Lecture 2
ELE 301: Signals and Systems

Prof. Paul Cuff

Princeton University

Fall 2011-12

Models of Continuous Time Signals

Today's topics:

- Signals
  - Sinuoidal signals
  - Exponential signals
  - Complex exponential signals
  - Unit step and unit ramp
  - Impulse functions
- Systems
  - Memory
  - Invertibility
  - Causality
  - Stability
  - Time invariance
  - Linearity

Sinusoidal Signals

- A sinusoidal signal is of the form

$$x(t) = \cos(\omega t + \theta).$$

where the *radian frequency* is ω, which has the units of radians/s.
- Also very commonly written as

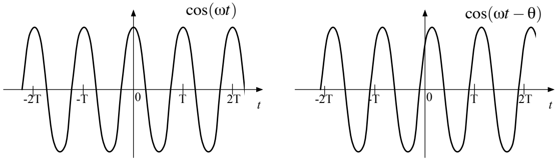
$$x(t) = A\cos(2\pi ft + \theta).$$

where f is the frequency in Hertz.
- We will often refer to ω as the frequency, but it must be kept in mind that it is really the *radian frequency*, and the *frequency* is actually f.

- The period of the sinuoid is

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

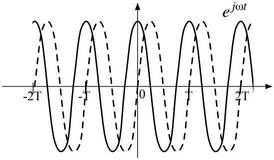
with the units of seconds.
- The *phase* or *phase angle* of the signal is θ, given in radians.



Complex Sinusoids

- The Euler relation defines $e^{j\phi} = \cos\phi + j\sin\phi$.
- A complex sinusoid is

$$Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + jA\sin(\omega t+\theta).$$



- Real sinusoid can be represented as the real part of a complex sinusoid

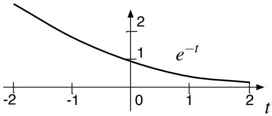
$$\Re\{Ae^{j(\omega t+\theta)}\} = A\cos(\omega t+\theta)$$

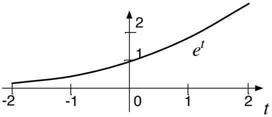
Exponential Signals

- An exponential signal is given by

$$x(t) = e^{\sigma t}$$

- If $\sigma < 0$ this is *exponential decay*.
- If $\sigma > 0$ this is *exponential growth*.



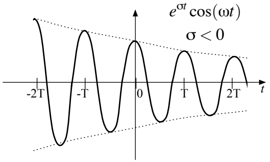


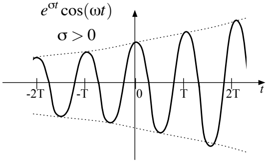
Damped or Growing Sinusoids

- A damped or growing sinusoid is given by

$$x(t) = e^{\sigma t}\cos(\omega t + \theta)$$

- Exponential growth ($\sigma > 0$) or decay ($\sigma < 0$), modulated by a sinusoid.



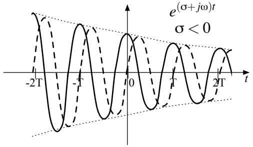


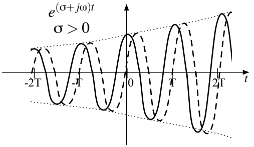
Complex Exponential Signals

- A complex exponential signal is given by

$$e^{(\sigma + j\omega)t + j\theta} = e^{\sigma t}(\cos(\omega t + \theta) + i\sin(\omega t + \theta))$$

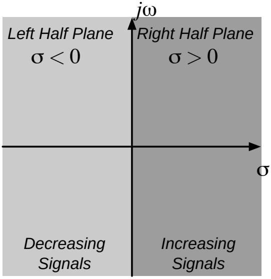
- A exponential growth or decay, modulated by a complex sinusoid.
- Includes all of the previous signals as special cases.





Complex Plane

Each complex frequency $s = \sigma + j\omega$ corresponds to a position in the complex plane.



Demonstration

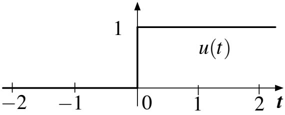
Take a look at complex exponentials in 3-dimensions by using "TheComplexExponential" at demonstrations.wolfram.com

Unit Step Functions

- The *unit step function* $u(t)$ is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Also known as the *Heaviside step function*.
- Alternate definitions of value exactly at zero, such as $1/2$.



