+
  Full22F4ϕ2F2ContributionNEEBunkBedNEϕλμRes+
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Full32F4ϕ2F2ContributionNEEBunkBedNEϕλμRes;

D12F4ϕ2F2ContributionNEEBunkBedEϕλμ=
 Simplify[(*dc*) (-I*Df p1^2 (p1^2+σ)+ Ω1)^0((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D21F4ϕ2F2ContributionNEEBunkBedEϕλμ=Simplify[(*dc*)
  (I*Df p1^2 (p1^2+σ)+ Ω1)^0(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D31F4ϕ2F2ContributionNEEBunkBedEϕλμ=
 Simplify[(*dc*) (I*Df p1^2 (p1^2+σ)+ Ω1)^0(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+
      3 DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
Full12F4ϕ2F2ContributionNEEBunkBedEϕλμ=
 NumF4ϕ2F2ContributionNEEBunkBedEϕλμ/D12F4ϕ2F2ContributionNEEBunkBedEϕλμ;
Full21F4ϕ2F2ContributionNEEBunkBedEϕλμ=
 NumF4ϕ2F2ContributionNEEBunkBedEϕλμ/D21F4ϕ2F2ContributionNEEBunkBedEϕλμ;
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Full31F4ϕ2F2ContributionNEEBunkBedEϕλμ=
 NumF4ϕ2F2ContributionNEEBunkBedEϕλμ/D31F4ϕ2F2ContributionNEEBunkBedEϕλμ;

Full12F4ϕ2F2ContributionNEEBunkBedEϕλμRes=
 0*2*π*I*D[Full12F4ϕ2F2ContributionNEEBunkBedEϕλμ,{Ω1,1}]/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full21F4ϕ2F2ContributionNEEBunkBedEϕλμRes=
 2*π*I*Full21F4ϕ2F2ContributionNEEBunkBedEϕλμ/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full31F4ϕ2F2ContributionNEEBunkBedEϕλμRes=
 2*π*I*Full31F4ϕ2F2ContributionNEEBunkBedEϕλμ/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullF4ϕ2F2ContributionNEEBunkBedEϕλμRes=
 (*Full12F4ϕ2F2ContributionNEEBunkBedEϕλμRes++*)
 Full21F4ϕ2F2ContributionNEEBunkBedEϕλμRes+
  Full31F4ϕ2F2ContributionNEEBunkBedEϕλμRes;


D12ϕF3ϕF3ContributionBunkBedSameλμ=
 Simplify[(*dc^2*)1*I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+
         3 DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D21ϕF3ϕF3ContributionBunkBedSameλμ=Simplify[(*dc^2*)
  1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
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            √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))];
D31ϕF3ϕF3ContributionBunkBedSameλμ=
 Simplify[(*dc^2*)1*(Df p1²  (p1²+σ)+ⅈ Ω1)*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))))*
   (*dc*)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
            √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))))^0
   (Df p1²  (p1²+σ)-ⅈ Ω1)^2  ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
         √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))))];
Full12ϕF3ϕF3ContributionBunkBedSameλμ=
 NumϕF3ϕF3ContributionBunkBedSameλμ/D12ϕF3ϕF3ContributionBunkBedSameλμ;
Full21ϕF3ϕF3ContributionBunkBedSameλμ=
 NumϕF3ϕF3ContributionBunkBedSameλμ/D21ϕF3ϕF3ContributionBunkBedSameλμ;
Full31ϕF3ϕF3ContributionBunkBedSameλμ=
 NumϕF3ϕF3ContributionBunkBedSameλμ/D31ϕF3ϕF3ContributionBunkBedSameλμ;


Full12ϕF3ϕF3ContributionBunkBedSameλμRes=
 2*π*I*Full12ϕF3ϕF3ContributionBunkBedSameλμ/.{Ω1→ⅈ Df p1²  (p1²+σ)};
Full21ϕF3ϕF3ContributionBunkBedSameλμRes=
 2*π*I*Full21ϕF3ϕF3ContributionBunkBedSameλμ/.{Ω1→1/(2(*dc*))  ⅈ (DDYuλ p1²+3 DDYuμ p1²+
       ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))};
Full31ϕF3ϕF3ContributionBunkBedSameλμRes=
 2*π*I*Full31ϕF3ϕF3ContributionBunkBedSameλμ/.
  {Ω1→1/(2(*dc*))  (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
       √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))};
FullϕF3ϕF3ContributionBunkBedSameλμRes=Full12ϕF3ϕF3ContributionBunkBedSameλμRes+
  Full21ϕF3ϕF3ContributionBunkBedSameλμRes+
  Full31ϕF3ϕF3ContributionBunkBedSameλμRes;

D12ϕF3ϕF3ContributionBunkBedOppositeλμ=Simplify[
  -I*1(*dc^2*)*(I^2)(Df p1²  (p1²+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))))^2
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
            √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))))^2
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(Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0];
D22ϕF3ϕF3ContributionBunkBedOppositeλμ=
 Simplify[-I*(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^2
  (**(dc^2)*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
  (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0];
D32ϕF3ϕF3ContributionBunkBedOppositeλμ=Simplify[
 -I*(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^2*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2*
  (*(dc^2)*)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
  (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0];
Full12ϕF3ϕF3ContributionBunkBedOppositeλμ=
 NumϕF3ϕF3ContributionBunkBedOppositeλμ/D12ϕF3ϕF3ContributionBunkBedOppositeλμ;
Full22ϕF3ϕF3ContributionBunkBedOppositeλμ=
 NumϕF3ϕF3ContributionBunkBedOppositeλμ/D22ϕF3ϕF3ContributionBunkBedOppositeλμ;
Full32ϕF3ϕF3ContributionBunkBedOppositeλμ=
 NumϕF3ϕF3ContributionBunkBedOppositeλμ/D32ϕF3ϕF3ContributionBunkBedOppositeλμ;


Full12ϕF3ϕF3ContributionBunkBedOppositeλμRes=
 2*π*I*D[Full12ϕF3ϕF3ContributionBunkBedOppositeλμ,{Ω1,1}]/.
  {Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22ϕF3ϕF3ContributionBunkBedOppositeλμRes=
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 2*π*I*D[Full22ϕF3ϕF3ContributionBunkBedOppositeλμ,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
Full32ϕF3ϕF3ContributionBunkBedOppositeλμRes=
 2*π*I*D[Full32ϕF3ϕF3ContributionBunkBedOppositeλμ,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
FullϕF3ϕF3ContributionBunkBedOppositeλμRes=
 Full12ϕF3ϕF3ContributionBunkBedOppositeλμRes+
  Full22ϕF3ϕF3ContributionBunkBedOppositeλμRes+
  Full32ϕF3ϕF3ContributionBunkBedOppositeλμRes;


D11ϕF3Extϕ2F2ContributionBunkBedλμ=
 Simplify[(*dc*)I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D22ϕF3Extϕ2F2ContributionBunkBedλμ=Simplify[(*dc*)
  (Df p1^2 (p1^2+σ)+ⅈ Ω1)(**(dc^2)*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D32ϕF3Extϕ2F2ContributionBunkBedλμ=Simplify[(*dc*)
  (Df p1^2 (p1^2+σ)+ⅈ Ω1)(**(dc^2)*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
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((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
  (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
Full11ϕF3Extϕ2F2ContributionBunkBedλμ=
 NumϕF3Extϕ2F2ContributionBunkBedλμ/D11ϕF3Extϕ2F2ContributionBunkBedλμ;
Full22ϕF3Extϕ2F2ContributionBunkBedλμ=
 NumϕF3Extϕ2F2ContributionBunkBedλμ/D22ϕF3Extϕ2F2ContributionBunkBedλμ;
Full32ϕF3Extϕ2F2ContributionBunkBedλμ=
 NumϕF3Extϕ2F2ContributionBunkBedλμ/D32ϕF3Extϕ2F2ContributionBunkBedλμ;


Full11ϕF3Extϕ2F2ContributionBunkBedλμRes=
 2*π*I*Full11ϕF3Extϕ2F2ContributionBunkBedλμ/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22ϕF3Extϕ2F2ContributionBunkBedλμRes=
 2*π*I*D[Full22ϕF3Extϕ2F2ContributionBunkBedλμ,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full32ϕF3Extϕ2F2ContributionBunkBedλμRes=
 2*π*I*D[Full32ϕF3Extϕ2F2ContributionBunkBedλμ,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullϕF3Extϕ2F2ContributionBunkBedλμRes=Full11ϕF3Extϕ2F2ContributionBunkBedλμRes+
  Full22ϕF3Extϕ2F2ContributionBunkBedλμRes+
  Full32ϕF3Extϕ2F2ContributionBunkBedλμRes;


D12ϕF3ϕ2F2ExtContributionBunkBedλμ=
 Simplify[-(*dc*) (Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
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        √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))^2];
D21ϕF3ϕ2F2ExtContributionBunkBedλμ=Simplify[
  -(*dc*)(Df p1²  (p1²+σ)+ⅈ Ω1)^0*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
        √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))
   (Df p1²  (p1²+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))^2
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
        √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))^2];
D31ϕF3ϕ2F2ExtContributionBunkBedλμ=
 Simplify[-(*dc*)(Df p1²  (p1²+σ)+ⅈ Ω1)^0*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+
        3 DDYuμ p1²+ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
        √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))^0
   (Df p1²  (p1²+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))^2
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
        √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²)))^2];
Full12ϕF3ϕ2F2ExtContributionBunkBedλμ=
 NumϕF3ϕ2F2ExtContributionBunkBedλμ/D12ϕF3ϕ2F2ExtContributionBunkBedλμ;
Full21ϕF3ϕ2F2ExtContributionBunkBedλμ=
 NumϕF3ϕ2F2ExtContributionBunkBedλμ/D21ϕF3ϕ2F2ExtContributionBunkBedλμ;
Full31ϕF3ϕ2F2ExtContributionBunkBedλμ=
 NumϕF3ϕ2F2ExtContributionBunkBedλμ/D31ϕF3ϕ2F2ExtContributionBunkBedλμ;


Full12ϕF3ϕ2F2ExtContributionBunkBedλμRes=
 0*2*π*I*D[Full12ϕF3ϕ2F2ExtContributionBunkBedλμ,Ω1]/.{Ω1→ⅈ Df p1²  (p1²+σ)};
Full21ϕF3ϕ2F2ExtContributionBunkBedλμRes=
 2*π*I*Full21ϕF3ϕ2F2ExtContributionBunkBedλμ/.{Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1²+3 DDYuμ p1²+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²  p1²))};
Full31ϕF3ϕ2F2ExtContributionBunkBedλμRes=
 2*π*I*Full31ϕF3ϕ2F2ExtContributionBunkBedλμ/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
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√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)}};

FullϕF3ϕ2F2ExtContributionBunkBedλμRes=Full12ϕF3ϕ2F2ExtContributionBunkBedλμRes+
  Full21ϕF3ϕ2F2ExtContributionBunkBedλμRes+
  Full31ϕF3ϕ2F2ExtContributionBunkBedλμRes;
ResF4F4ContributionOGSlimFishλμ=
 2*π*I*(Residue[FullF4F4ContributionOGSlimFishλμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]);
ResF4F4ContributionOGWideFishNϕλμ=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishNϕλμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]);
ResF4F4ContributionOGWideFishWϕ1λμ=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishWϕ1λμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]);
ResF4F4ContributionOGWideFishWϕ2λμ=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishWϕ2λμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]);
ResF4F4ContributionOGBunkBedSameλμ=
 2*π*I*(Residue[FullF4F4ContributionOGBunkBedSameλμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]);
ResF4F4ContributionOGBunkBedOppositeλμ=2*π*I*
  (Residue[FullF4F4ContributionOGBunkBedOppositeλμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]);

ResF4ϕF3ContributionNEEWideFishNϕλμ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishNϕλμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishNϕλμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1²+3 DDYuμ
         p1²+ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishNϕλμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1²+3 ⅈ
         DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)}]);

ResF4ϕF3ContributionNEEWideFishWϕ1λμ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishWϕ1λμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ1λμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1²+3 DDYuμ
         p1²+ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ1λμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1²+3 ⅈ
         DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)}]);
ResF4ϕF3ContributionNEEWideFishWϕ2λμ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishWϕ2λμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ2λμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1²+3 DDYuμ
         p1²+ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ2λμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1²+3 ⅈ
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DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕF3ContributionNEEBunkBedNEϕSameλμ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameλμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
   Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameλμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
       DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
   Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameλμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
       DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕF3ContributionNEEBunkBedNEϕOppositeλμ=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeλμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
   Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeλμ,
    {Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
   Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeλμ,
    {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕF3ContributionNEEBunkBedEϕSameλμ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameλμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
   Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameλμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
       DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
   Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameλμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
       DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕF3ContributionNEEBunkBedEϕOppositeλμ=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeλμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
   Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeλμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
       DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
   Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeλμ,
    {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕ2F2ContributionNEEWideFishNϕλμ=
 2*π*I*(Residue[FullF4ϕ2F2ContributionNEEWideFishNϕλμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
```

```
    Residue[FullF4ϕ2F2ContributionNEEWideFishNϕλμ,{Ω1,1/(2(*dc*)) i (DDYuλ p1^2+3 DDYuμ
        p1^2+i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishNϕλμ,{Ω1,1/(2(*dc*)) (i DDYuλ p1^2+3 i
        DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕ2F2ContributionNEEWideFishWϕλμ=
 2*π*I*(Residue[FullF4ϕ2F2ContributionNEEWideFishWϕλμ,{Ω1,i Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishWϕλμ,{Ω1,1/(2(*dc*)) i (DDYuλ p1^2+3 DDYuμ
        p1^2+i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishWϕλμ,{Ω1,1/(2(*dc*)) (i DDYuλ p1^2+3 i
        DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ContDfλμϕf3=
 2*(1/(2π)^3)(ResF4F4ContributionOGSlimFishλμ+ResF4F4ContributionOGWideFishNϕλμ+
    ResF4F4ContributionOGWideFishWϕ1λμ+ResF4F4ContributionOGWideFishWϕ2λμ+
    ResF4F4ContributionOGBunkBedSameλμ+ResF4F4ContributionOGBunkBedOppositeλμ+
    (ResF4ϕF3ContributionNEEWideFishNϕλμ+ResF4ϕF3ContributionNEEWideFishWϕ1λμ+
      ResF4ϕF3ContributionNEEWideFishWϕ2λμ+
      ResF4ϕF3ContributionNEEBunkBedNEϕSameλμ+
      ResF4ϕF3ContributionNEEBunkBedNEϕOppositeλμ+
      ResF4ϕF3ContributionNEEBunkBedEϕSameλμ+
      ResF4ϕF3ContributionNEEBunkBedEϕOppositeλμ+
      ResF4ϕ2F2ContributionNEEWideFishNϕλμ+ResF4ϕ2F2ContributionNEEWideFishWϕλμ+
      (FullF4ϕ2F2ContributionNEEBunkBedEϕλμRes+
        FullF4ϕ2F2ContributionNEEBunkBedNEϕλμRes+
        (FullϕF3ϕ2F2ExtContributionBunkBedλμRes+
          FullϕF3Extϕ2F2ContributionBunkBedλμRes+
          FullϕF3ϕF3ContributionBunkBedOppositeλμRes+
          FullϕF3ϕF3ContributionBunkBedSameλμRes))));*)
```

In[ ]:=
```
F4F4ContributionOGSlimFishλμ = Simplify[Together[
    ((1 / 4) D[(F4F4ContributionOGSlimFish /. {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ],
          Ω3 → 0, Ω2 → 0, ω → 0, p3[1] → p3, p3[2] → 0, p2[1] → p2, p2[2] → 0,
          q[1] → q, q[2] → 0}), {p3, 2}, {p2, 2}] /. {p3 → 0, q → 0, p2 → 0})]];
NumF4F4ContributionOGSlimFishλμ =
  Simplify[Integrate[Numerator[F4F4ContributionOGSlimFishλμ], {θ, 0, 2 π}]];
FullF4F4ContributionOGSlimFishλμ = NumF4F4ContributionOGSlimFishλμ /
   Simplify[Denominator[F4F4ContributionOGSlimFishλμ]];
ResF4F4ContributionOGSlimFishλμ =
  2 * π * I * (Residue[FullF4F4ContributionOGSlimFishλμ, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
ContDfλμϕf3 = 2 * (1 / (2 π) ^ 3) (ResF4F4ContributionOGSlimFishλμ);
```

Extract contribution of μ

In[ ]:=
```
(*F4F4ContributionOGSlimFishμ=
    Simplify[Together[((1/2)D[(F4F4ContributionOGSlimFish/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGWideFishNϕμ=
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishNϕ/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGWideFishWϕ1μ=
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishWϕ1/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];

F4F4ContributionOGWideFishWϕ2μ=
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishWϕ2/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];

F4F4ContributionOGBunkBedSameμ=
    Simplify[Together[((1/2)D[(F4F4ContributionOGBunkBedSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGBunkBedOppositeμ=Simplify[
     Together[((1/2)D[(F4F4ContributionOGBunkBedOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];
F4ϕF3ContributionNEEWideFishNϕμ=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishNϕ/.{p1[1]→p1*Cos[θ],
```

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            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEWideFishWϕ1μ=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishWϕ1/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEWideFishWϕ2μ=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishWϕ2/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];
F4ϕF3ContributionNEEBunkBedNEϕSameμ=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedNEϕSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedNEϕOppositeμ=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedNEϕOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedEϕSameμ=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedEϕSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedEϕOppositeμ=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedEϕOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})]];
F4ϕ2F2ContributionNEEWideFishNϕμ=Together[
     ((1/2)D[(F4ϕ2F2ContributionNEEWideFishNϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
            Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
         {p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEWideFishWϕμ=Together[
     ((1/2)D[(F4ϕ2F2ContributionNEEWideFishWϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
            Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
         {p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEBunkBedNEϕμ=Together[
     ((1/2)D[(F4ϕ2F2ContributionNEEBunkBedNEϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
            Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
         {p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEBunkBedEϕμ=Together[
```

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((1/2)D[(F4ϕ2F2ContributionNEEBunkBedEϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
        Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})];
ϕF3ϕF3ContributionBunkBedSameμ=Together[
    ((1/2)D[(ϕF3ϕF3ContributionBunkBedSame/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
        Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})];
ϕF3ϕF3ContributionBunkBedOppositeμ=
   Together[((1/2)D[(ϕF3ϕF3ContributionBunkBedOpposite/.{p1[1]→p1*Cos[θ],
        p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
        q[1]→0,q[2]→q}),{p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})];
ϕF3Extϕ2F2ContributionBunkBedμ=Together[
    ((1/2)D[(ϕF3Extϕ2F2ContributionBunkBed/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
        Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})];
ϕF3ϕ2F2ExtContributionBunkBedμ=Together[
    ((1/2)D[(ϕF3ϕ2F2ExtContributionBunkBed/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
        Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,1},{p2,1},{q,2}]/.{p3→0,q→0,p2→0})];
NumF4F4ContributionOGSlimFishμ=
   Simplify[Integrate[Numerator[F4F4ContributionOGSlimFishμ],{θ,0,2π}]];
 NumF4F4ContributionOGWideFishNϕμ=
   Simplify[Integrate[Numerator[F4F4ContributionOGWideFishNϕμ],{θ,0,2π}]];
 NumF4F4ContributionOGWideFishWϕ1μ=
   Simplify[Integrate[Numerator[F4F4ContributionOGWideFishWϕ1μ],{θ,0,2π}]];
 NumF4F4ContributionOGWideFishWϕ2μ=
   Simplify[Integrate[Numerator[F4F4ContributionOGWideFishWϕ2μ],{θ,0,2π}]];
 NumF4F4ContributionOGBunkBedSameμ=
   Simplify[Integrate[Numerator[F4F4ContributionOGBunkBedSameμ],{θ,0,2π}]];
 NumF4F4ContributionOGBunkBedOppositeμ=
   Simplify[Integrate[Numerator[F4F4ContributionOGBunkBedOppositeμ],{θ,0,2π}]];
 NumF4ϕF3ContributionNEEWideFishNϕμ=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishNϕμ],{θ,0,2π}]];
 NumF4ϕF3ContributionNEEWideFishWϕ1μ=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishWϕ1μ],{θ,0,2π}]];
NumF4ϕF3ContributionNEEWideFishWϕ2μ=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishWϕ2μ],{θ,0,2π}]];
NumF4ϕF3ContributionNEEBunkBedNEϕSameμ=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEBunkBedNEϕSameμ],{θ,0,2π}]];
 NumF4ϕF3ContributionNEEBunkBedNEϕOppositeμ=Simplify[
    Integrate[Numerator[F4ϕF3ContributionNEEBunkBedNEϕOppositeμ],{θ,0,2π}]];
 NumF4ϕF3ContributionNEEBunkBedEϕSameμ=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEBunkBedEϕSameμ],{θ,0,2π}]];
 NumF4ϕF3ContributionNEEBunkBedEϕOppositeμ=Simplify[
```

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      Integrate[Numerator[F4ϕF3ContributionNEEBunkBedEϕOppositeμ],{θ,0,2π}]];
   NumF4ϕ2F2ContributionNEEWideFishNϕμ=
     Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEWideFishNϕμ],{θ,0,2π}]];
   NumF4ϕ2F2ContributionNEEWideFishWϕμ=
     Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEWideFishWϕμ],{θ,0,2π}]];

   NumF4ϕ2F2ContributionNEEBunkBedNEϕμ=
     Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEBunkBedNEϕμ],{θ,0,2π}]];
 NumF4ϕ2F2ContributionNEEBunkBedEϕμ=
     Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEBunkBedEϕμ],{θ,0,2π}]];
 NumϕF3ϕF3ContributionBunkBedSameμ=
     Simplify[Integrate[Numerator[ϕF3ϕF3ContributionBunkBedSameμ],{θ,0,2π}]];
 NumϕF3ϕF3ContributionBunkBedOppositeμ=
     Simplify[Integrate[Numerator[ϕF3ϕF3ContributionBunkBedOppositeμ],{θ,0,2π}]];
 NumϕF3Extϕ2F2ContributionBunkBedμ=
     Integrate[Numerator[ϕF3Extϕ2F2ContributionBunkBedμ],{θ,0,2π}];
 NumϕF3ϕ2F2ExtContributionBunkBedμ=
     Integrate[Numerator[ϕF3ϕ2F2ExtContributionBunkBedμ],{θ,0,2π}];
 FullF4F4ContributionOGSlimFishμ=NumF4F4ContributionOGSlimFishμ/
      Simplify[Denominator[F4F4ContributionOGSlimFishμ]];
   FullF4F4ContributionOGWideFishNϕμ=NumF4F4ContributionOGWideFishNϕμ/
      Simplify[Denominator[F4F4ContributionOGWideFishNϕμ]];
   FullF4F4ContributionOGWideFishWϕ1μ=NumF4F4ContributionOGWideFishWϕ1μ/
      Simplify[Denominator[F4F4ContributionOGWideFishWϕ1μ]];
   FullF4F4ContributionOGWideFishWϕ2μ=NumF4F4ContributionOGWideFishWϕ2μ/
      Simplify[Denominator[F4F4ContributionOGWideFishWϕ2μ]];
   FullF4F4ContributionOGBunkBedSameμ=NumF4F4ContributionOGBunkBedSameμ/
      Simplify[Denominator[F4F4ContributionOGBunkBedSameμ]];
   FullF4F4ContributionOGBunkBedOppositeμ=NumF4F4ContributionOGBunkBedOppositeμ/
      Simplify[Denominator[F4F4ContributionOGBunkBedOppositeμ]];
   FullF4ϕF3ContributionNEEWideFishNϕμ=NumF4ϕF3ContributionNEEWideFishNϕμ/
      Simplify[Denominator[F4ϕF3ContributionNEEWideFishNϕμ]];
   FullF4ϕF3ContributionNEEWideFishWϕ1μ=NumF4ϕF3ContributionNEEWideFishWϕ1μ/
      Simplify[Denominator[F4ϕF3ContributionNEEWideFishWϕ1μ]];
   FullF4ϕF3ContributionNEEWideFishWϕ2μ=NumF4ϕF3ContributionNEEWideFishWϕ2μ/
      Simplify[Denominator[F4ϕF3ContributionNEEWideFishWϕ2μ]];
   FullF4ϕF3ContributionNEEBunkBedNEϕSameμ=NumF4ϕF3ContributionNEEBunkBedNEϕSameμ/
      Simplify[Denominator[F4ϕF3ContributionNEEBunkBedNEϕSameμ]];
   FullF4ϕF3ContributionNEEBunkBedNEϕOppositeμ=
     NumF4ϕF3ContributionNEEBunkBedNEϕOppositeμ/
      Simplify[Denominator[F4ϕF3ContributionNEEBunkBedNEϕOppositeμ]];
   FullF4ϕF3ContributionNEEBunkBedEϕSameμ=NumF4ϕF3ContributionNEEBunkBedEϕSameμ/
      Simplify[Denominator[F4ϕF3ContributionNEEBunkBedEϕSameμ]];
   FullF4ϕF3ContributionNEEBunkBedEϕOppositeμ=
```

```
  NumF4ϕF3ContributionNEEBunkBedEϕOppositeμ/
   Simplify[Denominator[F4ϕF3ContributionNEEBunkBedEϕOppositeμ]];
FullF4ϕ2F2ContributionNEEWideFishNϕμ=NumF4ϕ2F2ContributionNEEWideFishNϕμ/
   Simplify[Denominator[F4ϕ2F2ContributionNEEWideFishNϕμ]];
FullF4ϕ2F2ContributionNEEWideFishWϕμ=NumF4ϕ2F2ContributionNEEWideFishWϕμ/
   Simplify[Denominator[F4ϕ2F2ContributionNEEWideFishWϕμ]];

FullF4ϕ2F2ContributionNEEBunkBedNEϕμForIso=NumF4ϕ2F2ContributionNEEBunkBedNEϕμ/
   Simplify[Denominator[F4ϕ2F2ContributionNEEBunkBedNEϕμ]];
FullF4ϕ2F2ContributionNEEBunkBedEϕμForIso=NumF4ϕ2F2ContributionNEEBunkBedEϕμ/
   Simplify[Denominator[F4ϕ2F2ContributionNEEBunkBedEϕμ]];
FullϕF3ϕF3ContributionBunkBedSameμForIso=NumϕF3ϕF3ContributionBunkBedSameμ/
   Simplify[Denominator[ϕF3ϕF3ContributionBunkBedSameμ]];
FullϕF3ϕF3ContributionBunkBedOppositeμForIso=
  NumϕF3ϕF3ContributionBunkBedOppositeμ/
   Simplify[Denominator[ϕF3ϕF3ContributionBunkBedOppositeμ]];
FullϕF3Extϕ2F2ContributionBunkBedμForIso=NumϕF3Extϕ2F2ContributionBunkBedμ/
   Simplify[Denominator[ϕF3Extϕ2F2ContributionBunkBedμ]];
FullϕF3ϕ2F2ExtContributionBunkBedμForIso=NumϕF3ϕ2F2ExtContributionBunkBedμ/
   Simplify[Denominator[ϕF3ϕ2F2ExtContributionBunkBedμ]];


D11F4ϕ2F2ContributionNEEBunkBedNEϕμ=Simplify[Distribute[(*dc*)
    I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
D22F4ϕ2F2ContributionNEEBunkBedNEϕμ=
  Simplify[Distribute[(*dc*)(Df p1^2 (p1^2+σ)+ⅈ Ω1)*(*dc*)((*dc*)Ω1-
       (1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ
               DDYuμ-DDYuμ^2) p1^2)))^0((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
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((*dc*)Ω1+(1/2 (i DDYuλ p1^2+3 i DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
D32F4ϕ2F2ContributionNEEBunkBedNEϕμ=Simplify[
  Distribute[(*dc*)(Df p1^2 (p1^2+σ)+i Ω1)((*dc*)Ω1-(1/2 i (DDYuλ p1^2+3 DDYuμ p1^2+
      i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))*
    (*dc*)((*dc*)Ω1-(1/2 (i DDYuλ p1^2+3 i DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0*
    (Df p1^2 (p1^2+σ)-i Ω1) ((*dc*)Ω1+(1/2 i (DDYuλ p1^2+3 DDYuμ p1^2+
      i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
    ((*dc*)Ω1+(1/2 (i DDYuλ p1^2+3 i DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
Full11F4ϕ2F2ContributionNEEBunkBedNEϕμ=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕμ/D11F4ϕ2F2ContributionNEEBunkBedNEϕμ;
Full22F4ϕ2F2ContributionNEEBunkBedNEϕμ=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕμ/D22F4ϕ2F2ContributionNEEBunkBedNEϕμ;
Full32F4ϕ2F2ContributionNEEBunkBedNEϕμ=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕμ/D32F4ϕ2F2ContributionNEEBunkBedNEϕμ;


Full11F4ϕ2F2ContributionNEEBunkBedNEϕμRes=
  2*π*I*Full11F4ϕ2F2ContributionNEEBunkBedNEϕμ/.{Ω1→i Df p1^2 (p1^2+σ)};
Full22F4ϕ2F2ContributionNEEBunkBedNEϕμRes=
 2*π*I*Full22F4ϕ2F2ContributionNEEBunkBedNEϕμ/.
  {Ω1→1/(2(*dc*)) i (DDYuλ p1^2+3 DDYuμ p1^2+
      i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full32F4ϕ2F2ContributionNEEBunkBedNEϕμRes=
 2*π*I*Full32F4ϕ2F2ContributionNEEBunkBedNEϕμ/.
  {Ω1→1/(2(*dc*)) (i DDYuλ p1^2+3 i DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullF4ϕ2F2ContributionNEEBunkBedNEϕμRes=Full11F4ϕ2F2ContributionNEEBunkBedNEϕμRes+
  Full22F4ϕ2F2ContributionNEEBunkBedNEϕμRes+
  Full32F4ϕ2F2ContributionNEEBunkBedNEϕμRes;


D12F4ϕ2F2ContributionNEEBunkBedEϕμ=
 Simplify[(*dc*)(-I*Df p1^2 (p1^2+σ)+ Ω1)^0((*dc*)Ω1-(1/2 i (DDYuλ p1^2+3 DDYuμ p1^2+
      i Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
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((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
(I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D21F4ϕ2F2ContributionNEEBunkBedEϕμ=Simplify[(*dc*)
  (I*Df p1^2 (p1^2+σ)+ Ω1)^0(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
(I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D31F4ϕ2F2ContributionNEEBunkBedEϕμ=
 Simplify[(*dc*)(I*Df p1^2 (p1^2+σ)+ Ω1)^0(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+
      3 DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
(I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
Full12F4ϕ2F2ContributionNEEBunkBedEϕμ=
 NumF4ϕ2F2ContributionNEEBunkBedEϕμ/D12F4ϕ2F2ContributionNEEBunkBedEϕμ;
Full21F4ϕ2F2ContributionNEEBunkBedEϕμ=
 NumF4ϕ2F2ContributionNEEBunkBedEϕμ/D21F4ϕ2F2ContributionNEEBunkBedEϕμ;
Full31F4ϕ2F2ContributionNEEBunkBedEϕμ=
 NumF4ϕ2F2ContributionNEEBunkBedEϕμ/D31F4ϕ2F2ContributionNEEBunkBedEϕμ;

Full12F4ϕ2F2ContributionNEEBunkBedEϕμRes=
 0*2*π*I*D[Full12F4ϕ2F2ContributionNEEBunkBedEϕμ,{Ω1,1}]/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full21F4ϕ2F2ContributionNEEBunkBedEϕμRes=
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 2*π*I*Full21F4ϕ2F2ContributionNEEBunkBedEϕμ/.{Ω1→1/(2 (*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full31F4ϕ2F2ContributionNEEBunkBedEϕμRes=
 2*π*I*Full31F4ϕ2F2ContributionNEEBunkBedEϕμ/.
  {Ω1→1/(2 (*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullF4ϕ2F2ContributionNEEBunkBedEϕμRes=
 (*Full12F4ϕ2F2ContributionNEEBunkBedEϕμRes+*)
 Full21F4ϕ2F2ContributionNEEBunkBedEϕμRes+
  Full31F4ϕ2F2ContributionNEEBunkBedEϕμRes;


D12ϕF3ϕF3ContributionBunkBedSameμ=
 Simplify[(*dc^2*)1*I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+
         3 DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D21ϕF3ϕF3ContributionBunkBedSameμ=Simplify[(*dc^2*)
  1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D31ϕF3ϕF3ContributionBunkBedSameμ=
 Simplify[(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))*
   (*dc*)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
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√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)))^0
(Df p1² (p1²+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
    ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)))
((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
    √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)))];
Full12ϕF3ϕF3ContributionBunkBedSameμ=
 NumϕF3ϕF3ContributionBunkBedSameμ/D12ϕF3ϕF3ContributionBunkBedSameμ;
Full21ϕF3ϕF3ContributionBunkBedSameμ=
 NumϕF3ϕF3ContributionBunkBedSameμ/D21ϕF3ϕF3ContributionBunkBedSameμ;
Full31ϕF3ϕF3ContributionBunkBedSameμ=
 NumϕF3ϕF3ContributionBunkBedSameμ/D31ϕF3ϕF3ContributionBunkBedSameμ;


Full12ϕF3ϕF3ContributionBunkBedSameμRes=
 2*π*I*Full12ϕF3ϕF3ContributionBunkBedSameμ/.{Ω1→ⅈ Df p1² (p1²+σ)};
Full21ϕF3ϕF3ContributionBunkBedSameμRes=
 2*π*I*Full21ϕF3ϕF3ContributionBunkBedSameμ/.{Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1²+3 DDYuμ p1²+
    ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)};
Full31ϕF3ϕF3ContributionBunkBedSameμRes=
 2*π*I*Full31ϕF3ϕF3ContributionBunkBedSameμ/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
    √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)};
FullϕF3ϕF3ContributionBunkBedSameμRes=
 Full12ϕF3ϕF3ContributionBunkBedSameμRes+Full21ϕF3ϕF3ContributionBunkBedSameμRes+
  Full31ϕF3ϕF3ContributionBunkBedSameμRes;

D12ϕF3ϕF3ContributionBunkBedOppositeμ=Simplify[
  -I*(*dc^2*)1*(I^2)(Df p1² (p1²+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)))^2
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
      √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)))^2
  (Df p1² (p1²+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)))^0
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
      √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)))^0];
D22ϕF3ϕF3ContributionBunkBedOppositeμ=
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Simplify[-I*(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^2*
  ((*dc^2*)1)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2
  (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0];
D32ϕF3ϕF3ContributionBunkBedOppositeμ=Simplify[
 -I*(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^2*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2*
  ((*dc^2*)1)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
  (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0];
Full12ϕF3ϕF3ContributionBunkBedOppositeμ=
 NumϕF3ϕF3ContributionBunkBedOppositeμ/D12ϕF3ϕF3ContributionBunkBedOppositeμ;
Full22ϕF3ϕF3ContributionBunkBedOppositeμ=
 NumϕF3ϕF3ContributionBunkBedOppositeμ/D22ϕF3ϕF3ContributionBunkBedOppositeμ;
Full32ϕF3ϕF3ContributionBunkBedOppositeμ=
 NumϕF3ϕF3ContributionBunkBedOppositeμ/D32ϕF3ϕF3ContributionBunkBedOppositeμ;


Full12ϕF3ϕF3ContributionBunkBedOppositeμRes=
 2*π*I*D[Full12ϕF3ϕF3ContributionBunkBedOppositeμ,{Ω1,1}]/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22ϕF3ϕF3ContributionBunkBedOppositeμRes=
 2*π*I*D[Full22ϕF3ϕF3ContributionBunkBedOppositeμ,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full32ϕF3ϕF3ContributionBunkBedOppositeμRes=
 2*π*I*D[Full32ϕF3ϕF3ContributionBunkBedOppositeμ,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
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      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
Full϶F3϶F3ContributionBunkBedOppositeμRes=
 Full12϶F3϶F3ContributionBunkBedOppositeμRes+
  Full22϶F3϶F3ContributionBunkBedOppositeμRes+
  Full32϶F3϶F3ContributionBunkBedOppositeμRes;


D11϶F3Ext϶2F2ContributionBunkBedμ=
 Simplify[(*dc*)I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D22϶F3Ext϶2F2ContributionBunkBedμ=Simplify[(*dc*)
  (Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc^2*)1)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D32϶F3Ext϶2F2ContributionBunkBedμ=Simplify[(*dc*)
  (Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc^2*)1)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
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        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2  p1^2)))];
Full11ϕF3Extϕ2F2ContributionBunkBedμ=
 NumϕF3Extϕ2F2ContributionBunkBedμ/D11ϕF3Extϕ2F2ContributionBunkBedμ;
Full22ϕF3Extϕ2F2ContributionBunkBedμ=
 NumϕF3Extϕ2F2ContributionBunkBedμ/D22ϕF3Extϕ2F2ContributionBunkBedμ;
Full32ϕF3Extϕ2F2ContributionBunkBedμ=
 NumϕF3Extϕ2F2ContributionBunkBedμ/D32ϕF3Extϕ2F2ContributionBunkBedμ;


Full11ϕF3Extϕ2F2ContributionBunkBedμRes=
 2*π*I*Full11ϕF3Extϕ2F2ContributionBunkBedμ/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22ϕF3Extϕ2F2ContributionBunkBedμRes=
 2*π*I*D[Full22ϕF3Extϕ2F2ContributionBunkBedμ,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2  p1^2))};
Full32ϕF3Extϕ2F2ContributionBunkBedμRes=
 2*π*I*D[Full32ϕF3Extϕ2F2ContributionBunkBedμ,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2  p1^2))};
FullϕF3Extϕ2F2ContributionBunkBedμRes=
 Full11ϕF3Extϕ2F2ContributionBunkBedμRes+Full22ϕF3Extϕ2F2ContributionBunkBedμRes+
  Full32ϕF3Extϕ2F2ContributionBunkBedμRes;


D12ϕF3ϕ2F2ExtContributionBunkBedμ=
 Simplify[-(*dc*)(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2  p1^2))))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2  p1^2))))
   (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2  p1^2))))^2
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2  p1^2))))^2];
D21ϕF3ϕ2F2ExtContributionBunkBedμ=Simplify[
  -(*dc*)(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2  p1^2))))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
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√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))
  (Df p1² (p1²+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
      √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2];
D31ϕF3ϕ2F2ExtContributionBunkBedμ=
 Simplify[-(*dc*)(Df p1² (p1²+σ)+ⅈ Ω1)^0*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+
      3 DDYuμ p1²+ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
      √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0
  (Df p1² (p1²+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
      √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2];
Full12ϕF3ϕ2F2ExtContributionBunkBedμ=
 NumϕF3ϕ2F2ExtContributionBunkBedμ/D12ϕF3ϕ2F2ExtContributionBunkBedμ;
Full21ϕF3ϕ2F2ExtContributionBunkBedμ=
 NumϕF3ϕ2F2ExtContributionBunkBedμ/D21ϕF3ϕ2F2ExtContributionBunkBedμ;
Full31ϕF3ϕ2F2ExtContributionBunkBedμ=
 NumϕF3ϕ2F2ExtContributionBunkBedμ/D31ϕF3ϕ2F2ExtContributionBunkBedμ;


Full12ϕF3ϕ2F2ExtContributionBunkBedμRes=
 0*2*π*I*D[Full12ϕF3ϕ2F2ExtContributionBunkBedμ,Ω1]/.{Ω1→ⅈ Df p1² (p1²+σ)};
Full21ϕF3ϕ2F2ExtContributionBunkBedμRes=
 2*π*I*Full21ϕF3ϕ2F2ExtContributionBunkBedμ/.{Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1²+3 DDYuμ p1²+
     ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)};
Full31ϕF3ϕ2F2ExtContributionBunkBedμRes=
 2*π*I*Full31ϕF3ϕ2F2ExtContributionBunkBedμ/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
     √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)};

FullϕF3ϕ2F2ExtContributionBunkBedμRes=
 Full12ϕF3ϕ2F2ExtContributionBunkBedμRes+Full21ϕF3ϕ2F2ExtContributionBunkBedμRes+
  Full31ϕF3ϕ2F2ExtContributionBunkBedμRes;
ResF4F4ContributionOGSlimFishμ=
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 2*π*I*(Residue[FullF4F4ContributionOGSlimFishμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishNϕμ=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishNϕμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishWϕ1μ=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishWϕ1μ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishWϕ2μ=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishWϕ2μ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGBunkBedSameμ=
 2*π*I*(Residue[FullF4F4ContributionOGBunkBedSameμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGBunkBedOppositeμ=
 2*π*I*(Residue[FullF4F4ContributionOGBunkBedOppositeμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);

ResF4ϕF3ContributionNEEWideFishNϕμ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishNϕμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishNϕμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
        p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishNϕμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
        DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕF3ContributionNEEWideFishWϕ1μ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishWϕ1μ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ1μ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
        p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ1μ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
        DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕF3ContributionNEEWideFishWϕ2μ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishWϕ2μ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ2μ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
        p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ2μ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
        DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕF3ContributionNEEBunkBedNEϕSameμ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
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        DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
        DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕF3ContributionNEEBunkBedNEϕOppositeμ=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
        DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeμ,
     {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕF3ContributionNEEBunkBedEϕSameμ=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameμ,
     {Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+Residue[
     FullF4ϕF3ContributionNEEBunkBedEϕSameμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕF3ContributionNEEBunkBedEϕOppositeμ=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
        DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeμ,
     {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕ2F2ContributionNEEWideFishNϕμ=
 2*π*I*(Residue[FullF4ϕ2F2ContributionNEEWideFishNϕμ,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishNϕμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
        p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishNϕμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
        DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕ2F2ContributionNEEWideFishWϕμ=
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  2*π*I*(Residue[FullF4ϕ2F2ContributionNEEWideFishWϕμ,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]+
     Residue[FullF4ϕ2F2ContributionNEEWideFishWϕμ,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1²+3 DDYuμ
          p1²+ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)}]+
     Residue[FullF4ϕ2F2ContributionNEEWideFishWϕμ,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1²+3 ⅈ
          DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)}]);
ContDfμϕf3=
  2*(1/(2π)^3)(ResF4F4ContributionOGSlimFishμ+ResF4F4ContributionOGWideFishNϕμ+
     ResF4F4ContributionOGWideFishWϕ1μ+ResF4F4ContributionOGWideFishWϕ2μ+
     ResF4F4ContributionOGBunkBedSameμ+ResF4F4ContributionOGBunkBedOppositeμ+
     (ResF4ϕF3ContributionNEEWideFishNϕμ+ResF4ϕF3ContributionNEEWideFishWϕ1μ+
       ResF4ϕF3ContributionNEEWideFishWϕ2μ+ResF4ϕF3ContributionNEEBunkBedNEϕSameμ+
       ResF4ϕF3ContributionNEEBunkBedNEϕOppositeμ+
       ResF4ϕF3ContributionNEEBunkBedEϕSameμ+
       ResF4ϕF3ContributionNEEBunkBedEϕOppositeμ+
       ResF4ϕ2F2ContributionNEEWideFishNϕμ+ResF4ϕ2F2ContributionNEEWideFishWϕμ+
       (FullF4ϕ2F2ContributionNEEBunkBedEϕμRes+
         FullF4ϕ2F2ContributionNEEBunkBedNEϕμRes+
         (FullϕF3ϕ2F2ExtContributionBunkBedμRes+
           FullϕF3Extϕ2F2ContributionBunkBedμRes+
           FullϕF3ϕF3ContributionBunkBedOppositeμRes+
           FullϕF3ϕF3ContributionBunkBedSameμRes))));*)
```

In[ ]:=
```
F4F4ContributionOGSlimFishμ =
  Simplify[Together[((1 / 2) D[(F4F4ContributionOGSlimFish /.
          {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω3 → 0, Ω2 → 0, ω → 0,
           p3[1] → p3, p3[2] → 0, p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}),
        {p3, 1}, {p2, 1}, {q, 2}] /. {p3 → 0, q → 0, p2 → 0})]];
NumF4F4ContributionOGSlimFishμ =
  Simplify[Integrate[Numerator[F4F4ContributionOGSlimFishμ], {θ, 0, 2 π}]];
FullF4F4ContributionOGSlimFishμ = NumF4F4ContributionOGSlimFishμ /
   Simplify[Denominator[F4F4ContributionOGSlimFishμ]];
ResF4F4ContributionOGSlimFishμ =
  2 * π * I * (Residue[FullF4F4ContributionOGSlimFishμ, {Ω1, ⅈ Df * p1² (p1² κ + σ)}]);
ContDfμϕf3 = 2 * (1 / (2 π) ^ 3) (ResF4F4ContributionOGSlimFishμ);
```