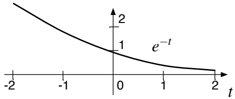
Uses for the unit step:

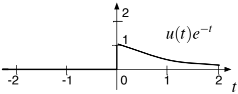
- Extracting part of another signal. For example, the piecewise-defined signal

$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

can be written as

$$x(t) = u(t)e^{-t}$$



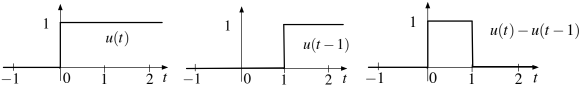


- Combinations of unit steps to create other signals. The offset rectangular signal

$$x(t) = \begin{cases} 0, & t \geq 1 \\ 1, & 0 \leq t < 1 \\ 0, & t < 0 \end{cases}$$

can be written as

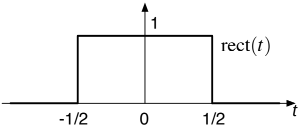
$$x(t) = u(t) - u(t-1).$$



Unit Rectangle

Unit rectangle signal:

$$\text{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$



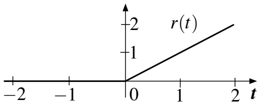
Unit Ramp

- The *unit ramp* is defined as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- The unit ramp is the integral of the unit step,

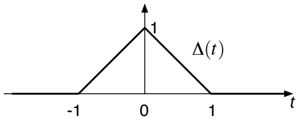
$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$



Unit Triangle

Unit Triangle Signal

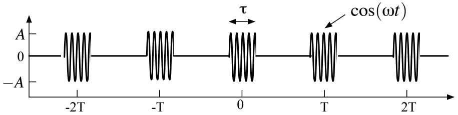
$$\Delta(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



More Complex Signals

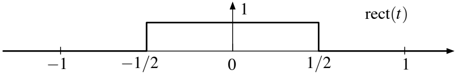
Many more interesting signals can be made up by combining these elements.

Example: Pulsed Doppler RF Waveform (we'll talk about this later!)

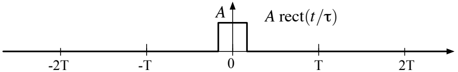


RF cosine gated on for τ μs, repeated every T μs, for a total of N pulses.

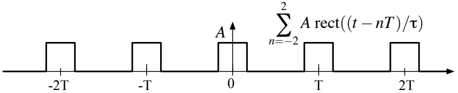
Start with a simple rect(t) pulse



Scale to the correct duration and amplitude for one subpulse

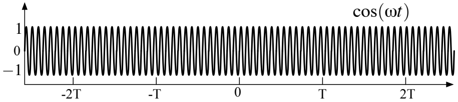


Combine shifted replicas

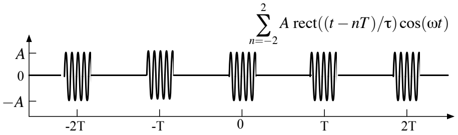


This is the *envelope* of the signal.

Then multiply by the RF carrier, shown below



to produce the pulsed Doppler waveform

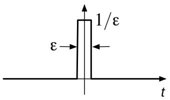


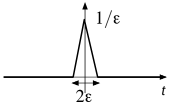
Impulsive signals

(Dirac's) **delta function** or **impulse** δ is an *idealization* of a signal that

- is very large near $t = 0$
- is very small away from $t = 0$
- has integral 1

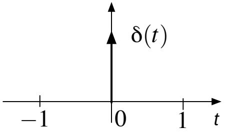
for example:

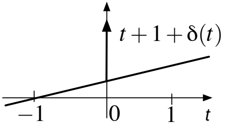




- the exact shape of the function doesn't matter
- ϵ is small (which depends on context)

On plots δ is shown as a solid arrow:





"Delta function" is not a function

Formal properties

Formally we **define** δ by the property that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0)$$

provided f is continuous at $t = 0$

idea: δ acts over a time interval very small, over which $f(t) \approx f(0)$

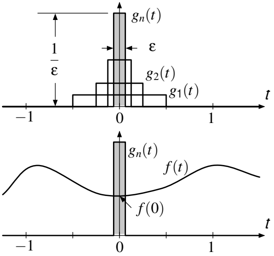
- $\delta(t)$ is not really defined for any t, only its behavior in an integral.
- Conceptually $\delta(t) = 0$ for $t \neq 0$, infinite at $t = 0$, but this doesn't make sense mathematically.