Extract contribution of K

In[ ]:=
```
(*F4F4ContributionOGSlimFishK=
    Simplify[Together[((1/2)D[(F4F4ContributionOGSlimFish/.{p1[1]→p1*Cos[θ],
             p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
             q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGWideFishNϕK=
```

```
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishNϕ/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGWideFishWϕ1K=
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishWϕ1/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];

F4F4ContributionOGWideFishWϕ2K=
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishWϕ2/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];

F4F4ContributionOGBunkBedSameK=
    Simplify[Together[((1/2)D[(F4F4ContributionOGBunkBedSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGBunkBedOppositeK=Simplify[
     Together[((1/2)D[(F4F4ContributionOGBunkBedOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];
F4ϕF3ContributionNEEWideFishNϕK=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishNϕ/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEWideFishWϕ1K=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishWϕ1/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEWideFishWϕ2K=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishWϕ2/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];
F4ϕF3ContributionNEEBunkBedNEϕSameK=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedNEϕSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedNEϕOppositeK=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedNEϕOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];
```

```
F4ϕF3ContributionNEEBunkBedEϕSameK=Simplify[
    Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedEϕSame/.{p1[1]→p1*Cos[θ],
          p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
          q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedEϕOppositeK=Simplify[
    Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedEϕOpposite/.{p1[1]→p1*Cos[θ],
          p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
          q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})]];
F4ϕ2F2ContributionNEEWideFishNϕK=Together[
    ((1/2)D[(F4ϕ2F2ContributionNEEWideFishNϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
         Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEWideFishWϕK=Together[
    ((1/2)D[(F4ϕ2F2ContributionNEEWideFishWϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
         Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEBunkBedNEϕK=Together[
    ((1/2)D[(F4ϕ2F2ContributionNEEBunkBedNEϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
         Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEBunkBedEϕK=Together[
    ((1/2)D[(F4ϕ2F2ContributionNEEBunkBedEϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
         Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})];
ϕF3ϕF3ContributionBunkBedSameK=Together[
    ((1/2)D[(ϕF3ϕF3ContributionBunkBedSame/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
         Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})];
ϕF3ϕF3ContributionBunkBedOppositeK=
   Together[((1/2)D[(ϕF3ϕF3ContributionBunkBedOpposite/.{p1[1]→p1*Cos[θ],
         p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
         q[1]→0,q[2]→q}),{p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})];
ϕF3Extϕ2F2ContributionBunkBedK=Together[
    ((1/2)D[(ϕF3Extϕ2F2ContributionBunkBed/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
         Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})];
ϕF3ϕ2F2ExtContributionBunkBedK=Together[
    ((1/2)D[(ϕF3ϕ2F2ExtContributionBunkBed/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
         Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
      {p3,2},{p2,1},{q,1}]/.{p3→0,q→0,p2→0})];
NumF4F4ContributionOGSlimFishK=
   Simplify[Integrate[Numerator[F4F4ContributionOGSlimFishK],{θ,0,2π}]];
```

```
NumF4F4ContributionOGWideFishNϕK=
  Simplify[Integrate[Numerator[F4F4ContributionOGWideFishNϕK],{θ,0,2π}]];
NumF4F4ContributionOGWideFishWϕ1K=
  Simplify[Integrate[Numerator[F4F4ContributionOGWideFishWϕ1K],{θ,0,2π}]];
NumF4F4ContributionOGWideFishWϕ2K=
  Simplify[Integrate[Numerator[F4F4ContributionOGWideFishWϕ2K],{θ,0,2π}]];
NumF4F4ContributionOGBunkBedSameK=
  Simplify[Integrate[Numerator[F4F4ContributionOGBunkBedSameK],{θ,0,2π}]];
NumF4F4ContributionOGBunkBedOppositeK=
  Simplify[Integrate[Numerator[F4F4ContributionOGBunkBedOppositeK],{θ,0,2π}]];
NumF4ϕF3ContributionNEEWideFishNϕK=
  Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishNϕK],{θ,0,2π}]];
NumF4ϕF3ContributionNEEWideFishWϕ1K=
  Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishWϕ1K],{θ,0,2π}]];
NumF4ϕF3ContributionNEEWideFishWϕ2K=
  Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishWϕ2K],{θ,0,2π}]];
NumF4ϕF3ContributionNEEBunkBedNEϕSameK=
  Simplify[Integrate[Numerator[F4ϕF3ContributionNEEBunkBedNEϕSameK],{θ,0,2π}]];
NumF4ϕF3ContributionNEEBunkBedNEϕOppositeK=Simplify[
   Integrate[Numerator[F4ϕF3ContributionNEEBunkBedNEϕOppositeK],{θ,0,2π}]];
NumF4ϕF3ContributionNEEBunkBedEϕSameK=
  Simplify[Integrate[Numerator[F4ϕF3ContributionNEEBunkBedEϕSameK],{θ,0,2π}]];
NumF4ϕF3ContributionNEEBunkBedEϕOppositeK=Simplify[
   Integrate[Numerator[F4ϕF3ContributionNEEBunkBedEϕOppositeK],{θ,0,2π}]];
NumF4ϕ2F2ContributionNEEWideFishNϕK=
  Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEWideFishNϕK],{θ,0,2π}]];
NumF4ϕ2F2ContributionNEEWideFishWϕK=
  Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEWideFishWϕK],{θ,0,2π}]];

NumF4ϕ2F2ContributionNEEBunkBedNEϕK=
  Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEBunkBedNEϕK],{θ,0,2π}]];
NumF4ϕ2F2ContributionNEEBunkBedEϕK=
  Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEBunkBedEϕK],{θ,0,2π}]];
NumϕF3ϕF3ContributionBunkBedSameK=
  Simplify[Integrate[Numerator[ϕF3ϕF3ContributionBunkBedSameK],{θ,0,2π}]];
NumϕF3ϕF3ContributionBunkBedOppositeK=
  Simplify[Integrate[Numerator[ϕF3ϕF3ContributionBunkBedOppositeK],{θ,0,2π}]];
NumϕF3Extϕ2F2ContributionBunkBedK=
  Integrate[Numerator[ϕF3Extϕ2F2ContributionBunkBedK],{θ,0,2π}];
NumϕF3ϕ2F2ExtContributionBunkBedK=
  Integrate[Numerator[ϕF3ϕ2F2ExtContributionBunkBedK],{θ,0,2π}];
FullF4F4ContributionOGSlimFishK=NumF4F4ContributionOGSlimFishK/
   Simplify[Denominator[F4F4ContributionOGSlimFishK]];
FullF4F4ContributionOGWideFishNϕK=NumF4F4ContributionOGWideFishNϕK/
```

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    Simplify[Denominator[F4F4ContributionOGWideFishNϕK]];
FullF4F4ContributionOGWideFishWϕ1K=NumF4F4ContributionOGWideFishWϕ1K/
    Simplify[Denominator[F4F4ContributionOGWideFishWϕ1K]];
FullF4F4ContributionOGWideFishWϕ2K=NumF4F4ContributionOGWideFishWϕ2K/
    Simplify[Denominator[F4F4ContributionOGWideFishWϕ2K]];
FullF4F4ContributionOGBunkBedSameK=NumF4F4ContributionOGBunkBedSameK/
    Simplify[Denominator[F4F4ContributionOGBunkBedSameK]];
FullF4F4ContributionOGBunkBedOppositeK=NumF4F4ContributionOGBunkBedOppositeK/
    Simplify[Denominator[F4F4ContributionOGBunkBedOppositeK]];
FullF4ϕF3ContributionNEEWideFishNϕK=NumF4ϕF3ContributionNEEWideFishNϕK/
    Simplify[Denominator[F4ϕF3ContributionNEEWideFishNϕK]];
FullF4ϕF3ContributionNEEWideFishWϕ1K=NumF4ϕF3ContributionNEEWideFishWϕ1K/
    Simplify[Denominator[F4ϕF3ContributionNEEWideFishWϕ1K]];
FullF4ϕF3ContributionNEEWideFishWϕ2K=NumF4ϕF3ContributionNEEWideFishWϕ2K/
    Simplify[Denominator[F4ϕF3ContributionNEEWideFishWϕ2K]];
FullF4ϕF3ContributionNEEBunkBedNEϕSameK=NumF4ϕF3ContributionNEEBunkBedNEϕSameK/
    Simplify[Denominator[F4ϕF3ContributionNEEBunkBedNEϕSameK]];
FullF4ϕF3ContributionNEEBunkBedNEϕOppositeK=
   NumF4ϕF3ContributionNEEBunkBedNEϕOppositeK/
    Simplify[Denominator[F4ϕF3ContributionNEEBunkBedNEϕOppositeK]];
FullF4ϕF3ContributionNEEBunkBedEϕSameK=NumF4ϕF3ContributionNEEBunkBedEϕSameK/
    Simplify[Denominator[F4ϕF3ContributionNEEBunkBedEϕSameK]];
FullF4ϕF3ContributionNEEBunkBedEϕOppositeK=
   NumF4ϕF3ContributionNEEBunkBedEϕOppositeK/
    Simplify[Denominator[F4ϕF3ContributionNEEBunkBedEϕOppositeK]];
FullF4ϕ2F2ContributionNEEWideFishNϕK=NumF4ϕ2F2ContributionNEEWideFishNϕK/
    Simplify[Denominator[F4ϕ2F2ContributionNEEWideFishNϕK]];
FullF4ϕ2F2ContributionNEEWideFishWϕK=NumF4ϕ2F2ContributionNEEWideFishWϕK/
    Simplify[Denominator[F4ϕ2F2ContributionNEEWideFishWϕK]];

FullF4ϕ2F2ContributionNEEBunkBedNEϕKForIso=NumF4ϕ2F2ContributionNEEBunkBedNEϕK/
    Simplify[Denominator[F4ϕ2F2ContributionNEEBunkBedNEϕK]];
FullF4ϕ2F2ContributionNEEBunkBedEϕKForIso=NumF4ϕ2F2ContributionNEEBunkBedEϕK/
    Simplify[Denominator[F4ϕ2F2ContributionNEEBunkBedEϕK]];
FullϕF3ϕF3ContributionBunkBedSameKForIso=NumϕF3ϕF3ContributionBunkBedSameK/
    Simplify[Denominator[ϕF3ϕF3ContributionBunkBedSameK]];
FullϕF3ϕF3ContributionBunkBedOppositeKForIso=
   NumϕF3ϕF3ContributionBunkBedOppositeK/
    Simplify[Denominator[ϕF3ϕF3ContributionBunkBedOppositeK]];
FullϕF3Extϕ2F2ContributionBunkBedKForIso=NumϕF3Extϕ2F2ContributionBunkBedK/
    Simplify[Denominator[ϕF3Extϕ2F2ContributionBunkBedK]];
FullϕF3ϕ2F2ExtContributionBunkBedKForIso=NumϕF3ϕ2F2ExtContributionBunkBedK/
    Simplify[Denominator[ϕF3ϕ2F2ExtContributionBunkBedK]];
```

```
D11F4ϕ2F2ContributionNEEBunkBedNEϕK=
  Simplify[Distribute[(*dc*)I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*
    ((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-
          2 DDYuλ DDYuμ-DDYuμ^2) p1^2))) ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ
        p1^2+√(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
    (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
    ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+√(4 DDYuA DDYuK+
          4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))]];
 D22F4ϕ2F2ContributionNEEBunkBedNEϕK=
  Simplify[Distribute[(*dc*)(Df p1^2 (p1^2+σ)+ⅈ Ω1)*dc*
    ((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+ⅈ √(4 DDYuA DDYuK+
           4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
    ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+√(4 DDYuA DDYuK+4 DDYuK^2-
          DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
    (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
    ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+√(4 DDYuA DDYuK+
          4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))]];
 D32F4ϕ2F2ContributionNEEBunkBedNEϕK=Simplify[
   Distribute[(*dc*)(Df p1^2 (p1^2+σ)+ⅈ Ω1) ((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))*
    (*dc*)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+√(4 DDYuA DDYuK+
           4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0*
    (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
    ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+√(4 DDYuA DDYuK+
          4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))]];
Full11F4ϕ2F2ContributionNEEBunkBedNEϕK=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕK/D11F4ϕ2F2ContributionNEEBunkBedNEϕK;
 Full22F4ϕ2F2ContributionNEEBunkBedNEϕK=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕK/D22F4ϕ2F2ContributionNEEBunkBedNEϕK;
 Full32F4ϕ2F2ContributionNEEBunkBedNEϕK=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕK/D32F4ϕ2F2ContributionNEEBunkBedNEϕK;
```

```
Full11F4ϕ2F2ContributionNEEBunkBedNEϕKRes=
  2*π*I*Full11F4ϕ2F2ContributionNEEBunkBedNEϕK/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22F4ϕ2F2ContributionNEEBunkBedNEϕKRes=
 2*π*I*Full22F4ϕ2F2ContributionNEEBunkBedNEϕK/.
 {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
Full32F4ϕ2F2ContributionNEEBunkBedNEϕKRes=
 2*π*I*Full32F4ϕ2F2ContributionNEEBunkBedNEϕK/.
 {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
FullF4ϕ2F2ContributionNEEBunkBedNEϕKRes=Full11F4ϕ2F2ContributionNEEBunkBedNEϕKRes+
  Full22F4ϕ2F2ContributionNEEBunkBedNEϕKRes+
  Full32F4ϕ2F2ContributionNEEBunkBedNEϕKRes;


D12F4ϕ2F2ContributionNEEBunkBedEϕK=
 Simplify[(*dc*)(-I*Df p1^2 (p1^2+σ)+ Ω1)^0((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D21F4ϕ2F2ContributionNEEBunkBedEϕK=Simplify[(*dc*)
  (I*Df p1^2 (p1^2+σ)+ Ω1)^0(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D31F4ϕ2F2ContributionNEEBunkBedEϕK=Simplify[(*dc*)
```

```
(I*Df p1^2 (p1^2+σ)+ Ω1)^0(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
  (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
Full12F4ϕ2F2ContributionNEEBunkBedEϕK=
 NumF4ϕ2F2ContributionNEEBunkBedEϕK/D12F4ϕ2F2ContributionNEEBunkBedEϕK;
Full21F4ϕ2F2ContributionNEEBunkBedEϕK=
 NumF4ϕ2F2ContributionNEEBunkBedEϕK/D21F4ϕ2F2ContributionNEEBunkBedEϕK;
Full31F4ϕ2F2ContributionNEEBunkBedEϕK=
 NumF4ϕ2F2ContributionNEEBunkBedEϕK/D31F4ϕ2F2ContributionNEEBunkBedEϕK;

Full12F4ϕ2F2ContributionNEEBunkBedEϕKRes=
 0*2*π*I*D[Full12F4ϕ2F2ContributionNEEBunkBedEϕK,{Ω1,1}]/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full21F4ϕ2F2ContributionNEEBunkBedEϕKRes=
 2*π*I*Full21F4ϕ2F2ContributionNEEBunkBedEϕK/.{Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
Full31F4ϕ2F2ContributionNEEBunkBedEϕKRes=
 2*π*I*Full31F4ϕ2F2ContributionNEEBunkBedEϕK/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
FullF4ϕ2F2ContributionNEEBunkBedEϕKRes=
 (*Full12F4ϕ2F2ContributionNEEBunkBedEϕKRes+*)
 Full21F4ϕ2F2ContributionNEEBunkBedEϕKRes+
  Full31F4ϕ2F2ContributionNEEBunkBedEϕKRes;


D12ϕF3ϕF3ContributionBunkBedSameK=Simplify[(*dc^2*)
  1*I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
  (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
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  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2))))];
D21ϕF3ϕF3ContributionBunkBedSameK=Simplify[(*dc^2*)
  1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
  (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2))))];
D31ϕF3ϕF3ContributionBunkBedSameK=
 Simplify[(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))*
  (*dc*)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
  (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2))))];
Full12ϕF3ϕF3ContributionBunkBedSameK=
 NumϕF3ϕF3ContributionBunkBedSameK/D12ϕF3ϕF3ContributionBunkBedSameK;
Full21ϕF3ϕF3ContributionBunkBedSameK=
 NumϕF3ϕF3ContributionBunkBedSameK/D21ϕF3ϕF3ContributionBunkBedSameK;
Full31ϕF3ϕF3ContributionBunkBedSameK=
 NumϕF3ϕF3ContributionBunkBedSameK/D31ϕF3ϕF3ContributionBunkBedSameK;


Full12ϕF3ϕF3ContributionBunkBedSameKRes=
 2*π*I*Full12ϕF3ϕF3ContributionBunkBedSameK/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full21ϕF3ϕF3ContributionBunkBedSameKRes=
 2*π*I*Full21ϕF3ϕF3ContributionBunkBedSameK/.{Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
Full31ϕF3ϕF3ContributionBunkBedSameKRes=
 2*π*I*Full31ϕF3ϕF3ContributionBunkBedSameK/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
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√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)};
FullϕF3ϕF3ContributionBunkBedSameKRes=
 Full12ϕF3ϕF3ContributionBunkBedSameKRes+Full21ϕF3ϕF3ContributionBunkBedSameKRes+
  Full31ϕF3ϕF3ContributionBunkBedSameKRes;

D12ϕF3ϕF3ContributionBunkBedOppositeK=Simplify[
  -I*(*dc^2*)1*(I^2)(Df p1² (p1²+σ)+i Ω1)^0*((*dc*)Ω1-(1/2 i (DDYuλ p1²+3 DDYuμ p1²+
          i √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2
   ((*dc*)Ω1-(1/2 (i DDYuλ p1²+3 i DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-
               DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2
   (Df p1² (p1²+σ)-i Ω1) ((*dc*)Ω1+(1/2 i (DDYuλ p1²+3 DDYuμ p1²+
          i √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0
   ((*dc*)Ω1+(1/2 (i DDYuλ p1²+3 i DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-
               DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0];
D22ϕF3ϕF3ContributionBunkBedOppositeK=
 Simplify[-I*(*dc^2*)1*(Df p1² (p1²+σ)+i Ω1)^2*
   ((*dc^2*)1)((*dc*)Ω1-(1/2 i (DDYuλ p1²+3 DDYuμ p1²+
          i √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0
   ((*dc*)Ω1-(1/2 (i DDYuλ p1²+3 i DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-
               DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2
   (Df p1² (p1²+σ)-i Ω1) ((*dc*)Ω1+(1/2 i (DDYuλ p1²+3 DDYuμ p1²+
          i √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0
   ((*dc*)Ω1+(1/2 (i DDYuλ p1²+3 i DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-
               DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0];
D32ϕF3ϕF3ContributionBunkBedOppositeK=Simplify[
  -I*(*dc^2*)1*(Df p1² (p1²+σ)+i Ω1)^2*((*dc*)Ω1-(1/2 i (DDYuλ p1²+3 DDYuμ p1²+
          i √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2*
   ((*dc^2*)1)((*dc*)Ω1-(1/2 (i DDYuλ p1²+3 i DDYuμ p1²+
          √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0
   (Df p1² (p1²+σ)-i Ω1) ((*dc*)Ω1+(1/2 i (DDYuλ p1²+3 DDYuμ p1²+
          i √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0
   ((*dc*)Ω1+(1/2 (i DDYuλ p1²+3 i DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-
               DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0];
Full12ϕF3ϕF3ContributionBunkBedOppositeK=
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 NumϕF3ϕF3ContributionBunkBedOppositeK/D12ϕF3ϕF3ContributionBunkBedOppositeK;
Full22ϕF3ϕF3ContributionBunkBedOppositeK=
 NumϕF3ϕF3ContributionBunkBedOppositeK/D22ϕF3ϕF3ContributionBunkBedOppositeK;
Full32ϕF3ϕF3ContributionBunkBedOppositeK=
 NumϕF3ϕF3ContributionBunkBedOppositeK/D32ϕF3ϕF3ContributionBunkBedOppositeK;


Full12ϕF3ϕF3ContributionBunkBedOppositeKRes=
 2*π*I*D[Full12ϕF3ϕF3ContributionBunkBedOppositeK,{Ω1,1}]/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22ϕF3ϕF3ContributionBunkBedOppositeKRes=
 2*π*I*D[Full22ϕF3ϕF3ContributionBunkBedOppositeK,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
Full32ϕF3ϕF3ContributionBunkBedOppositeKRes=
 2*π*I*D[Full32ϕF3ϕF3ContributionBunkBedOppositeK,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
FullϕF3ϕF3ContributionBunkBedOppositeKRes=
 Full12ϕF3ϕF3ContributionBunkBedOppositeKRes+
  Full22ϕF3ϕF3ContributionBunkBedOppositeKRes+
  Full32ϕF3ϕF3ContributionBunkBedOppositeKRes;


D11ϕF3Extϕ2F2ContributionBunkBedK=
 Simplify[(*dc*)I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+√(4 DDYuA DDYuK+4 DDYuK^2-
             DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
         √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D22ϕF3Extϕ2F2ContributionBunkBedK=Simplify[(*dc*)
  (Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc^2*)1)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+√(4 DDYuA DDYuK+4 DDYuK^2-
             DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
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    (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
    ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D32ϕF3Extϕ2F2ContributionBunkBedK=Simplify[(*dc*)
   (Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc^2*)1)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
    ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+√(4 DDYuA DDYuK+4 DDYuK^2-
                DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
    (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
    ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
Full11ϕF3Extϕ2F2ContributionBunkBedK=
 NumϕF3Extϕ2F2ContributionBunkBedK/D11ϕF3Extϕ2F2ContributionBunkBedK;
Full22ϕF3Extϕ2F2ContributionBunkBedK=
 NumϕF3Extϕ2F2ContributionBunkBedK/D22ϕF3Extϕ2F2ContributionBunkBedK;
Full32ϕF3Extϕ2F2ContributionBunkBedK=
 NumϕF3Extϕ2F2ContributionBunkBedK/D32ϕF3Extϕ2F2ContributionBunkBedK;


Full11ϕF3Extϕ2F2ContributionBunkBedKRes=
 2*π*I*Full11ϕF3Extϕ2F2ContributionBunkBedK/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22ϕF3Extϕ2F2ContributionBunkBedKRes=
 2*π*I*D[Full22ϕF3Extϕ2F2ContributionBunkBedK,{Ω1,1}]/.
  {Ω1→1/(2(**dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
Full32ϕF3Extϕ2F2ContributionBunkBedKRes=
 2*π*I*D[Full32ϕF3Extϕ2F2ContributionBunkBedK,{Ω1,1}]/.
  {Ω1→1/(2(**dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
FullϕF3Extϕ2F2ContributionBunkBedKRes=
 Full11ϕF3Extϕ2F2ContributionBunkBedKRes+Full22ϕF3Extϕ2F2ContributionBunkBedKRes+
  Full32ϕF3Extϕ2F2ContributionBunkBedKRes;


D12ϕF3ϕ2F2ExtContributionBunkBedK=
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 Simplify[-(*dc*) (Df p1² (p1²+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
        √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))
   (Df p1² (p1²+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-
          DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2];
D21ϕF3ϕ2F2ExtContributionBunkBedK=Simplify[
  -(*dc*) (Df p1² (p1²+σ)+ⅈ Ω1)^0*(*dc*) ((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
        √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))
   (Df p1² (p1²+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-
          DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2];
D31ϕF3ϕ2F2ExtContributionBunkBedK=Simplify[
  -(*dc*) (Df p1² (p1²+σ)+ⅈ Ω1)^0*(*dc*) ((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+
        √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^0
   (Df p1² (p1²+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1²+3 DDYuμ p1²+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1²+3 ⅈ DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-
          DDYuλ²-2 DDYuλ DDYuμ-DDYuμ²) p1²)))^2];
Full12ϕF3ϕ2F2ExtContributionBunkBedK=
 NumϕF3ϕ2F2ExtContributionBunkBedK/D12ϕF3ϕ2F2ExtContributionBunkBedK;
Full21ϕF3ϕ2F2ExtContributionBunkBedK=
 NumϕF3ϕ2F2ExtContributionBunkBedK/D21ϕF3ϕ2F2ExtContributionBunkBedK;
Full31ϕF3ϕ2F2ExtContributionBunkBedK=
 NumϕF3ϕ2F2ExtContributionBunkBedK/D31ϕF3ϕ2F2ExtContributionBunkBedK;


Full12ϕF3ϕ2F2ExtContributionBunkBedKRes=
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0*2*π*I*D[Full12ϕF3ϕ2F2ExtContributionBunkBedK,Ω1]/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full21ϕF3ϕ2F2ExtContributionBunkBedKRes=
 2*π*I*Full21ϕF3ϕ2F2ExtContributionBunkBedK/.{Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
Full31ϕF3ϕ2F2ExtContributionBunkBedKRes=
 2*π*I*Full31ϕF3ϕ2F2ExtContributionBunkBedK/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};

FullϕF3ϕ2F2ExtContributionBunkBedKRes=
 Full12ϕF3ϕ2F2ExtContributionBunkBedKRes+Full21ϕF3ϕ2F2ExtContributionBunkBedKRes+
  Full31ϕF3ϕ2F2ExtContributionBunkBedKRes;
ResF4F4ContributionOGSlimFishK=
 2*π*I*(Residue[FullF4F4ContributionOGSlimFishK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishNϕK=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishNϕK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishWϕ1K=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishWϕ1K,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishWϕ2K=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishWϕ2K,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGBunkBedSameK=
 2*π*I*(Residue[FullF4F4ContributionOGBunkBedSameK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGBunkBedOppositeK=
 2*π*I*(Residue[FullF4F4ContributionOGBunkBedOppositeK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);

ResF4ϕF3ContributionNEEWideFishNϕK=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishNϕK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishNϕK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
          p1^2+ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishNϕK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ
          p1^2+√(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);

ResF4ϕF3ContributionNEEWideFishWϕ1K=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEWideFishWϕ1K,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+Residue[
     FullF4ϕF3ContributionNEEWideFishWϕ1K,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
         ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+Residue[
     FullF4ϕF3ContributionNEEWideFishWϕ1K,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
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        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);
ResF4ϕF3ContributionNEEWideFishWϕ2K=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEWideFishWϕ2K,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+Residue[
     FullF4ϕF3ContributionNEEWideFishWϕ2K,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+Residue[
     FullF4ϕF3ContributionNEEWideFishWϕ2K,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);

ResF4ϕF3ContributionNEEBunkBedNEϕSameK=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameK,
     {Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameK,
     {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);
ResF4ϕF3ContributionNEEBunkBedNEϕOppositeK=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeK,
     {Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeK,
     {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);

ResF4ϕF3ContributionNEEBunkBedEϕSameK=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameK,
     {Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+Residue[
     FullF4ϕF3ContributionNEEBunkBedEϕSameK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);
ResF4ϕF3ContributionNEEBunkBedEϕOppositeK=2*π*I*
```

```
(Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
  Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeK,
   {Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+
  Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeK,
   {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);

ResF4ϕ2F2ContributionNEEWideFishNϕK=2*π*I*
 (Residue[FullF4ϕ2F2ContributionNEEWideFishNϕK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+Residue[
    FullF4ϕ2F2ContributionNEEWideFishNϕK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+Residue[
    FullF4ϕ2F2ContributionNEEWideFishNϕK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);
ResF4ϕ2F2ContributionNEEWideFishWϕK=2*π*I*
 (Residue[FullF4ϕ2F2ContributionNEEWideFishWϕK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+Residue[
    FullF4ϕ2F2ContributionNEEWideFishWϕK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]+Residue[
    FullF4ϕ2F2ContributionNEEWideFishWϕK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}]);
ContDfKϕf3=
 2*(1/(2π)^3)(ResF4F4ContributionOGSlimFishK+ResF4F4ContributionOGWideFishNϕK+
   ResF4F4ContributionOGWideFishWϕ1K+ResF4F4ContributionOGWideFishWϕ2K+
   ResF4F4ContributionOGBunkBedSameK+ResF4F4ContributionOGBunkBedOppositeK+
   (ResF4ϕF3ContributionNEEWideFishNϕK+ResF4ϕF3ContributionNEEWideFishWϕ1K+
     ResF4ϕF3ContributionNEEWideFishWϕ2K+ResF4ϕF3ContributionNEEBunkBedNEϕSameK+
     ResF4ϕF3ContributionNEEBunkBedNEϕOppositeK+
     ResF4ϕF3ContributionNEEBunkBedEϕSameK+
     ResF4ϕF3ContributionNEEBunkBedEϕOppositeK+
     ResF4ϕ2F2ContributionNEEWideFishNϕK+ResF4ϕ2F2ContributionNEEWideFishWϕK+
     (FullF4ϕ2F2ContributionNEEBunkBedEϕKRes+
       FullF4ϕ2F2ContributionNEEBunkBedNEϕKRes+
       (FullϕF3ϕ2F2ExtContributionBunkBedKRes+
         FullϕF3Extϕ2F2ContributionBunkBedKRes+
         FullϕF3ϕF3ContributionBunkBedOppositeKRes+
         FullϕF3ϕF3ContributionBunkBedSameKRes))));*)
```

In[∘]:=
```
F4F4ContributionOGSlimFishK =
  Simplify[Together[((1 / 2) D[(F4F4ContributionOGSlimFish /.
         {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω3 → 0, Ω2 → 0, ω → 0,
          p3[1] → p3, p3[2] → 0, p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}),
       {p3, 2}, {p2, 1}, {q, 1}] /. {p3 → 0, q → 0, p2 → 0})]];
NumF4F4ContributionOGSlimFishK =
  Simplify[Integrate[Numerator[F4F4ContributionOGSlimFishK], {θ, 0, 2 π}]];
FullF4F4ContributionOGSlimFishK = NumF4F4ContributionOGSlimFishK /
   Simplify[Denominator[F4F4ContributionOGSlimFishK]];
ResF4F4ContributionOGSlimFishK =
  2 * π * I * (Residue[FullF4F4ContributionOGSlimFishK, {Ω1, ⅈ Df * p1^2 (p1^2 κ + σ)}]);
ContDfKϕf3 = 2 * (1 / (2 π) ^3) (ResF4F4ContributionOGSlimFishK);
```

Extract contribution of AK

In[∘]:=
```
(*F4F4ContributionOGSlimFishAK=
    Simplify[Together[((1/2)D[(F4F4ContributionOGSlimFish/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGWideFishNϕAK=
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishNϕ/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGWideFishWϕ1AK=
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishWϕ1/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];

F4F4ContributionOGWideFishWϕ2AK=
    Simplify[Together[((1/2)D[(F4F4ContributionOGWideFishWϕ2/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];

F4F4ContributionOGBunkBedSameAK=
    Simplify[Together[((1/2)D[(F4F4ContributionOGBunkBedSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];
F4F4ContributionOGBunkBedOppositeAK=Simplify[
     Together[((1/2)D[(F4F4ContributionOGBunkBedOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];
F4ϕF3ContributionNEEWideFishNϕAK=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishNϕ/.{p1[1]→p1*Cos[θ],
```

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            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEWideFishWϕ1AK=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishWϕ1/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEWideFishWϕ2AK=
    Simplify[Together[((1/2)D[(F4ϕF3ContributionNEEWideFishWϕ2/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];
F4ϕF3ContributionNEEBunkBedNEϕSameAK=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedNEϕSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedNEϕOppositeAK=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedNEϕOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedEϕSameAK=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedEϕSame/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];

F4ϕF3ContributionNEEBunkBedEϕOppositeAK=Simplify[
     Together[((1/2)D[(F4ϕF3ContributionNEEBunkBedEϕOpposite/.{p1[1]→p1*Cos[θ],
            p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
            q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})]];
F4ϕ2F2ContributionNEEWideFishNϕAK=Together[
     ((1/2)D[(F4ϕ2F2ContributionNEEWideFishNϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
           Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
        {p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEWideFishWϕAK=Together[
     ((1/2)D[(F4ϕ2F2ContributionNEEWideFishWϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
           Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
        {p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEBunkBedNEϕAK=Together[
     ((1/2)D[(F4ϕ2F2ContributionNEEBunkBedNEϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
           Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
        {p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})];
F4ϕ2F2ContributionNEEBunkBedEϕAK=Together[
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    ((1/2)D[(F4ϕ2F2ContributionNEEBunkBedEϕ/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
           Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
        {p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})];
ϕF3ϕF3ContributionBunkBedSameAK=Together[
    ((1/2)D[(ϕF3ϕF3ContributionBunkBedSame/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
           Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
        {p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})];
ϕF3ϕF3ContributionBunkBedOppositeAK=
   Together[((1/2)D[(ϕF3ϕF3ContributionBunkBedOpposite/.{p1[1]→p1*Cos[θ],
           p1[2]→p1*Sin[θ],Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,
           q[1]→0,q[2]→q}),{p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})];
ϕF3Extϕ2F2ContributionBunkBedAK=Together[
    ((1/2)D[(ϕF3Extϕ2F2ContributionBunkBed/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
           Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
        {p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})];
ϕF3ϕ2F2ExtContributionBunkBedAK=Together[
    ((1/2)D[(ϕF3ϕ2F2ExtContributionBunkBed/.{p1[1]→p1*Cos[θ],p1[2]→p1*Sin[θ],
           Ω3→0,Ω2→0,ω→0,p3[1]→p3,p3[2]→0,p2[1]→p2,p2[2]→0,q[1]→0,q[2]→q}),
        {p3,1},{p2,2},{q,1}]/.{p3→0,q→0,p2→0})];
NumF4F4ContributionOGSlimFishAK=
   Simplify[Integrate[Numerator[F4F4ContributionOGSlimFishAK],{θ,0,2π}]];
  NumF4F4ContributionOGWideFishNϕAK=
   Simplify[Integrate[Numerator[F4F4ContributionOGWideFishNϕAK],{θ,0,2π}]];
  NumF4F4ContributionOGWideFishWϕ1AK=
   Simplify[Integrate[Numerator[F4F4ContributionOGWideFishWϕ1AK],{θ,0,2π}]];
  NumF4F4ContributionOGWideFishWϕ2AK=
   Simplify[Integrate[Numerator[F4F4ContributionOGWideFishWϕ2AK],{θ,0,2π}]];
  NumF4F4ContributionOGBunkBedSameAK=
   Simplify[Integrate[Numerator[F4F4ContributionOGBunkBedSameAK],{θ,0,2π}]];
  NumF4F4ContributionOGBunkBedOppositeAK=
   Simplify[Integrate[Numerator[F4F4ContributionOGBunkBedOppositeAK],{θ,0,2π}]];
  NumF4ϕF3ContributionNEEWideFishNϕAK=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishNϕAK],{θ,0,2π}]];
  NumF4ϕF3ContributionNEEWideFishWϕ1AK=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishWϕ1AK],{θ,0,2π}]];
NumF4ϕF3ContributionNEEWideFishWϕ2AK=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEWideFishWϕ2AK],{θ,0,2π}]];
NumF4ϕF3ContributionNEEBunkBedNEϕSameAK=Simplify[
    Integrate[Numerator[F4ϕF3ContributionNEEBunkBedNEϕSameAK],{θ,0,2π}]];
  NumF4ϕF3ContributionNEEBunkBedNEϕOppositeAK=Simplify[
    Integrate[Numerator[F4ϕF3ContributionNEEBunkBedNEϕOppositeAK],{θ,0,2π}]];
  NumF4ϕF3ContributionNEEBunkBedEϕSameAK=
   Simplify[Integrate[Numerator[F4ϕF3ContributionNEEBunkBedEϕSameAK],{θ,0,2π}]];
  NumF4ϕF3ContributionNEEBunkBedEϕOppositeAK=Simplify[
```

```
      Integrate[Numerator[F4ϕF3ContributionNEEBunkBedEϕOppositeAK],{θ,0,2π}]];
   NumF4ϕ2F2ContributionNEEWideFishNϕAK=
     Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEWideFishNϕAK],{θ,0,2π}]];
   NumF4ϕ2F2ContributionNEEWideFishWϕAK=
     Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEWideFishWϕAK],{θ,0,2π}]];

   NumF4ϕ2F2ContributionNEEBunkBedNEϕAK=
     Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEBunkBedNEϕAK],{θ,0,2π}]];
 NumF4ϕ2F2ContributionNEEBunkBedEϕAK=
     Simplify[Integrate[Numerator[F4ϕ2F2ContributionNEEBunkBedEϕAK],{θ,0,2π}]];
 NumϕF3ϕF3ContributionBunkBedSameAK=
     Simplify[Integrate[Numerator[ϕF3ϕF3ContributionBunkBedSameAK],{θ,0,2π}]];
 NumϕF3ϕF3ContributionBunkBedOppositeAK=
     Simplify[Integrate[Numerator[ϕF3ϕF3ContributionBunkBedOppositeAK],{θ,0,2π}]];
 NumϕF3Extϕ2F2ContributionBunkBedAK=
     Integrate[Numerator[ϕF3Extϕ2F2ContributionBunkBedAK],{θ,0,2π}];
 NumϕF3ϕ2F2ExtContributionBunkBedAK=
     Integrate[Numerator[ϕF3ϕ2F2ExtContributionBunkBedAK],{θ,0,2π}];
 FullF4F4ContributionOGSlimFishAK=NumF4F4ContributionOGSlimFishAK/
      Simplify[Denominator[F4F4ContributionOGSlimFishAK]];
   FullF4F4ContributionOGWideFishNϕAK=NumF4F4ContributionOGWideFishNϕAK/
      Simplify[Denominator[F4F4ContributionOGWideFishNϕAK]];
   FullF4F4ContributionOGWideFishWϕ1AK=NumF4F4ContributionOGWideFishWϕ1AK/
      Simplify[Denominator[F4F4ContributionOGWideFishWϕ1AK]];
   FullF4F4ContributionOGWideFishWϕ2AK=NumF4F4ContributionOGWideFishWϕ2AK/
      Simplify[Denominator[F4F4ContributionOGWideFishWϕ2AK]];
   FullF4F4ContributionOGBunkBedSameAK=NumF4F4ContributionOGBunkBedSameAK/
      Simplify[Denominator[F4F4ContributionOGBunkBedSameAK]];
   FullF4F4ContributionOGBunkBedOppositeAK=NumF4F4ContributionOGBunkBedOppositeAK/
      Simplify[Denominator[F4F4ContributionOGBunkBedOppositeAK]];
   FullF4ϕF3ContributionNEEWideFishNϕAK=NumF4ϕF3ContributionNEEWideFishNϕAK/
      Simplify[Denominator[F4ϕF3ContributionNEEWideFishNϕAK]];
   FullF4ϕF3ContributionNEEWideFishWϕ1AK=NumF4ϕF3ContributionNEEWideFishWϕ1AK/
      Simplify[Denominator[F4ϕF3ContributionNEEWideFishWϕ1AK]];
   FullF4ϕF3ContributionNEEWideFishWϕ2AK=NumF4ϕF3ContributionNEEWideFishWϕ2AK/
      Simplify[Denominator[F4ϕF3ContributionNEEWideFishWϕ2AK]];
   FullF4ϕF3ContributionNEEBunkBedNEϕSameAK=
     NumF4ϕF3ContributionNEEBunkBedNEϕSameAK/
      Simplify[Denominator[F4ϕF3ContributionNEEBunkBedNEϕSameAK]];
   FullF4ϕF3ContributionNEEBunkBedNEϕOppositeAK=
     NumF4ϕF3ContributionNEEBunkBedNEϕOppositeAK/
      Simplify[Denominator[F4ϕF3ContributionNEEBunkBedNEϕOppositeAK]];
   FullF4ϕF3ContributionNEEBunkBedEϕSameAK=NumF4ϕF3ContributionNEEBunkBedEϕSameAK/
      Simplify[Denominator[F4ϕF3ContributionNEEBunkBedEϕSameAK]];
```

```
FullF4ϕF3ContributionNEEBunkBedEϕOppositeAK=
  NumF4ϕF3ContributionNEEBunkBedEϕOppositeAK/
   Simplify[Denominator[F4ϕF3ContributionNEEBunkBedEϕOppositeAK]];
FullF4ϕ2F2ContributionNEEWideFishNϕAK=NumF4ϕ2F2ContributionNEEWideFishNϕAK/
   Simplify[Denominator[F4ϕ2F2ContributionNEEWideFishNϕAK]];
FullF4ϕ2F2ContributionNEEWideFishWϕAK=NumF4ϕ2F2ContributionNEEWideFishWϕAK/
   Simplify[Denominator[F4ϕ2F2ContributionNEEWideFishWϕAK]];

FullF4ϕ2F2ContributionNEEBunkBedNEϕAKForIso=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕAK/
   Simplify[Denominator[F4ϕ2F2ContributionNEEBunkBedNEϕAK]];
FullF4ϕ2F2ContributionNEEBunkBedEϕAKForIso=NumF4ϕ2F2ContributionNEEBunkBedEϕAK/
   Simplify[Denominator[F4ϕ2F2ContributionNEEBunkBedEϕAK]];
FullϕF3ϕF3ContributionBunkBedSameAKForIso=NumϕF3ϕF3ContributionBunkBedSameAK/
   Simplify[Denominator[ϕF3ϕF3ContributionBunkBedSameAK]];
FullϕF3ϕF3ContributionBunkBedOppositeAKForIso=
  NumϕF3ϕF3ContributionBunkBedOppositeAK/
   Simplify[Denominator[ϕF3ϕF3ContributionBunkBedOppositeAK]];
FullϕF3Extϕ2F2ContributionBunkBedAKForIso=NumϕF3Extϕ2F2ContributionBunkBedAK/
   Simplify[Denominator[ϕF3Extϕ2F2ContributionBunkBedAK]];
FullϕF3ϕ2F2ExtContributionBunkBedAKForIso=NumϕF3ϕ2F2ExtContributionBunkBedAK/
   Simplify[Denominator[ϕF3ϕ2F2ExtContributionBunkBedAK]];


D11F4ϕ2F2ContributionNEEBunkBedNEϕAK=Simplify[Distribute[(*dc*)
    I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
     ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
D22F4ϕ2F2ContributionNEEBunkBedNEϕAK=
  Simplify[Distribute[(*dc*)(Df p1^2 (p1^2+σ)+ⅈ Ω1)*(*dc*)((*dc*)Ω1-
       (1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ
                DDYuμ-DDYuμ^2) p1^2)))^0((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
```

```
(Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
D32F4ϕ2F2ContributionNEEBunkBedNEϕAK=Simplify[
   Distribute[(*dc*)(Df p1^2 (p1^2+σ)+ⅈ Ω1) ((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))*
  (*dc*)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0*
  (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))]];
Full11F4ϕ2F2ContributionNEEBunkBedNEϕAK=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕAK/D11F4ϕ2F2ContributionNEEBunkBedNEϕAK;
Full22F4ϕ2F2ContributionNEEBunkBedNEϕAK=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕAK/D22F4ϕ2F2ContributionNEEBunkBedNEϕAK;
Full32F4ϕ2F2ContributionNEEBunkBedNEϕAK=
  NumF4ϕ2F2ContributionNEEBunkBedNEϕAK/D32F4ϕ2F2ContributionNEEBunkBedNEϕAK;


Full11F4ϕ2F2ContributionNEEBunkBedNEϕAKRes=
  2*π*I*Full11F4ϕ2F2ContributionNEEBunkBedNEϕAK/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22F4ϕ2F2ContributionNEEBunkBedNEϕAKRes=
 2*π*I*Full22F4ϕ2F2ContributionNEEBunkBedNEϕAK/.
 {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full32F4ϕ2F2ContributionNEEBunkBedNEϕAKRes=
 2*π*I*Full32F4ϕ2F2ContributionNEEBunkBedNEϕAK/.
 {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullF4ϕ2F2ContributionNEEBunkBedNEϕAKRes=
 Full11F4ϕ2F2ContributionNEEBunkBedNEϕAKRes+
  Full22F4ϕ2F2ContributionNEEBunkBedNEϕAKRes+
  Full32F4ϕ2F2ContributionNEEBunkBedNEϕAKRes;
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D12F4ϕ2F2ContributionNEEBunkBedEϕAK=
 Simplify[(*dc*) (-I*Df p1^2 (p1^2+σ)+ Ω1)^0((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D21F4ϕ2F2ContributionNEEBunkBedEϕAK=Simplify[(*dc*)
  (I*Df p1^2 (p1^2+σ)+ Ω1)^0(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D31F4ϕ2F2ContributionNEEBunkBedEϕAK=
 Simplify[(*dc*) (I*Df p1^2 (p1^2+σ)+ Ω1)^0(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+
        3 DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   (I*Df p1^2 (p1^2+σ)+ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
Full12F4ϕ2F2ContributionNEEBunkBedEϕAK=
 NumF4ϕ2F2ContributionNEEBunkBedEϕAK/D12F4ϕ2F2ContributionNEEBunkBedEϕAK;
Full21F4ϕ2F2ContributionNEEBunkBedEϕAK=
 NumF4ϕ2F2ContributionNEEBunkBedEϕAK/D21F4ϕ2F2ContributionNEEBunkBedEϕAK;
Full31F4ϕ2F2ContributionNEEBunkBedEϕAK=
 NumF4ϕ2F2ContributionNEEBunkBedEϕAK/D31F4ϕ2F2ContributionNEEBunkBedEϕAK;
```

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Full12F4ϕ2F2ContributionNEEBunkBedEϕAKRes=
 0*2*π*I*D[Full12F4ϕ2F2ContributionNEEBunkBedEϕAK,{Ω1,1}]/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full21F4ϕ2F2ContributionNEEBunkBedEϕAKRes=
 2*π*I*Full21F4ϕ2F2ContributionNEEBunkBedEϕAK/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full31F4ϕ2F2ContributionNEEBunkBedEϕAKRes=
 2*π*I*Full31F4ϕ2F2ContributionNEEBunkBedEϕAK/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullF4ϕ2F2ContributionNEEBunkBedEϕAKRes=
 (*Full12F4ϕ2F2ContributionNEEBunkBedEϕAKRes+*)
 Full21F4ϕ2F2ContributionNEEBunkBedEϕAKRes+
  Full31F4ϕ2F2ContributionNEEBunkBedEϕAKRes;


D12ϕF3ϕF3ContributionBunkBedSameAK=
 Simplify[(*dc^2*)1*I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+
        3 DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D21ϕF3ϕF3ContributionBunkBedSameAK=Simplify[(*dc^2*)
  1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)*(*dc*)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
D31ϕF3ϕF3ContributionBunkBedSameAK=
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Simplify[(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))*
   (*dc*)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2 ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
Full12ϕF3ϕF3ContributionBunkBedSameAK=
 NumϕF3ϕF3ContributionBunkBedSameAK/D12ϕF3ϕF3ContributionBunkBedSameAK;
Full21ϕF3ϕF3ContributionBunkBedSameAK=
 NumϕF3ϕF3ContributionBunkBedSameAK/D21ϕF3ϕF3ContributionBunkBedSameAK;
Full31ϕF3ϕF3ContributionBunkBedSameAK=
 NumϕF3ϕF3ContributionBunkBedSameAK/D31ϕF3ϕF3ContributionBunkBedSameAK;


Full12ϕF3ϕF3ContributionBunkBedSameAKRes=
 2*π*I*Full12ϕF3ϕF3ContributionBunkBedSameAK/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full21ϕF3ϕF3ContributionBunkBedSameAKRes=
 2*π*I*Full21ϕF3ϕF3ContributionBunkBedSameAK/.{Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
       ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full31ϕF3ϕF3ContributionBunkBedSameAKRes=
 2*π*I*Full31ϕF3ϕF3ContributionBunkBedSameAK/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
       Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullϕF3ϕF3ContributionBunkBedSameAKRes=Full12ϕF3ϕF3ContributionBunkBedSameAKRes+
  Full21ϕF3ϕF3ContributionBunkBedSameAKRes+
  Full31ϕF3ϕF3ContributionBunkBedSameAKRes;

D12ϕF3ϕF3ContributionBunkBedOppositeAK=Simplify[
  -I*(*dc^2*)1*(I^2)(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
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   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0];
D22ϕF3ϕF3ContributionBunkBedOppositeAK=
 Simplify[-I*(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^2*
   ((*dc^2*)1)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0];
D32ϕF3ϕF3ContributionBunkBedOppositeAK=Simplify[
  -I*(*dc^2*)1*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^2*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2*
   ((*dc^2*)1)((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0];
Full12ϕF3ϕF3ContributionBunkBedOppositeAK=
 NumϕF3ϕF3ContributionBunkBedOppositeAK/D12ϕF3ϕF3ContributionBunkBedOppositeAK;
Full22ϕF3ϕF3ContributionBunkBedOppositeAK=
 NumϕF3ϕF3ContributionBunkBedOppositeAK/D22ϕF3ϕF3ContributionBunkBedOppositeAK;
Full32ϕF3ϕF3ContributionBunkBedOppositeAK=
 NumϕF3ϕF3ContributionBunkBedOppositeAK/D32ϕF3ϕF3ContributionBunkBedOppositeAK;


Full12ϕF3ϕF3ContributionBunkBedOppositeAKRes=
 2*π*I*D[Full12ϕF3ϕF3ContributionBunkBedOppositeAK,{Ω1,1}]/.
  {Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22ϕF3ϕF3ContributionBunkBedOppositeAKRes=
 2*π*I*D[Full22ϕF3ϕF3ContributionBunkBedOppositeAK,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
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      ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)}};
Full32ϕF3ϕF3ContributionBunkBedOppositeAKRes=
 2*π*I*D[Full32ϕF3ϕF3ContributionBunkBedOppositeAK,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)};
FullϕF3ϕF3ContributionBunkBedOppositeAKRes=
 Full12ϕF3ϕF3ContributionBunkBedOppositeAKRes+
  Full22ϕF3ϕF3ContributionBunkBedOppositeAKRes+
  Full32ϕF3ϕF3ContributionBunkBedOppositeAKRes;


D11ϕF3Extϕ2F2ContributionBunkBedAK=
 Simplify[(*dc*)I*(Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D22ϕF3Extϕ2F2ContributionBunkBedAK=Simplify[(*dc*)
  (Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc^2*)1)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   (Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))
   ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))];
D32ϕF3Extϕ2F2ContributionBunkBedAK=Simplify[(*dc*)
  (Df p1^2 (p1^2+σ)+ⅈ Ω1)*((*dc^2*)1)((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
          ⅈ √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^2
   ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
          √(4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2) p1^2)))^0
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(Df p1^2 (p1^2+σ)-ⅈ Ω1) ((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))];
Full11ϕF3Extϕ2F2ContributionBunkBedAK=
 NumϕF3Extϕ2F2ContributionBunkBedAK/D11ϕF3Extϕ2F2ContributionBunkBedAK;
Full22ϕF3Extϕ2F2ContributionBunkBedAK=
 NumϕF3Extϕ2F2ContributionBunkBedAK/D22ϕF3Extϕ2F2ContributionBunkBedAK;
Full32ϕF3Extϕ2F2ContributionBunkBedAK=
 NumϕF3Extϕ2F2ContributionBunkBedAK/D32ϕF3Extϕ2F2ContributionBunkBedAK;


Full11ϕF3Extϕ2F2ContributionBunkBedAKRes=
 2*π*I*Full11ϕF3Extϕ2F2ContributionBunkBedAK/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full22ϕF3Extϕ2F2ContributionBunkBedAKRes=
 2*π*I*D[Full22ϕF3Extϕ2F2ContributionBunkBedAK,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full32ϕF3Extϕ2F2ContributionBunkBedAKRes=
 2*π*I*D[Full32ϕF3Extϕ2F2ContributionBunkBedAK,{Ω1,1}]/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
FullϕF3Extϕ2F2ContributionBunkBedAKRes=Full11ϕF3Extϕ2F2ContributionBunkBedAKRes+
  Full22ϕF3Extϕ2F2ContributionBunkBedAKRes+
  Full32ϕF3Extϕ2F2ContributionBunkBedAKRes;


D12ϕF3ϕ2F2ExtContributionBunkBedAK=
 Simplify[-(*dc*) (Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2];
D21ϕF3ϕ2F2ExtContributionBunkBedAK=Simplify[
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-(*dc*) (Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*(*dc*) ((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2];
D31ϕF3ϕ2F2ExtContributionBunkBedAK=
 Simplify[-(*dc*) (Df p1^2 (p1^2+σ)+ⅈ Ω1)^0*(*dc*) ((*dc*)Ω1-(1/2 ⅈ (DDYuλ p1^2+
      3 DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))
  ((*dc*)Ω1-(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^0
  (Df p1^2 (p1^2+σ)-ⅈ Ω1)^2((*dc*)Ω1+(1/2 ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2
  ((*dc*)Ω1+(1/2 (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)))^2];
Full12ϕF3ϕ2F2ExtContributionBunkBedAK=
 NumϕF3ϕ2F2ExtContributionBunkBedAK/D12ϕF3ϕ2F2ExtContributionBunkBedAK;
Full21ϕF3ϕ2F2ExtContributionBunkBedAK=
 NumϕF3ϕ2F2ExtContributionBunkBedAK/D21ϕF3ϕ2F2ExtContributionBunkBedAK;
Full31ϕF3ϕ2F2ExtContributionBunkBedAK=
 NumϕF3ϕ2F2ExtContributionBunkBedAK/D31ϕF3ϕ2F2ExtContributionBunkBedAK;


Full12ϕF3ϕ2F2ExtContributionBunkBedAKRes=
 0*2*π*I*D[Full12ϕF3ϕ2F2ExtContributionBunkBedAK,Ω1]/.{Ω1→ⅈ Df p1^2 (p1^2+σ)};
Full21ϕF3ϕ2F2ExtContributionBunkBedAKRes=
 2*π*I*Full21ϕF3ϕ2F2ExtContributionBunkBedAK/.{Ω1→1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
      ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
Full31ϕF3ϕ2F2ExtContributionBunkBedAKRes=
 2*π*I*Full31ϕF3ϕ2F2ExtContributionBunkBedAK/.
  {Ω1→1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
      Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)};
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FullϕF3ϕ2F2ExtContributionBunkBedAKRes=Full12ϕF3ϕ2F2ExtContributionBunkBedAKRes+
  Full21ϕF3ϕ2F2ExtContributionBunkBedAKRes+
  Full31ϕF3ϕ2F2ExtContributionBunkBedAKRes;
ResF4F4ContributionOGSlimFishAK=
 2*π*I*(Residue[FullF4F4ContributionOGSlimFishAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishNϕAK=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishNϕAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishWϕ1AK=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishWϕ1AK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGWideFishWϕ2AK=
 2*π*I*(Residue[FullF4F4ContributionOGWideFishWϕ2AK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGBunkBedSameAK=
 2*π*I*(Residue[FullF4F4ContributionOGBunkBedSameAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);
ResF4F4ContributionOGBunkBedOppositeAK=2*π*I*
  (Residue[FullF4F4ContributionOGBunkBedOppositeAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]);

ResF4ϕF3ContributionNEEWideFishNϕAK=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishNϕAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishNϕAK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
          p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishNϕAK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
          DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕF3ContributionNEEWideFishWϕ1AK=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishWϕ1AK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ1AK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
          p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ1AK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
          DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕF3ContributionNEEWideFishWϕ2AK=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEWideFishWϕ2AK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ2AK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
          p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEWideFishWϕ2AK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
          DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕF3ContributionNEEBunkBedNEϕSameAK=
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 2*π*I*(Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameAK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
         DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕSameAK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
         DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕF3ContributionNEEBunkBedNEϕOppositeAK=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeAK,
     {Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ p1^2+
        ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedNEϕOppositeAK,
     {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕF3ContributionNEEBunkBedEϕSameAK=
 2*π*I*(Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameAK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
         DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedEϕSameAK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
         DDYuμ p1^2+Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);
ResF4ϕF3ContributionNEEBunkBedEϕOppositeAK=2*π*I*
  (Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeAK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3
         DDYuμ p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕF3ContributionNEEBunkBedEϕOppositeAK,
     {Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ DDYuμ p1^2+
        Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]);

ResF4ϕ2F2ContributionNEEWideFishNϕAK=
 2*π*I*(Residue[FullF4ϕ2F2ContributionNEEWideFishNϕAK,{Ω1,ⅈ Df*p1^2 (p1^2 κ+σ)}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishNϕAK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1^2+3 DDYuμ
         p1^2+ⅈ Sqrt[4 DDYuA DDYuK+4 DDYuK^2-DDYuλ^2-2 DDYuλ DDYuμ-DDYuμ^2] p1^2)}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishNϕAK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1^2+3 ⅈ
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            DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²)}]);
ResF4ϕ2F2ContributionNEEWideFishWϕAK=
 2*π*I*(Residue[FullF4ϕ2F2ContributionNEEWideFishWϕAK,{Ω1,ⅈ Df*p1² (p1² κ+σ)}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishWϕAK,{Ω1,1/(2(*dc*)) ⅈ (DDYuλ p1²+3 DDYuμ
          p1²+ⅈ √(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²))}]+
    Residue[FullF4ϕ2F2ContributionNEEWideFishWϕAK,{Ω1,1/(2(*dc*)) (ⅈ DDYuλ p1²+3 ⅈ
          DDYuμ p1²+√(4 DDYuA DDYuK+4 DDYuK²-DDYuλ²-2 DDYuλ DDYuμ-DDYuμ² p1²))}]);
ContDfAKϕf3=
 2*(1/(2π)^3)(ResF4F4ContributionOGSlimFishAK+ResF4F4ContributionOGWideFishNϕAK+
    ResF4F4ContributionOGWideFishWϕ1AK+ResF4F4ContributionOGWideFishWϕ2AK+
    ResF4F4ContributionOGBunkBedSameAK+ResF4F4ContributionOGBunkBedOppositeAK+
    (ResF4ϕF3ContributionNEEWideFishNϕAK+ResF4ϕF3ContributionNEEWideFishWϕ1AK+
      ResF4ϕF3ContributionNEEWideFishWϕ2AK+
      ResF4ϕF3ContributionNEEBunkBedNEϕSameAK+
      ResF4ϕF3ContributionNEEBunkBedNEϕOppositeAK+
      ResF4ϕF3ContributionNEEBunkBedEϕSameAK+
      ResF4ϕF3ContributionNEEBunkBedEϕOppositeAK+
      ResF4ϕ2F2ContributionNEEWideFishNϕAK+ResF4ϕ2F2ContributionNEEWideFishWϕAK+
      (FullF4ϕ2F2ContributionNEEBunkBedEϕAKRes+
        FullF4ϕ2F2ContributionNEEBunkBedNEϕAKRes+
        (FullϕF3ϕ2F2ExtContributionBunkBedAKRes+
          FullϕF3Extϕ2F2ContributionBunkBedAKRes+
          FullϕF3ϕF3ContributionBunkBedOppositeAKRes+
          FullϕF3ϕF3ContributionBunkBedSameAKRes))));*)
```

In[◦]:=

```
F4F4ContributionOGSlimFishAK =
  Simplify[Together[((1 / 2) D[(F4F4ContributionOGSlimFish /.
          {p1[1] → p1 * Cos[θ], p1[2] → p1 * Sin[θ], Ω3 → 0, Ω2 → 0, ω → 0,
           p3[1] → p3, p3[2] → 0, p2[1] → p2, p2[2] → 0, q[1] → 0, q[2] → q}),
        {p3, 1}, {p2, 2}, {q, 1}] /. {p3 → 0, q → 0, p2 → 0})]];
NumF4F4ContributionOGSlimFishAK =
  Simplify[Integrate[Numerator[F4F4ContributionOGSlimFishAK], {θ, 0, 2 π}]];
FullF4F4ContributionOGSlimFishAK = NumF4F4ContributionOGSlimFishAK /
   Simplify[Denominator[F4F4ContributionOGSlimFishAK]];
ResF4F4ContributionOGSlimFishAK =
  2 * π * I * (Residue[FullF4F4ContributionOGSlimFishAK, {Ω1, ⅈ Df * p1² (p1² κ + σ)}]);
ContDfAKϕf3 = 2 * (1 / (2 π) ^3) (ResF4F4ContributionOGSlimFishAK);
```

Simplifying the Expressions and Replacing Diffusion of Lambda Second Version

Adjust this one Simplify[(ContDfAKϕf3+ContDfKϕf3)]

```
In[∘ ]:= ContDfλμC = - (1 / I) ContDfλμ;
ContDfμC = (1 / I) ContDfμ;
ContDfKC = - (1 / I) ContDfK;
ContDfAC = (1 / I) (ContDfAK - ContDfK);
```

```
In[∘ ]:= ContDfλμϕf3C = -ContDfλμϕf3;
ContDfμϕf3C = -ContDfμϕf3;
ContDfKϕf3C = ContDfKϕf3;
ContDfAϕf3C = (ContDfAKϕf3 + ContDfKϕf3);
```

```
In[∘ ]:= ContDλμYFFC = (- (1 / I) ContDλμYFF);
ContDμYFFC = ((1 / I) ContDμYFF);
ContDAYFFC = (-1 / I) (ContDAKYFF - ContDKYFF);
ContDKYFFC = ((1 / I) ContDKYFF);
```

```
In[∘ ]:= ContλμC = Contλμ;
ContμC = Contμ;
ContAC = ContA;
ContKC = ContK;
```

```
In[∘ ]:= F4BFinalC = F4BFinal;
F4SFinalC = F4SFinal;
```

Numerical Renormalization Functions

```
In[∘ ]:= Clear[Λ, dc]
```

```
In[∘ ]:= F4BFinalF[x] = (q^2 F4BFinalC) /. {Df → DfR[x], DDYuμ → DDYuμR[x], DDYf2μ → DDYf2μR[x],
    Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x], DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x],
    Dfϕufλ → DfϕufλR[x], Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
    DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
    DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
    σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ};
```

Stress contribution

```
In[∘ ]:= StressCont[x] = (q^2(*(1/Df)*) (F4SFinalC)) /.
   {Df → DfR[x], DDYuμ → DDYuμR[x], DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x],
    Dfϕf3μ → Dfϕf3μR[x], DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
    Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
    DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
    DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
    σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ};
```

```
In[∘ ]:= Clear[Λ, dc]
```

In[ ]:=
```
FullSimplify[StressCont[x]]
```
Out[ ]=
```
$Aborted
```
In[ ]:=
```
StressCont[x] = StressCont[x](*-(σR[x]/(DfR[x]))F4BFinalF[x]*);
```
Shear contribution

In[ ]:=
```
DfufϕShearCont[x] =
  ((((q^2(*(1/Df)*) (ContDfμC))) /. {Df → DfR[x], DDYuμ → DDYuμR[x],
       DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x],
       DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
       Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
       DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
       DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
       σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    1 * (DfϕufμR[x] / DfR[x]) F4BFinalF[x]);
Dfϕf3ShearCont[x] =
  ((((q^2(*(1/Df)*) (ContDfμϕf3C))) /. {Df → DfR[x], DDYuμ → DDYuμR[x],
       DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x],
       DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
       Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
       DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
       DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
       σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    1 * (Dfϕf3μR[x] / DfR[x]) F4BFinalF[x]);
DYuShearCont[x] =
  ((((q^2(*(1/Df)*) (ContμC))) /. {Df → DfR[x], DDYuμ → DDYuμR[x], DDYf2μ →
        DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x], DDYuλ → DDYuλR[x],
       DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x], Dfϕf3λ → Dfϕf3λR[x],
       DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x], DfϕufA → DfϕufAR[x],
       Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x], DDYf2K → DDYf2KR[x],
       DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x], σ → (σR[x] / DfR[x]),
       u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    0 * (DDYuμR[x] / DfR[x]) F4BFinalF[x]);
DYFFShearCont[x] =
  ((((q^2(*(1/Df)*) (ContDμYFFC))) /. {Df → DfR[x], DDYuμ → DDYuμR[x],
       DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x],
       DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
       Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
       DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
       DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
       σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    0 * (DDYf2μR[x] / DfR[x]) F4BFinalF[x]);
```
Lambda contribution

```
In[*]:= DfufϕLambdaCont[x] =
   (((q^2 ((*(1/Df)*)(ContDfλμC - 2 * ContDfμC))) /. {Df → DfR[x], DDYuμ → DDYuμR[x],
        DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x],
        DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
        Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
        DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
        DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
        σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
     1 * (DfϕufλR[x] / DfR[x]) F4BFinalF[x]);
Dfϕf3LambdaCont[x] =
   (((q^2 ((*(1/Df)*)(ContDfλμϕf3C - 2 * ContDfμϕf3C))) /. {Df → DfR[x],
        DDYuμ → DDYuμR[x], DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ →
         Dfϕf3μR[x], DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
        Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
        DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
        DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
        σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
     1 * (Dfϕf3λR[x] / DfR[x]) F4BFinalF[x]);
DYuLambdaCont[x] =
   (((q^2 ((*(1/Df)*)(ContλμC - 2 * ContμC))) /. {Df → DfR[x], DDYuμ → DDYuμR[x],
        DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x],
        DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
        Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
        DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
        DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
        σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
     0 * (DDYuλR[x] / DfR[x]) F4BFinalF[x]);
DYFFLambdaCont[x] =
   (((q^2 ((*(1/Df)*)(ContDλμYFFC - 2 * ContDμYFFC))) /. {Df → DfR[x],
        DDYuμ → DDYuμR[x], DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ →
         Dfϕf3μR[x], DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
        Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
        DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
        DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
        σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
     0 * (DDYf2λR[x] / DfR[x]) F4BFinalF[x]);
```

K Contribution

In[ ]:=
```
DfufϕKCont[x] =
  (((((*(1/Df)*)q^2 (ContDfKC))) /. {Df → DfR[x], DDYuμ → DDYuμR[x], DDYf2μ →
        DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x], DDYuλ → DDYuλR[x],
      DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x], Dfϕf3λ → Dfϕf3λR[x],
      DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x], DfϕufA → DfϕufAR[x],
      Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x], DDYf2K → DDYf2KR[x],
      DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x], σ → (σR[x] / DfR[x]),
      u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    1 * (DfϕufKR[x] / DfR[x]) F4BFinalF[x]);
Dfϕf3KCont[x] =
  (((((*(1/Df)*)q^2 (ContDfKϕf3C))) /. {Df → DfR[x], DDYuμ → DDYuμR[x],
      DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x],
      DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
      Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
      DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
      DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
      σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    1 * (Dfϕf3KR[x] / DfR[x]) F4BFinalF[x]);
DYuKCont[x] =
  (((((*(1/Df)*)q^2 (ContKC))) /. {Df → DfR[x], DDYuμ → DDYuμR[x], DDYf2μ →
        DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x], DDYuλ → DDYuλR[x],
      DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x], Dfϕf3λ → Dfϕf3λR[x],
      DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x], DfϕufA → DfϕufAR[x],
      Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x], DDYf2K → DDYf2KR[x],
      DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x], σ → (σR[x] / DfR[x]),
      u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    0 * (DDYuKR[x] / DfR[x]) F4BFinalF[x]);
DYFFKCont[x] =
  (((((*(1/Df)*)q^2 (ContDKYFFC))) /. {Df → DfR[x], DDYuμ → DDYuμR[x], DDYf2μ →
        DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x], DDYuλ → DDYuλR[x],
      DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x], Dfϕf3λ → Dfϕf3λR[x],
      DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x], DfϕufA → DfϕufAR[x],
      Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x], DDYf2K → DDYf2KR[x],
      DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x], σ → (σR[x] / DfR[x]),
      u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    0 * (DDYf2KR[x] / DfR[x]) F4BFinalF[x]);
```

A contribution

```
In[ ]:= DfufϕACont[x] =
  (((((*(1/Df)*) (q^2 (ContDfAC) (*+q^2(ContDfKC)*)))) /. {Df → DfR[x],
       DDYuμ → DDYuμR[x], DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ →
        Dfϕf3μR[x], DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
       Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
       DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
       DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
       σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    1 * (DfϕufAR[x] / DfR[x]) F4BFinalF[x]);
DfϕF3ACont[x] =
  (((((*(1/Df)*) (q^2 (ContDfAϕf3C) (*+q^2(ContDfKC)*)))) /. {Df → DfR[x],
       DDYuμ → DDYuμR[x], DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ →
        Dfϕf3μR[x], DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
       Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
       DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
       DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
       σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    1 * (Dfϕf3AR[x] / DfR[x]) F4BFinalF[x]);
DYuACont[x] =
  (((((*(1/Df)*) (q^2 (ContAC) (*+q^2(ContDfKC)*)))) /. {Df → DfR[x], DDYuμ →
        DDYuμR[x], DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x],
       DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
       Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
       DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
       DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
       σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    0 * (DDYuAR[x] / DfR[x]) F4BFinalF[x]);
DYFFACont[x] =
  (((((*(1/Df)*) (q^2 (ContDAYFFC) (*+q^2(ContDfKC)*)))) /. {Df → DfR[x],
       DDYuμ → DDYuμR[x], DDYf2μ → DDYf2μR[x], Dfϕufμ → DfϕufμR[x], Dfϕf3μ →
        Dfϕf3μR[x], DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x], Dfϕufλ → DfϕufλR[x],
       Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
       DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
       DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
       σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
    0 * (DDYf2AR[x] / DfR[x]) F4BFinalF[x]);
```

LL Contribution

In[◦]:=
```
LCont[x] =
  (2 * q^2 (L / Df) F4BFinalC) /. {Df → DfR[x], DDYuμ → DDYuμR[x], DDYf2μ → DDYf2μR[x],
    DfϕufμR → DfϕufμR[x], Dfϕf3μ → Dfϕf3μR[x], DDYuλ → DDYuλR[x], DDYf2λ → DDYf2λR[x],
    Dfϕufλ → DfϕufλR[x], Dfϕf3λ → Dfϕf3λR[x], DDYuA → DDYuAR[x], DDYf2A → DDYf2AR[x],
    DfϕufA → DfϕufAR[x], Dfϕf3A → Dfϕf3AR[x], DDYuK → DDYuKR[x],
    DDYf2K → DDYf2KR[x], DfϕufK → DfϕufKR[x], Dfϕf3K → Dfϕf3KR[x],
    σ → (σR[x] / DfR[x]), u → (uuR[x]), L → LLR[x], p1 → Λ, z → Λ, q → Λ, p2 → Λ};
```

Derivation of Lth Assuming Fluctuation Dissipation

```
LTHDerivation =
  (DfR + F4BFinalF[x]) /. {DDYuλR[x] → DD λ, DDYf2λR[x] → DD λ, Dfϕf3λR[x] → λ DfR,
    DfϕufλR[x] → λ DfR, DDYuμR[x] → DD μ, DDYf2μR[x] → DD μ, Dfϕf3μR[x] → μ DfR,
    DfϕufμR[x] → μ DfR, DDYuAR[x] → DD * A, DDYf2AR[x] → DD * A, Dfϕf3AR[x] → DfR * A,
    DfϕufAR[x] → DfR * A, DDYuKR[x] → DD * K, DDYf2KR[x] → DD * K, Dfϕf3KR[x] → DfR * K,
    DfϕufKR[x] → DfR * K, LLR[x] → DD, DfR[x] → DfR, σR[x] → 0};
LTHDerivationTN = Numerator[Together[LTHDerivation]];
LTHDerivationTD = Denominator[Together[LTHDerivation]];
Simplify[Solve[Simplify[Coefficient[LTHDerivationTN, Λ^2]] * Λ^2 +
    Simplify[LTHDerivationTN /. {Λ → 0}] == 0, Λ]]
```

In[◦]:=
```
Simplify[Solve[Simplify[Coefficient[LTHDerivationTN, Λ^26]] * Λ^2 +
    Simplify[Coefficient[LTHDerivationTN, Λ^24]] == 0, Λ],
 Assumptions → {μ > 0, λ > 0, A > 0, K > 0}]
```

Run Numerical Code

In[◦]:=
```
Clear[Λ, κo, dc]
```

```
(*Clear[Λ,κo,dc]
  kBT=10^(0);
dc=1;
κo=(*(1/kBT)*)1;
μo=(*(1/kBT)*)kBT*(1/κo^2)*1;
λo=(*(1/kBT)*)kBT*(1/κo^2)*1;
Ao=(*(1/kBT)*)kBT*(1/κo^2)*1/10;
Ko=(*(1/kBT)*)kBT*(1/κo^2)*0*1/10000;
σo=-1000/(400005*κo);
LLo=((1/kBT)^2)*κo^2;
DDAo=(*-*)(1/kBT)Ao*κo^2;
DDKo=(*-*)(1/kBT)Ko*κo^2;
DDλμo=(1/kBT)(λo)*κo^2;
DDμo=(1/kBT)μo*κo^2;
Dfo=κo;*)
```

```
(*Clear[Λ,κo,dc]
  kBT=10^(0);
dc=1;
κo=(*(1/kBT)*)1;
μo=(*(1/kBT)*)kBT*(1/κo^2)*1;
λo=(*(1/kBT)*)kBT*(1/κo^2)*1;
Ao=(*(1/kBT)*)kBT*(1/κo^2)*0*1/10;
Ko=(*(1/kBT)*)kBT*(1/κo^2)*1/10;
σo=97/(10000*κo);
LLo=((1/kBT)^2)*κo^2;
DDAo=(*-*)(1/kBT)Ao*κo^2;
DDKo=(*-*)(1/kBT)Ko*κo^2;
DDλμo=(1/kBT)(λo)*κo^2;
DDμo=(1/kBT)μo*κo^2;
Dfo=κo;*)
```

In[ ]:=
```
Clear[Λ, κo, dc]
kBT = 10^(0);
dc = 1;
κo = (*(1/kBT)*)1;
μo = (*(1/kBT)*)kBT * (1 / κo^2) * 1;
λo = (*(1/kBT)*)kBT * (1 / κo^2) * 1;
Ao = (*(1/kBT)*)kBT * (1 / κo^2) * 0 * 1 / 10;
Ko = (*(1/kBT)*)kBT * (1 / κo^2) * 0 / 10;
σo = 0 / (10 000 * κo);
LLo = ((1 / kBT)^2) * κo^2;
DDAo = (*-*)(1 / kBT) Ao * κo^2;
DDKo = (*-*)(1 / kBT) Ko * κo^2;
DDλμo = (1 / kBT) (λo) * κo^2;
DDμo = (1 / kBT) μo * κo^2;
Dfo = κo;

(*Clear[Λ,κo,dc]
  kBT=10^(0);
dc=1;
κo=(*(1/kBT)*)1;
μo=(*(1/kBT)*)kBT*(1/κo^2)*1;
λo=(*(1/kBT)*)kBT*(1/κo^2)*1;
Ao=(*(1/kBT)*)kBT*(1/κo^2)*0*1/10;
Ko=(*(1/kBT)*)kBT*(1/κo^2)*1/10;
σo=57/(10000*κo);
LLo=((1/kBT)^2)*κo^2;
DDAo=(*-*)(1/kBT)Ao*κo^2;
DDKo=(*-*)(1/kBT)Ko*κo^2;
DDλμo=(1/kBT)(λo)*κo^2;
DDμo=(1/kBT)μo*κo^2;
Dfo=κo;*)
```

In[•]:=
```
DDYuAo = DDAo;
DDYf2Ao = DDAo;
DfϕufAo = Dfo * Ao;
Dfϕf3Ao = Dfo * Ao;
DDYuKo = DDKo;
DDYf2Ko = DDKo;
DfϕufKo = Dfo * Ko;
Dfϕf3Ko = Dfo * Ko;
DDYuλμo = 10 * DDλμo;
DDYf2λμo = DDλμo;
Dfϕufλμo = Dfo * (λo);
Dfϕf3λμo = Dfo * (λo);
DDYuμo = DDμo;
DDYf2μo = DDμo;
Dfϕufμo = Dfo * (μo);
Dfϕf3μo = Dfo * (μo);
Dfo = Dfo;
LLo = LLo;
σo = Dfo * σo;
```

In[•]:=
```
WP = 30;
```
(*lth= $\frac{2\ \pi^{3/2}\ \sqrt{\kappa o}}{\sqrt{3}\ \sqrt{\frac{\mu o\ (\lambda o+\mu o)}{\kappa o\ (\lambda o+2\ \mu o)}}}$;*)
```
lth = π / ((√ (4 Ao^4 Ko^2 - 16 Ko^6 - 3 μo^2 (λo + μo)^3 (λo + 3 μo) -
        Ao^3 Ko (4 Ko^2 + λo^2 - 2 λo μo - 19 μo^2) - 8 Ko^4 (λo^2 + λo μo + 2 μo^2) +
        Ko^2 (3 λo^4 + 10 λo^3 μo + 28 λo^2 μo^2 + 62 λo μo^3 + 41 μo^4) -
        Ao^2 (48 Ko^4 + μo (λo + μo)^2 (λo + 5 μo) - 2 Ko^2 (λo^2 + 4 λo μo - μo^2)) +
        Ao (-56 Ko^5 + 6 Ko μo^2 (3 λo^2 + 10 λo μo + 7 μo^2) -
          2 Ko^3 (λo^2 - 2 λo μo + 17 μo^2)))) /
    (2 Sqrt[π] √ ((-λo - 3 μo) (-4 Ao^2 Ko^2 - 4 Ko^4 + μo (λo + μo)^2 (λo + 2 μo) +
          Ko^2 (λo^2 - 2 λo μo - 7 μo^2) + Ao Ko (-8 Ko^2 + λo^2 - 2 λo μo - 7 μo^2))));
(*This definition uses lth derived assuming
 fluctuation
 dissipation*)
ao = 10^-6 lth;
LR = 10^12 lth;
Λ = Pi / ao;
xmax = Log[LR / ao];
ξf = 1;
ξu = 1;
ξtf = 4;
ξtu = 4;
ξx = 1;
```

```
ξP = -3;
ξY = -3;
dim = 2;
N[Ko^2 + Ao * Ko + μo (λo + 3 μo) / 2](*>0  stability condition for intial moduli*)
Timing[s = NDSolve[{
    DDYuAR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYuAR[x] - (DYuACont[x]),
    DDYf2AR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYf2AR[x] - (DYFFACont[x]),
    DfϕufAR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DfϕufAR[x] - (DfufϕACont[x]),
    Dfϕf3AR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * Dfϕf3AR[x] - (Dfϕf3ACont[x]),

    DDYuKR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYuKR[x] - (DYuKCont[x]),
    DDYf2KR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYf2KR[x] - (DYFFKCont[x]),
    DfϕufKR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DfϕufKR[x] - (DfufϕKCont[x]),
    Dfϕf3KR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * Dfϕf3KR[x] - (Dfϕf3KCont[x]),


    DDYuλR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYuλR[x] - (DYuLambdaCont[x]),
    DDYf2λR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYf2λR[x] - (DYFFLambdaCont[x]),
    DfϕufλR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DfϕufλR[x] - (DfufϕLambdaCont[x]),
    Dfϕf3λR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * Dfϕf3λR[x] - (Dfϕf3LambdaCont[x]),


    DDYuμR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYuμR[x] - (DYuShearCont[x]),
    DDYf2μR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYf2μR[x] - (DYFFShearCont[x]),
    DfϕufμR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DfϕufμR[x] - (DfufϕShearCont[x]),
    Dfϕf3μR'[x] == (ξY + ξu - 2 ξx + 2 ξx + ξtu) * Dfϕf3μR[x] - (Dfϕf3ShearCont[x]),

    σR'[x] == ((2 ξx + ξtf - 2 ξx + ξP + ξf) * σR[x] - (StressCont[x])),
    LLR'[x] == (ξu + ξY + 2 ξx) * LLR[x] - LCont[x],
    DfR'[x] == (2 ξP + 2 ξx + ξtf) * DfR[x] - (F4BFinalF[x]),
    DDYuAR[0] == DDYuAo, DDYf2AR[0] == DDYf2Ao, DfϕufAR[0] == DfϕufAo,
    Dfϕf3AR[0] == Dfϕf3Ao, DDYuKR[0] == DDYuKo, DDYf2KR[0] == DDYf2Ko,
    DfϕufKR[0] == DfϕufKo, Dfϕf3KR[0] == Dfϕf3Ko, DDYuλR[0] == DDYuλμo,
    DDYf2λR[0] == DDYf2λμo, DfϕufλR[0] == Dfϕufλμo, Dfϕf3λR[0] == Dfϕf3λμo,
    DDYuμR[0] == DDYuμo, DDYf2μR[0] == DDYf2μo, DfϕufμR[0] == Dfϕufμo,
    Dfϕf3μR[0] == Dfϕf3μo, DfR[0] == Dfo, LLR[0] == LLo, σR[0] == σo},
   {DDYuAR, DDYf2AR, DfϕufAR, Dfϕf3AR, DDYuKR, DDYf2KR, DfϕufKR, Dfϕf3KR, DDYuλR,
    DDYf2λR, DfϕufλR, Dfϕf3λR, DDYuμR, DDYf2μR, DfϕufμR, Dfϕf3μR, DfR, LLR, σR},
   {x, 0, xmax}, WorkingPrecision → WP, Method → {"EquationSimplification" → "Solve"}
   (*,Method→{"StiffnessSwitching"}*)(*,MaxSteps→10^6*)];]
```

Out[ ]=

```
2.01
```

Out[ ]=

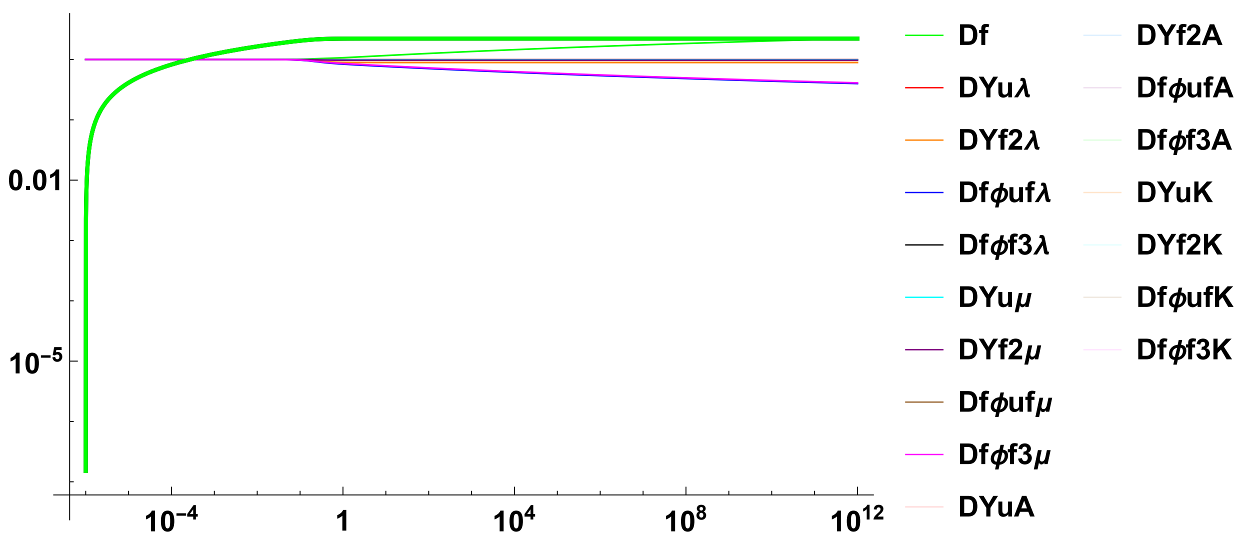
```
{81.3194, Null}
```

In[ ]:=

```
LogLogPlot[{(E^(-(ξP + ξf - 4 ξx + dim * ξx + ξtf) * Log[x * lth / ao]))
    Evaluate[DfR[Log[x * lth / ao]] /. s] / Evaluate[DfR[0] /. s],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DDYuλR[Log[x * lth / ao]] /. s] / Evaluate[DDYuλR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DDYf2λR[Log[x * lth / ao]] /. s] / Evaluate[DDYf2λR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DfϕufλR[Log[x * lth / ao]] /. s] / Evaluate[DfϕufλR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[Dfϕf3λR[Log[x * lth / ao]] /. s] / Evaluate[Dfϕf3R[0] /. s]]],
  (E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
   Abs[Evaluate[DDYuμR[Log[x * lth / ao]] /. s] / Evaluate[DDYuμR[0] /. s]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DDYf2μR[Log[x * lth / ao]] /. s] / Evaluate[DDYf2μR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DfϕufμR[Log[x * lth / ao]] /. s] / Evaluate[DfϕufR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[Dfϕf3μR[Log[x * lth / ao]] /. s] / Evaluate[Dfϕf3μR[0] /. s]]],
  (E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
   Abs[Evaluate[DDYuAR[Log[x * lth / ao]] /. s] / Evaluate[DDYuμR[0] /. s]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DDYf2AR[Log[x * lth / ao]] /. s] / Evaluate[DDYuμR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DfϕufAR[Log[x * lth / ao]] /. s] / Evaluate[DfϕufμR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[Dfϕf3AR[Log[x * lth / ao]] /. s] / Evaluate[DfϕufμR[0] /. s]]],
  (E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
   Abs[Evaluate[DDYuKR[Log[x * lth / ao]] /. s] / Evaluate[DDYuμR[0] /. s]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DDYf2KR[Log[x * lth / ao]] /. s] / Evaluate[DDYuμR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[DfϕufKR[Log[x * lth / ao]] /. s] / Evaluate[DfϕufμR[0] /. s]]],
  Abs[(E^(-(ξY + ξu - 2 ξx + 2 ξx + ξtu) * Log[x * lth / ao]))
    Abs[Evaluate[Dfϕf3KR[Log[x * lth / ao]] /. s] / Evaluate[DfϕufμR[0] /. s]]],
  (*(((E^(-(ξP+ξf-4ξx+dim*ξx+ξtf)*Log[x*lth/ao]))
      Evaluate[DfR[Log[x*lth/ao]]/.s]/Evaluate[DfR[0]/.s])^(1))*)
  (E^((-(2 ξx + ξtf - 2 ξx + ξP + ξf)) * Log[x * lth / ao]))
   (Abs[Evaluate[σR[Log[x * lth / ao]] /. s]]),
  (*(((E^(-(ξP+ξf-4ξx+dim*ξx+ξtf)*Log[x*lth/ao]))
      Evaluate[DfR[Log[x*lth/ao]]/.s]/Evaluate[DfR[0]/.s])^(1))*)
```

```
(E ^ ((- (2 ξx + ξtf - 2 ξx + ξP + ξf)) * Log[x * lth / ao]))
   (Evaluate[σR[Log[x * lth / ao]] /. s])},
 {x, ao / lth, LR / lth}, PlotRange → All, PlotStyle →
  {{Green, Thickness[.002]}, {Red, Thickness[.002]}, {Orange, Thickness[.002]},
   {Blue, Thickness[.002]}, {Black, Thickness[.002]}, {Cyan, Thickness[.002]},
   {Purple, Thickness[.002]}, {Brown, Thickness[.002]}, {Magenta, Thickness[.002]},
   {LightRed, Thickness[.002]}, {LightBlue, Thickness[.002]},
   {LightPurple, Thickness[.002]}, {LightGreen, Thickness[.002]},
   {LightOrange, Thickness[.002]}, {LightCyan, Thickness[.002]},
   {LightBrown, Thickness[.002]}, {LightMagenta, Thickness[.002]},
   {Black, Thickness[.005]}, {Green, Thickness[.005]}},
 PlotLegends → {"Df", "DYuλ", "DYf2λ", "Dfϕufλ", "Dfϕf3λ", "DYuμ",
   "DYf2μ", "Dfϕufμ", "Dfϕf3μ", "DYuA", "DYf2A", "DfϕufA",
   "Dfϕf3A", "DYuK", "DYf2K", "DfϕufK", "Dfϕf3K"},
 LabelStyle → {FontSize → 15 , Black, Bold}]
```

Out[ ]=



In[ ]:=

```
(Log[Abs[Evaluate[DfϕufμR[Log[(LR / (1 * lth)) * lth / ao]]] /. s][[1]]] -
    Log[Abs[Evaluate[DfϕufμR[Log[(LR / (10 lth)) * lth / ao]]] /. s][[1]]]) /
  (Log[(LR / (1 * lth)) * lth / ao] - Log[(LR / (10 lth)) * lth / ao]) - 2
```

Out[ ]=

```
-1.1999900472935643380068945887
```

In[ ]:=

```
(Log[Abs[Evaluate[DfϕufKR[Log[(LR / (1 * lth)) * lth / ao]]] /. s][[1]]] -
    Log[Abs[Evaluate[DfϕufKR[Log[(LR / (10 lth)) * lth / ao]]] /. s][[1]]]) /
  (Log[(LR / (1 * lth)) * lth / ao] - Log[(LR / (10 lth)) * lth / ao]) - 2
```

Out[ ]=

```
-1.5999800959758835889755405842
```

In[∘]:=
```
(Log[Abs[Evaluate[DDYuμR[Log[(LR / (10 * lth)) * lth / ao]]] /. s][[1]]] -
     Log[Abs[Evaluate[DDYuμR[Log[(LR / (100 lth)) * lth / ao]]] /. s][[1]]]) /
   (Log[(LR / (10 * lth)) * lth / ao] - Log[(LR / (100 lth)) * lth / ao]) - 2
```

Out[∘]=
```
-0.3999750091097553744925498267
```

In[∘]:=
```
(Log[Abs[Evaluate[DDYuKR[Log[(LR / (10 * lth)) * lth / ao]]] /. s][[1]]] -
     Log[Abs[Evaluate[DDYuKR[Log[(LR / (100 lth)) * lth / ao]]] /. s][[1]]]) /
   (Log[(LR / (10 * lth)) * lth / ao] - Log[(LR / (100 lth)) * lth / ao]) - 2
```

Out[∘]=
```
-0.7999500075302968253816384273
```

Stability Analysis

In[∘]:=
```
Clear[dc, Λ]
```

In[∘]:=
```
F4BFinalFS = (q^2 F4BFinalC) /.
```

$$\left\{Df \to DfR,\ DDYu\mu \to DDR * \frac{16 * \pi\, \Lambda^2}{4 + dc} + DDYu\mu R,\ DDYf2\mu \to DDR * \frac{16 * \pi\, \Lambda^2}{4 + dc} + DDYf2\mu R,\right.$$

$$Df\phi uf\mu \to DfR * \frac{16 * \pi\, \Lambda^2}{4 + dc} + Df\phi uf\mu R,\ Df\phi f3\mu \to DfR * \frac{16 * \pi\, \Lambda^2}{4 + dc} + Df\phi f3\mu R,$$

$$DDYu\lambda \to DDR * \frac{-8 * \pi\, \Lambda^2}{4 + dc} + DDYu\lambda R,\ DDYf2\lambda \to DDR * \frac{-8 * \pi\, \Lambda^2}{4 + dc} + DDYf2\lambda R,$$

$$Df\phi uf\lambda \to DfR * \frac{-8 * \pi\, \Lambda^2}{4 + dc} + Df\phi uf\lambda R,\ Df\phi f3\lambda \to DfR * \frac{-8 * \pi\, \Lambda^2}{4 + dc} + Df\phi f3\lambda R,$$

$$DDYuA \to DDYuAR,\ DDYf2A \to DDYf2AR,\ Df\phi ufA \to Df\phi ufAR,$$

$$Df\phi f3A \to Df\phi f3AR,\ DDYuK \to DDYuKR,\ DDYf2K \to DDYf2KR,\ Df\phi ufK \to Df\phi ufKR,$$

$$\left. Df\phi f3K \to Df\phi f3KR,\ \sigma \to (\sigma R / DfR),\ L \to DDR,\ p1 \to \Lambda,\ z \to \Lambda,\ q \to \Lambda,\ p2 \to \Lambda\right\};$$

Stress contribution

In[∘]:=
```
Simplify[StressContS]
```

Simplify : Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification.

Simplify : Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification.

Simplify : Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification.

General : Further output of Simplify::time will be suppressed during this calculation.

In[ ]:=
```
StressContS = (q^2(*(1/Df)*) (F4SFinalC) - 0 * (σ / (Df)) (q^2 F4BFinalC)) /.
  {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
   Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
   DDYuλ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYf2λR,
   Dfϕufλ → DfR * (- 8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (- 8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
   DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
   Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
   Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ};
```

Shear contribution

In[ ]:=
```
DfufϕShearContS = (((q^2(*(1/Df)*) (ContDfμC) - 1 * (Dfϕufμ / Df) q^2 F4BFinalC)) /.
    {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
     Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
     DDYuλ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYf2λR,
     Dfϕufλ → DfR * (- 8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (- 8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
     DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR,
     DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR,
     σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ}));

Dfϕf3ShearContS =
  ((((*(1/Df)*) (q^2 ContDfμϕf3C - 1 * (Dfϕf3μ / Df) q^2 F4BFinalC))) /.
    {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
     Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
     DDYuλ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYf2λR,
     Dfϕufλ → DfR * (- 8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (- 8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
     DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
```

```
        Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
        Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ})
      (*-1*(Dfϕf3μR[x]/DfR[x])F4BFinalF[x]*));
DYuShearContS =
  (((q^2(*(1/Df)*) (ContμC))) /. {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR,
        DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR, Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR,
        Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR, DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR,
        DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR, Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR,
        Dfϕf3λ → DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR, DDYuA → DDYuAR, DDYf2A → DDYf2AR,
        DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR,
        DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ,
        z → Λ, q → Λ, p2 → Λ}) - 0 * (DDYuμR[x] / DfR[x]) F4BFinalF[x]);
DYFFShearContS =
  (((q^2(*(1/Df)*) (ContDμYFFC))) /. {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR,
        DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR, Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR,
        Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR, DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR,
        DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR, Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR,
        Dfϕf3λ → DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR, DDYuA → DDYuAR, DDYf2A → DDYf2AR,
        DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR,
        DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ,
        z → Λ, q → Λ, p2 → Λ}) - 0 * (DDYf2μR[x] / DfR[x]) F4BFinalF[x]);
```

Lambda contribution

```
In[◦]:= DfufϕLambdaContS =
  (((q^2 ((*(1/Df)*)(ContDfλμC - 2 * ContDfμC)) - 1 * (Dfϕufλ / Df) (q^2)
```

```
F4BFinalC)) /. {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR,
    DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR, Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR,
    Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR, DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR,
    DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR, Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ →
     DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR, DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
    Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
    Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ})
  (*-1*(DfϕufλR[x]/DfR[x])F4BFinalF[x]*));
Dfϕf3LambdaContS =
  (((q^2 ((*(1/Df)*) (ContDfλμϕf3C - 2 * ContDfμϕf3C)) - 1 * (Dfϕf3λ / Df)
       (q^2) F4BFinalC)) /. {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR,
    DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR, Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR,
    Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR, DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR,
    DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR, Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ →
     DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR, DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
    Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
    Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ})
  (*-1*(Dfϕf3λR[x]/DfR[x])F4BFinalF[x]*));
DYuLambdaContS =
  (((q^2 ((*(1/Df)*) (ContλμC - 2 * ContμC))) /. {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) +
       DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR, Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR,
```

```
Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR, DDYuλ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYuλR,
DDYf2λ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYf2λR, Dfϕufλ → DfR * (- 8 * π Λ^2)/(4 + dc) + DfϕufλR,
Dfϕf3λ → DfR * (- 8 * π Λ^2)/(4 + dc) + Dfϕf3λR, DDYuA → DDYuAR, DDYf2A → DDYf2AR,
DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR,
DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ,
z → Λ, q → Λ, p2 → Λ}) - 0 * (DDYuλR[x] / DfR[x]) F4BFinalF[x]);

DYFFLambdaContS = (((q^2 ((*(1/Df)*) (ContDλμYFFC - 2 * ContDμYFFC))) /.
  {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
   Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
   DDYuλ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYf2λR,
   Dfϕufλ → DfR * (- 8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (- 8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
   DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR,
   DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR,
   σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
  0 * (DDYf2λR[x] / DfR[x]) F4BFinalF[x]);
```

K Contribution

```
In[ ]:= DfufϕKContS = (((((*(1/Df)*)q^2 (ContDfKC)) - 1 * (DfϕufK / Df) (q^2) F4BFinalC) /.
  {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
   Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
   DDYuλ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYf2λR,
   Dfϕufλ → DfR * (- 8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (- 8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
   DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
   Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
```

```
    Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ})
  (*-1*(DfϕufKR[x]/DfR[x])F4BFinalF[x]*));
Dfϕf3KContS = (((((*(1/Df)*)q^2 (ContDfKϕf3C) - 1 * (Dfϕf3K / Df) (q^2) F4BFinalC)) /.
    {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
     Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
     DDYuλ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYf2λR,
     Dfϕufλ → DfR * (- 8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (- 8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
     DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
     Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
     Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ})
  (*-1*(Dfϕf3KR[x]/DfR[x])F4BFinalF[x]*));
DYuKContS = ((((*(1/Df)*)q^2 (ContKC))) /.
    {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
     Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
     DDYuλ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (- 8 * π Λ^2)/(4 + dc) + DDYf2λR,
     Dfϕufλ → DfR * (- 8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (- 8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
     DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR,
     DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR,
     σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
  0 * (DDYuKR[x] / DfR[x]) F4BFinalF[x]);
DYFFKContS =
 ((((*(1/Df)*)q^2 (ContDKYFFC))) /. {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR,
```

```
DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR, Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR,
Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR, DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR,
DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR, Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR,
Dfϕf3λ → DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR, DDYuA → DDYuAR, DDYf2A → DDYf2AR,
DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR,
DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ,
z → Λ, q → Λ, p2 → Λ}) - 0 * (DDYf2KR[x] / DfR[x]) F4BFinalF[x]);
```

A contribution

```
In[ ]:= DfufϕAContS = ((((((*(1/Df)*)
    (q^2 (ContDfAC) (*+q^2(ContDfKC)*)) - 1 * (DfϕufA / Df) (q^2) F4BFinalC))) /.
  {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
   Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
   DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR,
   Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
   DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
   Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
   Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ})
  (*-1*(DfϕufAR[x]/DfR[x])F4BFinalF[x]*));

Dfϕf3AContS = (((((*(1/Df)*)
    (q^2 (ContDfAϕf3C) (*+q^2(ContDfKC)*)) - 1 * (Dfϕf3A / Df) (q^2) F4BFinalC)) /.
  {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
   Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
   DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR,
```

```
        Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
        DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
        Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
        Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ})
      (*-1*(Dfϕf3AR[x]/DfR[x])F4BFinalF[x]*));

DYuAContS = ((((((*(1/Df)*)(q^2 (ContAC)(*+q^2(ContDfKC)*)))) /.
      {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
       Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
       DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR,
       Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
       DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR,
       DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR,
       σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
      0 * (DDYuAR[x] / DfR[x]) F4BFinalF[x]);

DYFFAContS = ((((((*(1/Df)*)(q^2 (ContDAYFFC)(*+q^2(ContDfKC)*)))) /.
      {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
       Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
       DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR,
       Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
       DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR, Dfϕf3A → Dfϕf3AR,
       DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR, Dfϕf3K → Dfϕf3KR,
       σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ}) -
```

```
0 * (DDYf2AR[x] / DfR[x]) F4BFinalF[x]);
```

LL Contribution

```
In[ ]:= LContS = (2 * q^2 (L / Df) F4BFinalC) /.
  {Df → DfR, DDYuμ → DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR, DDYf2μ → DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR,
   Dfϕufμ → DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR, Dfϕf3μ → DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR,
   DDYuλ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR, DDYf2λ → DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR,
   Dfϕufλ → DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR, Dfϕf3λ → DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR,
   DDYuA → DDYuAR, DDYf2A → DDYf2AR, DfϕufA → DfϕufAR,
   Dfϕf3A → Dfϕf3AR, DDYuK → DDYuKR, DDYf2K → DDYf2KR, DfϕufK → DfϕufKR,
   Dfϕf3K → Dfϕf3KR, σ → (σR / DfR), L → DDR, p1 → Λ, z → Λ, q → Λ, p2 → Λ};
```

```
In[ ]:= βDDYuAR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYuAR - (DYuAContS);
βDDYf2AR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYf2AR - (DYFFAContS);
βDfϕufAR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DfϕufAR - (DfufϕAContS);
βDfϕf3AR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * Dfϕf3AR - (Dfϕf3AContS);

βDDYuKR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYuKR - (DYuKContS);
βDDYf2KR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DDYf2KR - (DYFFKContS);
βDfϕufKR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * DfϕufKR - (DfufϕKContS);
βDfϕf3KR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * Dfϕf3KR - (Dfϕf3KContS);


βDDYuλR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * (DDR * (-8 * π Λ^2)/(4 + dc) + DDYuλR) - (DYuLambdaContS);
βDDYf2λR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * (DDR * (-8 * π Λ^2)/(4 + dc) + DDYf2λR) - (DYFFLambdaContS);
βDfϕufλR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * (DfR * (-8 * π Λ^2)/(4 + dc) + DfϕufλR) - (DfufϕLambdaContS);
βDfϕf3λR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * (DfR * (-8 * π Λ^2)/(4 + dc) + Dfϕf3λR) - (Dfϕf3LambdaContS);


βDDYuμR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * (DDR * (16 * π Λ^2)/(4 + dc) + DDYuμR) - (DYuShearContS);
βDDYf2μR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * (DDR * (16 * π Λ^2)/(4 + dc) + DDYf2μR) - (DYFFShearContS);
βDfϕufμR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * (DfR * (16 * π Λ^2)/(4 + dc) + DfϕufμR) - (DfufϕShearContS);
βDfϕf3μR = (ξY + ξu - 2 ξx + 2 ξx + ξtu) * (DfR * (16 * π Λ^2)/(4 + dc) + Dfϕf3μR) - (Dfϕf3ShearContS);

βσR = ((2 ξx + ξtf - 2 ξx + ξP + ξf) * σR - (StressContS));
βLLR = (ξu + ξY + 2 ξx) * DDR - LContS;
βDfR = (2 ξP + 2 ξx + ξtf) * DfR - (F4BFinalFS);
```

```
In[•]:= βDDYuARS = (βDDYuAR * DDR - DDYuAR * βLLR) / DDR ^ 2;
βDDYf2ARS = (βDDYf2AR * DDR - DDYf2AR * βLLR) / DDR ^ 2;
βDfϕufARS = (βDfϕufAR * DfR - DfϕufAR * βDfR) / DfR ^ 2;
βDfϕf3ARS = (βDfϕf3AR * DfR - Dfϕf3AR * βDfR) / DfR ^ 2;

βDDYuKRS = (βDDYuKR * DDR - DDYuKR * βLLR) / DDR ^ 2;
βDDYf2KRS = (βDDYf2KR * DDR - DDYf2KR * βLLR) / DDR ^ 2;
βDfϕufKRS = (βDfϕufKR * DfR - DfϕufKR * βDfR) / DfR ^ 2;
βDfϕf3KRS = (βDfϕf3KR * DfR - Dfϕf3KR * βDfR) / DfR ^ 2;
```

$$\beta\text{DDYu}\lambda\text{RS} = \left(\beta\text{DDYu}\lambda\text{R} * \text{DDR} - \left(\text{DDR} * \frac{-8 * \pi\,\Lambda^2}{4 + \text{dc}} + \text{DDYu}\lambda\text{R}\right) * \beta\text{LLR}\right) \Big/ \text{DDR}\,\hat{}\,2;$$

$$\beta\text{DDYf2}\lambda\text{RS} = \left(\beta\text{DDYf2}\lambda\text{R} * \text{DDR} - \left(\text{DDR} * \frac{-8 * \pi\,\Lambda^2}{4 + \text{dc}} + \text{DDYf2}\lambda\text{R}\right) * \beta\text{LLR}\right) \Big/ \text{DDR}\,\hat{}\,2;$$

$$\beta\text{Df}\phi\text{uf}\lambda\text{RS} = \left(\beta\text{Df}\phi\text{uf}\lambda\text{R} * \text{DfR} - \left(\text{DfR} * \frac{-8 * \pi\,\Lambda^2}{4 + \text{dc}} + \text{Df}\phi\text{uf}\lambda\text{R}\right) * \beta\text{DfR}\right) \Big/ \text{DfR}\,\hat{}\,2;$$

$$\beta\text{Df}\phi\text{f3}\lambda\text{RS} = \left(\beta\text{Df}\phi\text{f3}\lambda\text{R} * \text{DfR} - \left(\text{DfR} * \frac{-8 * \pi\,\Lambda^2}{4 + \text{dc}} + \text{Df}\phi\text{f3}\lambda\text{R}\right) * \beta\text{DfR}\right) \Big/ \text{DfR}\,\hat{}\,2;$$

$$\beta\text{DDYu}\mu\text{RS} = \left(\beta\text{DDYu}\mu\text{R} * \text{DDR} - \left(\text{DDR} * \frac{16 * \pi\,\Lambda^2}{4 + \text{dc}} + \text{DDYu}\mu\text{R}\right) * \beta\text{LLR}\right) \Big/ \text{DDR}\,\hat{}\,2;$$

$$\beta\text{DDYf2}\mu\text{RS} = \left(\beta\text{DDYf2}\mu\text{R} * \text{DDR} - \left(\text{DDR} * \frac{16 * \pi\,\Lambda^2}{4 + \text{dc}} + \text{DDYf2}\mu\text{R}\right) * \beta\text{LLR}\right) \Big/ \text{DDR}\,\hat{}\,2;$$

$$\beta\text{Df}\phi\text{uf}\mu\text{RS} = \left(\beta\text{Df}\phi\text{uf}\mu\text{R} * \text{DfR} - \left(\text{DfR} * \frac{16 * \pi\,\Lambda^2}{4 + \text{dc}} + \text{Df}\phi\text{uf}\mu\text{R}\right) * \beta\text{DfR}\right) \Big/ \text{DfR}\,\hat{}\,2;$$

$$\beta\text{Df}\phi\text{f3}\mu\text{RS} = \left(\beta\text{Df}\phi\text{f3}\mu\text{R} * \text{DfR} - \left(\text{DfR} * \frac{16 * \pi\,\Lambda^2}{4 + \text{dc}} + \text{Df}\phi\text{f3}\mu\text{R}\right) * \beta\text{DfR}\right) \Big/ \text{DfR}\,\hat{}\,2;$$

```
βσRS = ((1 / DfR) βσR - (σR / DfR ^ 2) * βDfR);
(*βσRS=(βσR*DfR-0*σR*βDfR)/DfR^2;*)
```

```
In[•]:= LinearDDYuARDDYuAR =
  Simplify[D[βDDYuARS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDDYf2AR =
  Simplify[D[βDDYuARS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
```

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        DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDfϕufAR =
  Simplify[D[βDDYuARS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDfϕf3AR =
  Simplify[D[βDDYuARS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDDYuKR =
  Simplify[D[βDDYuARS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDDYf2KR =
  Simplify[D[βDDYuARS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDfϕufKR =
  Simplify[D[βDDYuARS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDfϕf3KR =
  Simplify[D[βDDYuARS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDDYuλR =
  Simplify[D[βDDYuARS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDDYf2λR =
  Simplify[D[βDDYuARS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDfϕufλR =
  Simplify[D[βDDYuARS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
```

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    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDfϕf3λR =
  Simplify[D[βDDYuARS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDDYuμR =
  Simplify[D[βDDYuARS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDDYf2μR =
  Simplify[D[βDDYuARS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDfϕufμR =
  Simplify[D[βDDYuARS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARDfϕf3μR =
  Simplify[D[βDDYuARS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuARσR =
  Simplify[D[βDDYuARS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
    DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0,
    DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDDYf2ARDDYuAR =
  Simplify[D[βDDYf2ARS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDDYf2AR =
  Simplify[D[βDDYf2ARS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
```

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LinearDDYf2ARDfϕufAR =
  Simplify[D[βDDYf2ARS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDfϕf3AR =
  Simplify[D[βDDYf2ARS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDDYuKR =
  Simplify[D[βDDYf2ARS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDDYf2KR =
  Simplify[D[βDDYf2ARS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDfϕufKR =
  Simplify[D[βDDYf2ARS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDfϕf3KR =
  Simplify[D[βDDYf2ARS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDDYuλR =
  Simplify[D[βDDYf2ARS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDDYf2λR =
  Simplify[D[βDDYf2ARS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDfϕufλR =
  Simplify[D[βDDYf2ARS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
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    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDfϕf3λR =
  Simplify[D[βDDYf2ARS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDDYuμR =
  Simplify[D[βDDYf2ARS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDDYf2μR =
  Simplify[D[βDDYf2ARS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDfϕufμR =
  Simplify[D[βDDYf2ARS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARDfϕf3μR =
  Simplify[D[βDDYf2ARS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2ARσR =
  Simplify[D[βDDYf2ARS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDfϕufARDDYuAR =
  Simplify[D[βDfϕufARS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDDYf2AR =
  Simplify[D[βDfϕufARS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDfϕufAR =
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  Simplify[D[βDfϕufARS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDfϕf3AR =
  Simplify[D[βDfϕufARS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDDYuKR =
  Simplify[D[βDfϕufARS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDDYf2KR =
  Simplify[D[βDfϕufARS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDfϕufKR =
  Simplify[D[βDfϕufARS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDfϕf3KR =
  Simplify[D[βDfϕufARS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDDYuλR =
  Simplify[D[βDfϕufARS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDDYf2λR =
  Simplify[D[βDfϕufARS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDfϕufλR =
  Simplify[D[βDfϕufARS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDfϕufARDfϕf3λR =
  Simplify[D[βDfϕufARS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDDYuμR =
  Simplify[D[βDfϕufARS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDDYf2μR =
  Simplify[D[βDfϕufARS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDfϕufμR =
  Simplify[D[βDfϕufARS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARDfϕf3μR =
  Simplify[D[βDfϕufARS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufARσR =
  Simplify[D[βDfϕufARS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDfϕf3ARDDYuAR =
  Simplify[D[βDfϕf3ARS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDDYf2AR =
  Simplify[D[βDfϕf3ARS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDfϕufAR =
  Simplify[D[βDfϕf3ARS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
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      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDfϕf3AR =
  Simplify[D[βDfϕf3ARS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDDYuKR =
  Simplify[D[βDfϕf3ARS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDDYf2KR =
  Simplify[D[βDfϕf3ARS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDfϕufKR =
  Simplify[D[βDfϕf3ARS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDfϕf3KR =
  Simplify[D[βDfϕf3ARS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDDYuλR =
  Simplify[D[βDfϕf3ARS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDDYf2λR =
  Simplify[D[βDfϕf3ARS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDfϕufλR =
  Simplify[D[βDfϕf3ARS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDfϕf3λR =
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  Simplify[D[βDfϕf3ARS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDDYuμR =
  Simplify[D[βDfϕf3ARS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDDYf2μR =
  Simplify[D[βDfϕf3ARS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDfϕufμR =
  Simplify[D[βDfϕf3ARS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARDfϕf3μR =
  Simplify[D[βDfϕf3ARS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3ARσR =
  Simplify[D[βDfϕf3ARS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDDYuKRDDYuAR =
  Simplify[D[βDDYuKRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDDYf2AR =
  Simplify[D[βDDYuKRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDfϕufAR =
  Simplify[D[βDDYuKRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
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      DfϕfЗAR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDfϕf3AR =
  Simplify[D[βDDYuKRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDDYuKR =
  Simplify[D[βDDYuKRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDDYf2KR =
  Simplify[D[βDDYuKRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDfϕufKR =
  Simplify[D[βDDYuKRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDfϕf3KR =
  Simplify[D[βDDYuKRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDDYuλR =
  Simplify[D[βDDYuKRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDDYf2λR =
  Simplify[D[βDDYuKRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDfϕufλR =
  Simplify[D[βDDYuKRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDfϕf3λR =
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  Simplify[D[βDDYuKRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDDYuμR =
  Simplify[D[βDDYuKRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDDYf2μR =
  Simplify[D[βDDYuKRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDfϕufμR =
  Simplify[D[βDDYuKRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRDfϕf3μR =
  Simplify[D[βDDYuKRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuKRσR =
  Simplify[D[βDDYuKRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
     DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0,
     DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDDYf2KRDDYuAR =
  Simplify[D[βDDYf2KRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDDYf2AR =
  Simplify[D[βDDYf2KRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDfϕufAR =
  Simplify[D[βDDYf2KRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
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      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDfϕf3AR =
  Simplify[D[βDDYf2KRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDDYuKR =
  Simplify[D[βDDYf2KRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDDYf2KR =
  Simplify[D[βDDYf2KRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDfϕufKR =
  Simplify[D[βDDYf2KRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDfϕf3KR =
  Simplify[D[βDDYf2KRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDDYuλR =
  Simplify[D[βDDYf2KRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDDYf2λR =
  Simplify[D[βDDYf2KRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDfϕufλR =
  Simplify[D[βDDYf2KRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDfϕf3λR =
  Simplify[D[βDDYf2KRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
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      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDDYuμR =
  Simplify[D[βDDYf2KRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDDYf2μR =
  Simplify[D[βDDYf2KRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDfϕufμR =
  Simplify[D[βDDYf2KRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRDfϕf3μR =
  Simplify[D[βDDYf2KRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2KRσR =
  Simplify[D[βDDYf2KRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDfϕufKRDDYuAR =
  Simplify[D[βDfϕufKRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDDYf2AR =
  Simplify[D[βDfϕufKRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDfϕufAR =
  Simplify[D[βDfϕufKRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
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      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDfϕf3AR =
  Simplify[D[βDfϕufKRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDDYuKR =
  Simplify[D[βDfϕufKRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDDYf2KR =
  Simplify[D[βDfϕufKRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDfϕufKR =
  Simplify[D[βDfϕufKRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDfϕf3KR =
  Simplify[D[βDfϕufKRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDDYuλR =
  Simplify[D[βDfϕufKRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDDYf2λR =
  Simplify[D[βDfϕufKRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDfϕufλR =
  Simplify[D[βDfϕufKRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDfϕf3λR =
  Simplify[D[βDfϕufKRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
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      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDDYuμR =
  Simplify[D[βDfϕufKRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDDYf2μR =
  Simplify[D[βDfϕufKRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDfϕufμR =
  Simplify[D[βDfϕufKRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRDfϕf3μR =
  Simplify[D[βDfϕufKRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufKRσR =
  Simplify[D[βDfϕufKRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDfϕf3KRDDYuAR =
  Simplify[D[βDfϕf3KRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDDYf2AR =
  Simplify[D[βDfϕf3KRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDfϕufAR =
  Simplify[D[βDfϕf3KRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDfϕf3KRDfϕf3AR =
  Simplify[D[βDfϕf3KRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDDYuKR =
  Simplify[D[βDfϕf3KRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDDYf2KR =
  Simplify[D[βDfϕf3KRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDfϕufKR =
  Simplify[D[βDfϕf3KRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDfϕf3KR =
  Simplify[D[βDfϕf3KRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDDYuλR =
  Simplify[D[βDfϕf3KRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDDYf2λR =
  Simplify[D[βDfϕf3KRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDfϕufλR =
  Simplify[D[βDfϕf3KRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDfϕf3λR =
  Simplify[D[βDfϕf3KRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
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      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDDYuμR =
  Simplify[D[βDfϕf3KRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDDYf2μR =
  Simplify[D[βDfϕf3KRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDfϕufμR =
  Simplify[D[βDfϕf3KRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRDfϕf3μR =
  Simplify[D[βDfϕf3KRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3KRσR =
  Simplify[D[βDfϕf3KRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDDYuλRDDYuAR =
  Simplify[D[βDDYuλRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDDYf2AR =
  Simplify[D[βDDYuλRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDfϕufAR =
  Simplify[D[βDDYuλRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDfϕf3AR =
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  Simplify[D[βDDYuλRS, {DfϕfЗAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDDYuKR =
  Simplify[D[βDDYuλRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDDYf2KR =
  Simplify[D[βDDYuλRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDfϕufKR =
  Simplify[D[βDDYuλRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDfϕf3KR =
  Simplify[D[βDDYuλRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDDYuλR =
  Simplify[D[βDDYuλRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDDYf2λR =
  Simplify[D[βDDYuλRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDfϕufλR =
  Simplify[D[βDDYuλRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDfϕf3λR =
  Simplify[D[βDDYuλRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDDYuλRDDYuμR =
  Simplify[D[βDDYuλRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDDYf2μR =
  Simplify[D[βDDYuλRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDfϕufμR =
  Simplify[D[βDDYuλRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRDfϕf3μR =
  Simplify[D[βDDYuλRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuλRσR =
  Simplify[D[βDDYuλRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
     DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0,
     DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDDYf2λRDDYuAR =
  Simplify[D[βDDYf2λRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDDYf2AR =
  Simplify[D[βDDYf2λRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDfϕufAR =
  Simplify[D[βDDYf2λRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDfϕf3AR =
  Simplify[D[βDDYf2λRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
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    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDDYuKR =
  Simplify[D[βDDYf2λRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDDYf2KR =
  Simplify[D[βDDYf2λRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDfϕufKR =
  Simplify[D[βDDYf2λRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDfϕf3KR =
  Simplify[D[βDDYf2λRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDDYuλR =
  Simplify[D[βDDYf2λRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDDYf2λR =
  Simplify[D[βDDYf2λRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDfϕufλR =
  Simplify[D[βDDYf2λRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDfϕf3λR =
  Simplify[D[βDDYf2λRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDDYuμR =
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  Simplify[D[βDDYf2λRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDDYf2μR =
  Simplify[D[βDDYf2λRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDfϕufμR =
  Simplify[D[βDDYf2λRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRDfϕf3μR =
  Simplify[D[βDDYf2λRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2λRσR =
  Simplify[D[βDDYf2λRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDfϕufλRDDYuAR =
  Simplify[D[βDfϕufλRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDDYf2AR =
  Simplify[D[βDfϕufλRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDfϕufAR =
  Simplify[D[βDfϕufλRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDfϕf3AR =
  Simplify[D[βDfϕufλRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
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      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDDYuKR =
  Simplify[D[βDfϕufλRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDDYf2KR =
  Simplify[D[βDfϕufλRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDfϕufKR =
  Simplify[D[βDfϕufλRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDfϕf3KR =
  Simplify[D[βDfϕufλRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDDYuλR =
  Simplify[D[βDfϕufλRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDDYf2λR =
  Simplify[D[βDfϕufλRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDfϕufλR =
  Simplify[D[βDfϕufλRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDfϕf3λR =
  Simplify[D[βDfϕufλRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDDYuμR =
  Simplify[D[βDfϕufλRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
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      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDDYf2μR =
   Simplify[D[βDfϕufλRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDfϕufμR =
   Simplify[D[βDfϕufλRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRDfϕf3μR =
   Simplify[D[βDfϕufλRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufλRσR =
   Simplify[D[βDfϕufλRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearDfϕf3λRDDYuAR =
   Simplify[D[βDfϕf3λRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDDYf2AR =
   Simplify[D[βDfϕf3λRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDfϕufAR =
   Simplify[D[βDfϕf3λRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDfϕf3AR =
   Simplify[D[βDfϕf3λRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
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      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDDYuKR =
  Simplify[D[βDfϕf3λRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDDYf2KR =
  Simplify[D[βDfϕf3λRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDfϕufKR =
  Simplify[D[βDfϕf3λRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDfϕf3KR =
  Simplify[D[βDfϕf3λRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDDYuλR =
  Simplify[D[βDfϕf3λRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDDYf2λR =
  Simplify[D[βDfϕf3λRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDfϕufλR =
  Simplify[D[βDfϕf3λRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDfϕf3λR =
  Simplify[D[βDfϕf3λRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDDYuμR =
  Simplify[D[βDfϕf3λRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
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      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDDYf2μR =
  Simplify[D[βDfϕf3λRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDfϕufμR =
  Simplify[D[βDfϕf3λRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRDfϕf3μR =
  Simplify[D[βDfϕf3λRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3λRσR =
  Simplify[D[βDfϕf3λRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDDYuμRDDYuAR =
  Simplify[D[βDDYuμRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDDYf2AR =
  Simplify[D[βDDYuμRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDfϕufAR =
  Simplify[D[βDDYuμRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDfϕf3AR =
  Simplify[D[βDDYuμRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDDYuμRDDYuKR =
  Simplify[D[βDDYuμRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDDYf2KR =
  Simplify[D[βDDYuμRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDfϕufKR =
  Simplify[D[βDDYuμRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDfϕf3KR =
  Simplify[D[βDDYuμRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDDYuλR =
  Simplify[D[βDDYuμRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDDYf2λR =
  Simplify[D[βDDYuμRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDfϕufλR =
  Simplify[D[βDDYuμRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDfϕf3λR =
  Simplify[D[βDDYuμRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDDYuμR =
  Simplify[D[βDDYuμRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
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      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDDYf2μR =
  Simplify[D[βDDYuμRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDfϕufμR =
  Simplify[D[βDDYuμRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRDfϕf3μR =
  Simplify[D[βDDYuμRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYuμRσR =
  Simplify[D[βDDYuμRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
      DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0,
      DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDDYf2μRDDYuAR =
  Simplify[D[βDDYf2μRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDDYf2AR =
  Simplify[D[βDDYf2μRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDfϕufAR =
  Simplify[D[βDDYf2μRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDfϕf3AR =
  Simplify[D[βDDYf2μRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDDYuKR =
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  Simplify[D[βDDYf2μRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDDYf2KR =
  Simplify[D[βDDYf2μRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDfϕufKR =
  Simplify[D[βDDYf2μRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDfϕf3KR =
  Simplify[D[βDDYf2μRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDDYuλR =
  Simplify[D[βDDYf2μRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDDYf2λR =
  Simplify[D[βDDYf2μRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDfϕufλR =
  Simplify[D[βDDYf2μRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDfϕf3λR =
  Simplify[D[βDDYf2μRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDDYuμR =
  Simplify[D[βDDYf2μRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDDYf2μRDDYf2μR =
  Simplify[D[βDDYf2μRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDfϕufμR =
  Simplify[D[βDDYf2μRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRDfϕf3μR =
  Simplify[D[βDDYf2μRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDDYf2μRσR =
  Simplify[D[βDDYf2μRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDfϕufμRDDYuAR =
  Simplify[D[βDfϕufμRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDDYf2AR =
  Simplify[D[βDfϕufμRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDfϕufAR =
  Simplify[D[βDfϕufμRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDfϕf3AR =
  Simplify[D[βDfϕufμRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDDYuKR =
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  Simplify[D[βDfϕufμRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDDYf2KR =
  Simplify[D[βDfϕufμRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDfϕufKR =
  Simplify[D[βDfϕufμRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDfϕf3KR =
  Simplify[D[βDfϕufμRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDDYuλR =
  Simplify[D[βDfϕufμRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDDYf2λR =
  Simplify[D[βDfϕufμRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDfϕufλR =
  Simplify[D[βDfϕufμRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDfϕf3λR =
  Simplify[D[βDfϕufμRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDDYuμR =
  Simplify[D[βDfϕufμRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDfϕufμRDDYf2μR =
  Simplify[D[βDfϕufμRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDfϕufμR =
  Simplify[D[βDfϕufμRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRDfϕf3μR =
  Simplify[D[βDfϕufμRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕufμRσR =
  Simplify[D[βDfϕufμRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
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LinearDfϕf3μRDDYuAR =
  Simplify[D[βDfϕf3μRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDDYf2AR =
  Simplify[D[βDfϕf3μRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDfϕufAR =
  Simplify[D[βDfϕf3μRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDfϕf3AR =
  Simplify[D[βDfϕf3μRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDDYuKR =
  Simplify[D[βDfϕf3μRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
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      DfϕF3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDDYf2KR =
  Simplify[D[βDfϕf3μRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDfϕufKR =
  Simplify[D[βDfϕf3μRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDfϕf3KR =
  Simplify[D[βDfϕf3μRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDDYuλR =
  Simplify[D[βDfϕf3μRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDDYf2λR =
  Simplify[D[βDfϕf3μRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDfϕufλR =
  Simplify[D[βDfϕf3μRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDfϕf3λR =
  Simplify[D[βDfϕf3μRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDDYuμR =
  Simplify[D[βDfϕf3μRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDDYf2μR =
```

```
  Simplify[D[βDfϕf3μRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDfϕufμR =
  Simplify[D[βDfϕf3μRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRDfϕf3μR =
  Simplify[D[βDfϕf3μRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearDfϕf3μRσR =
  Simplify[D[βDfϕf3μRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];


LinearσRDDYuAR =
  Simplify[D[βσRS, {DDYuAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
     DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0,
     DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDDYf2AR =
  Simplify[D[βσRS, {DDYf2AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDfϕufAR =
  Simplify[D[βσRS, {DfϕufAR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDfϕf3AR =
  Simplify[D[βσRS, {Dfϕf3AR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
     Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
     DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
     DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDDYuKR =
  Simplify[D[βσRS, {DDYuKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
     DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0,
```

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      DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDDYf2KR =
  Simplify[D[βσRS, {DDYf2KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDfϕufKR =
  Simplify[D[βσRS, {DfϕufKR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDfϕf3KR =
  Simplify[D[βσRS, {Dfϕf3KR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDDYuλR =
  Simplify[D[βσRS, {DDYuλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
      DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0,
      DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDDYf2λR =
  Simplify[D[βσRS, {DDYf2λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDfϕufλR =
  Simplify[D[βσRS, {DfϕufλR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDfϕf3λR =
  Simplify[D[βσRS, {Dfϕf3λR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
      Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
      DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDDYuμR =
  Simplify[D[βσRS, {DDYuμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
      DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0,
      DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
      DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDDYf2μR =
  Simplify[D[βσRS, {DDYf2μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
```

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    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDfϕufμR =
  Simplify[D[βσRS, {DfϕufμR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRDfϕf3μR =
  Simplify[D[βσRS, {Dfϕf3μR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0,
    Dfϕf3AR → 0, DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0,
    DDYf2λR → 0, DDYuλR → 0, DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0,
    DfϕufμR → 0, Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
LinearσRσR =
  Simplify[D[βσRS, {σR, 1}] /. {DDYf2AR → 0, DDYuAR → 0, DfϕufAR → 0, Dfϕf3AR → 0,
    DDYf2KR → 0, DDYuKR → 0, DfϕufKR → 0, Dfϕf3KR → 0, DDYf2λR → 0, DDYuλR → 0,
    DfϕufλR → 0, Dfϕf3λR → 0, DDYf2μR → 0, DDYuμR → 0, DfϕufμR → 0,
    Dfϕf3μR → 0, σR → 0}, Assumptions → {DDR > 0, Λ > 0, dc > 0}];
```

In[•]:=
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MStability = {{LinearDDYuARDDYuAR, LinearDDYuARDDYf2AR,
    LinearDDYuARDfϕufAR, LinearDDYuARDfϕf3AR, LinearDDYuARDDYuKR,
    LinearDDYuARDDYf2KR, LinearDDYuARDfϕufKR, LinearDDYuARDfϕf3KR,
    LinearDDYuARDDYuλR, LinearDDYuARDDYf2λR, LinearDDYuARDfϕufλR,
    LinearDDYuARDfϕf3λR, LinearDDYuARDDYuμR, LinearDDYuARDDYf2μR,
    LinearDDYuARDfϕufμR, LinearDDYuARDfϕf3μR, LinearDDYuARσR},
   {LinearDDYf2ARDDYuAR, LinearDDYf2ARDDYf2AR, LinearDDYf2ARDfϕufAR,
    LinearDDYf2ARDfϕf3AR, LinearDDYf2ARDDYuKR, LinearDDYf2ARDDYf2KR,
    LinearDDYf2ARDfϕufKR, LinearDDYf2ARDfϕf3KR,
    LinearDDYf2ARDDYuλR, LinearDDYf2ARDDYf2λR, LinearDDYf2ARDfϕufλR,
    LinearDDYf2ARDfϕf3λR, LinearDDYf2ARDDYuμR, LinearDDYf2ARDDYf2μR,
    LinearDDYf2ARDfϕufμR, LinearDDYf2ARDfϕf3μR, LinearDDYf2ARσR},
   {LinearDfϕufARDDYuAR, LinearDfϕufARDDYf2AR, LinearDfϕufARDfϕufAR,
    LinearDfϕufARDfϕf3AR, LinearDfϕufARDDYuKR, LinearDfϕufARDDYf2KR,
    LinearDfϕufARDfϕufKR, LinearDfϕufARDfϕf3KR,
    LinearDfϕufARDDYuλR, LinearDfϕufARDDYf2λR, LinearDfϕufARDfϕufλR,
    LinearDfϕufARDfϕf3λR, LinearDfϕufARDDYuμR, LinearDfϕufARDDYf2μR,
    LinearDfϕufARDfϕufμR, LinearDfϕufARDfϕf3μR, LinearDfϕufARσR},
   {LinearDfϕf3ARDDYuAR, LinearDfϕf3ARDDYf2AR, LinearDfϕf3ARDfϕufAR,
    LinearDfϕf3ARDfϕf3AR, LinearDfϕf3ARDDYuKR, LinearDfϕf3ARDDYf2KR,
    LinearDfϕf3ARDfϕufKR, LinearDfϕf3ARDfϕf3KR, LinearDfϕf3ARDDYuλR,
    LinearDfϕf3ARDDYf2λR, LinearDfϕf3ARDfϕufλR, LinearDfϕf3ARDfϕf3λR,
    LinearDfϕf3ARDDYuμR, LinearDfϕf3ARDDYf2μR, LinearDfϕf3ARDfϕufμR,
```

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 LinearDfϕf3ARDfϕf3μR, LinearDfϕf3ARσR}, {LinearDDYuKRDDYuAR,
 LinearDDYuKRDDYf2AR, LinearDDYuKRDfϕufAR, LinearDDYuKRDfϕf3AR,
 LinearDDYuKRDDYuKR, LinearDDYuKRDDYf2KR, LinearDDYuKRDfϕufKR,
 LinearDDYuKRDfϕf3KR, LinearDDYuKRDDYuλR, LinearDDYuKRDDYf2λR,
 LinearDDYuKRDfϕufλR, LinearDDYuKRDfϕf3λR, LinearDDYuKRDDYuμR,
 LinearDDYuKRDDYf2μR, LinearDDYuKRDfϕufμR, LinearDDYuKRDfϕf3μR, LinearDDYuKRσR},
{LinearDDYf2KRDDYuAR, LinearDDYf2KRDDYf2AR, LinearDDYf2KRDfϕufAR,
 LinearDDYf2KRDfϕf3AR, LinearDDYf2KRDDYuKR, LinearDDYf2KRDDYf2KR,
 LinearDDYf2KRDfϕufKR, LinearDDYf2KRDfϕf3KR,
 LinearDDYf2KRDDYuλR, LinearDDYf2KRDDYf2λR, LinearDDYf2KRDfϕufλR,
 LinearDDYf2KRDfϕf3λR, LinearDDYf2KRDDYuμR, LinearDDYf2KRDDYf2μR,
 LinearDDYf2KRDfϕufμR, LinearDDYf2KRDfϕf3μR, LinearDDYf2KRσR},
{LinearDfϕufKRDDYuAR, LinearDfϕufKRDDYf2AR, LinearDfϕufKRDfϕufAR,
 LinearDfϕufKRDfϕf3AR, LinearDfϕufKRDDYuKR, LinearDfϕufKRDDYf2KR,
 LinearDfϕufKRDfϕufKR, LinearDfϕufKRDfϕf3KR,
 LinearDfϕufKRDDYuλR, LinearDfϕufKRDDYf2λR, LinearDfϕufKRDfϕufλR,
 LinearDfϕufKRDfϕf3λR, LinearDfϕufKRDDYuμR, LinearDfϕufKRDDYf2μR,
 LinearDfϕufKRDfϕufμR, LinearDfϕufKRDfϕf3μR, LinearDfϕufKRσR},
{LinearDfϕf3KRDDYuAR, LinearDfϕf3KRDDYf2AR, LinearDfϕf3KRDfϕufAR,
 LinearDfϕf3KRDfϕf3AR, LinearDfϕf3KRDDYuKR, LinearDfϕf3KRDDYf2KR,
 LinearDfϕf3KRDfϕufKR, LinearDfϕf3KRDfϕf3KR, LinearDfϕf3KRDDYuλR,
 LinearDfϕf3KRDDYf2λR, LinearDfϕf3KRDfϕufλR, LinearDfϕf3KRDfϕf3λR,
 LinearDfϕf3KRDDYuμR, LinearDfϕf3KRDDYf2μR, LinearDfϕf3KRDfϕufμR,
 LinearDfϕf3KRDfϕf3μR, LinearDfϕf3KRσR}, {LinearDDYuλRDDYuAR,
 LinearDDYuλRDDYf2AR, LinearDDYuλRDfϕufAR, LinearDDYuλRDfϕf3AR,
 LinearDDYuλRDDYuKR, LinearDDYuλRDDYf2KR, LinearDDYuλRDfϕufKR,
 LinearDDYuλRDfϕf3KR, LinearDDYuλRDDYuλR, LinearDDYuλRDDYf2λR,
 LinearDDYuλRDfϕufλR, LinearDDYuλRDfϕf3λR, LinearDDYuλRDDYuμR,
 LinearDDYuλRDDYf2μR, LinearDDYuλRDfϕufμR, LinearDDYuλRDfϕf3μR, LinearDDYuλRσR},
{LinearDDYf2λRDDYuAR, LinearDDYf2λRDDYf2AR, LinearDDYf2λRDfϕufAR,
 LinearDDYf2λRDfϕf3AR, LinearDDYf2λRDDYuKR, LinearDDYf2λRDDYf2KR,
 LinearDDYf2λRDfϕufKR, LinearDDYf2λRDfϕf3KR,
 LinearDDYf2λRDDYuλR, LinearDDYf2λRDDYf2λR, LinearDDYf2λRDfϕufλR,
 LinearDDYf2λRDfϕf3λR, LinearDDYf2λRDDYuμR, LinearDDYf2λRDDYf2μR,
 LinearDDYf2λRDfϕufμR, LinearDDYf2λRDfϕf3μR, LinearDDYf2λRσR},
{LinearDfϕufλRDDYuAR, LinearDfϕufλRDDYf2AR, LinearDfϕufλRDfϕufAR,
 LinearDfϕufλRDfϕf3AR, LinearDfϕufλRDDYuKR, LinearDfϕufλRDDYf2KR,
 LinearDfϕufλRDfϕufKR, LinearDfϕufλRDfϕf3KR,
 LinearDfϕufλRDDYuλR, LinearDfϕufλRDDYf2λR, LinearDfϕufλRDfϕufλR,
 LinearDfϕufλRDfϕf3λR, LinearDfϕufλRDDYuμR, LinearDfϕufλRDDYf2μR,
 LinearDfϕufλRDfϕufμR, LinearDfϕufλRDfϕf3μR, LinearDfϕufλRσR},
{LinearDfϕf3λRDDYuAR, LinearDfϕf3λRDDYf2AR, LinearDfϕf3λRDfϕufAR,
 LinearDfϕf3λRDfϕf3AR, LinearDfϕf3λRDDYuKR, LinearDfϕf3λRDDYf2KR,
 LinearDfϕf3λRDfϕufKR, LinearDfϕf3λRDfϕf3KR, LinearDfϕf3λRDDYuλR,
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  LinearDfϕf3λRDDYf2λR, LinearDfϕf3λRDfϕufλR, LinearDfϕf3λRDfϕf3λR,
  LinearDfϕf3λRDDYuμR, LinearDfϕf3λRDDYf2μR, LinearDfϕf3λRDfϕufμR,
  LinearDfϕf3λRDfϕf3μR, LinearDfϕf3λRσR}, {LinearDDYuμRDDYuAR,
  LinearDDYuμRDDYf2AR, LinearDDYuμRDfϕufAR, LinearDDYuμRDfϕf3AR,
  LinearDDYuμRDDYuKR, LinearDDYuμRDDYf2KR, LinearDDYuμRDfϕufKR,
  LinearDDYuμRDfϕf3KR, LinearDDYuμRDDYuλR, LinearDDYuμRDDYf2λR,
  LinearDDYuμRDfϕufλR, LinearDDYuμRDfϕf3λR, LinearDDYuμRDDYuμR,
  LinearDDYuμRDDYf2μR, LinearDDYuμRDfϕufμR, LinearDDYuμRDfϕf3μR, LinearDDYuμRσR},
 {LinearDDYf2μRDDYuAR, LinearDDYf2μRDDYf2AR, LinearDDYf2μRDfϕufAR,
  LinearDDYf2μRDfϕf3AR, LinearDDYf2μRDDYuKR, LinearDDYf2μRDDYf2KR,
  LinearDDYf2μRDfϕufKR, LinearDDYf2μRDfϕf3KR,
  LinearDDYf2μRDDYuλR, LinearDDYf2μRDDYf2λR, LinearDDYf2μRDfϕufλR,
  LinearDDYf2μRDfϕf3λR, LinearDDYf2μRDDYuμR, LinearDDYf2μRDDYf2μR,
  LinearDDYf2μRDfϕufμR, LinearDDYf2μRDfϕf3μR, LinearDDYf2μRσR},
 {LinearDfϕufμRDDYuAR, LinearDfϕufμRDDYf2AR, LinearDfϕufμRDfϕufAR,
  LinearDfϕufμRDfϕf3AR, LinearDfϕufμRDDYuKR, LinearDfϕufμRDDYf2KR,
  LinearDfϕufμRDfϕufKR, LinearDfϕufμRDfϕf3KR,
  LinearDfϕufμRDDYuλR, LinearDfϕufμRDDYf2λR, LinearDfϕufμRDfϕufλR,
  LinearDfϕufμRDfϕf3λR, LinearDfϕufμRDDYuμR, LinearDfϕufμRDDYf2μR,
  LinearDfϕufμRDfϕufμR, LinearDfϕufμRDfϕf3μR, LinearDfϕufμRσR},
 {LinearDfϕf3μRDDYuAR, LinearDfϕf3μRDDYf2AR, LinearDfϕf3μRDfϕufAR,
  LinearDfϕf3μRDfϕf3AR, LinearDfϕf3μRDDYuKR, LinearDfϕf3μRDDYf2KR,
  LinearDfϕf3μRDfϕufKR, LinearDfϕf3μRDfϕf3KR, LinearDfϕf3μRDDYuλR,
  LinearDfϕf3μRDDYf2λR, LinearDfϕf3μRDfϕufλR, LinearDfϕf3μRDfϕf3λR,
  LinearDfϕf3μRDDYuμR, LinearDfϕf3μRDDYf2μR, LinearDfϕf3μRDfϕufμR,
  LinearDfϕf3μRDfϕf3μR, LinearDfϕf3μRσR}, {LinearσRDDYuAR,
  LinearσRDDYf2AR, LinearσRDfϕufAR, LinearσRDfϕf3AR, LinearσRDDYuKR,
  LinearσRDDYf2KR, LinearσRDfϕufKR, LinearσRDfϕf3KR, LinearσRDDYuλR,
  LinearσRDDYf2λR, LinearσRDfϕufλR, LinearσRDfϕf3λR, LinearσRDDYuμR,
  LinearσRDDYf2μR, LinearσRDfϕufμR, LinearσRDfϕf3μR, LinearσRσR}};
```

In[ ]:=

```
MStabilityT = Transpose[MStability];
MStabilityTD = {DDR * MStabilityT[[1]], DDR * MStabilityT[[2]],
   DfR * MStabilityT[[3]], DfR * MStabilityT[[4]], DDR * MStabilityT[[5]],
   DDR * MStabilityT[[6]], DfR * MStabilityT[[7]], DfR * MStabilityT[[8]],
   DDR * MStabilityT[[9]], DDR * MStabilityT[[10]], DfR * MStabilityT[[11]],
   DfR * MStabilityT[[12]], DDR * MStabilityT[[13]], DDR * MStabilityT[[14]],
   DfR * MStabilityT[[15]], DfR * MStabilityT[[16]], DfR * MStabilityT[[17]]};
MStabilityTDT = Simplify[Transpose[MStabilityTD]];
```

In[ ]:=
```
ξf = 1;
ξu = 1;
ξtf = 4;
ξtu = 4;
ξx = 1;
ξP = -3;
ξY = -3;
dim = 2;

Eigensystem[MStabilityTDT]
```

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