Example: Model $\delta(t)$ as

$$g_n(t) = n\,\mathrm{rect}(nt)$$

as $n \to \infty$. This has an area $(n)(1/n) = 1$. If $f(t)$ is continuous at $t = 0$, then

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)g_n(t)\, dt = f(0) \int_{-\infty}^{\infty} g_n(t)\, dt = f(0)$$



Scaled impulses

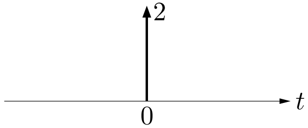
$\alpha\delta(t)$ is an impulse at time T, with *magnitude* or *strength* α

We have

$$\int_{-\infty}^{\infty} \alpha\delta(t)f(t)\,dt = \alpha f(0)$$

provided f is continuous at 0

On plots: write area next to the arrow, *e.g.*, for $2\delta(t)$,



Multiplication of a Function by an Impulse

- Consider a function $\phi(x)$ multiplied by an impulse $\delta(t)$,

$$\phi(t)\delta(t)$$

  If $\phi(t)$ is continuous at $t = 0$, can this be simplified?
- Substitute into the formal definition with a continuous $f(t)$ and evaluate,

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\left[\phi(t)\delta(t)\right]\,dt &= \int_{-\infty}^{\infty} \left[f(t)\phi(t)\right]\delta(t)\,dt \\ &= f(0)\phi(0)\end{aligned}$$

- Hence

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

  is a scaled impulse, with strength $\phi(0)$.

Sifting property

- The signal $x(t) = \delta(t - T)$ is an impulse function with impulse at $t = T$.
- For f continuous at $t = T$,
$$\int_{-\infty}^{\infty} f(t)\delta(t - T)\, dt = f(T)$$
- Multiplying by a function $f(t)$ by an impulse at time T and integrating, extracts the value of $f(T)$.
- This will be important in modeling sampling later in the course.

Limits of Integration

The integral of a δ is non-zero only if it is in the integration interval:

- If $a < 0$ and $b > 0$ then
$$\int_a^b \delta(t)\, dt = 1$$
because the δ is within the limits.
- If $a > 0$ or $b < 0$, and $a < b$ then
$$\int_a^b \delta(t)\, dt = 0$$
because the δ is outside the integration interval.
- **Ambiguous** if $a = 0$ or $b = 0$

Our convention: to avoid confusion we use limits such as $a-$ or $b+$ to denote whether we include the impulse or not.

$$\int_{0+}^{1} \delta(t)\, dt = 0, \quad \int_{0-}^{1} \delta(t)\, dt = 1, \quad \int_{-1}^{0-} \delta(t)\, dt = 0, \quad \int_{-1}^{0+} \delta(t)\, dt = 1$$

example:

$$\begin{aligned}
&\int_{-2}^{3} f(t)(2 + \delta(t+1) - 3\delta(t-1) + 2\delta(t+3))\, dt \\
&= 2\int_{-2}^{3} f(t)\, dt + \int_{-2}^{3} f(t)\delta(t+1)\, dt - 3\int_{-2}^{3} f(t)\delta(t-1)\, dt \\
&\quad + 2\int_{-2}^{3} f(t)\delta(t+3))\, dt \\
&= 2\int_{-2}^{3} f(t)\, dt + f(-1) - 3f(1)
\end{aligned}$$

Physical interpretation

Impulse functions are used to model physical signals

- that act over short time intervals
- whose effect depends on integral of signal

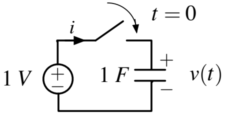
example: hammer blow, or bat hitting ball, at $t = 2$

- force f acts on mass m between $t = 1.999$ sec and $t = 2.001$ sec
- $\int_{1.999}^{2.001} f(t)\, dt = I$ (mechanical impulse, $\mathrm{N} \cdot \mathrm{sec}$)
- blow induces change in velocity of

$$v(2.001) - v(1.999) = \frac{1}{m}\int_{1.999}^{2.001} f(\tau)\, d\tau = I/m$$

For most applications, model force as impulse at $t = 2$, with magnitude I.

example: rapid charging of capacitor

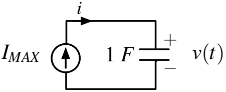


assuming $v(0) = 0$, what is $v(t)$, $i(t)$ for $t > 0$?

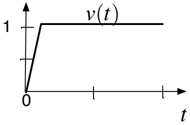
- $i(t)$ is very large, for a very short time
- a unit charge is transferred to the capacitor 'almost instantaneously'
- $v(t)$ increases to $v(t) = 1$ 'almost instantaneously'

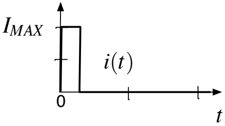
To calculate i, v, we need a more detailed model.

For example, assume the current delivered by the source is limited: if $v(t) < 1$, the source acts as a current source $i(t) = I_{\max}$



$$i(t) = \frac{dv(t)}{dt} = I_{\max}, \quad v(0) = 0$$





As $I_{\max} \to \infty$, i approaches an impulse, v approaches a unit step

In conclusion,

- large current i acts over very short time between $t = 0$ and ϵ
- total charge transfer is $\int_0^{\epsilon} i(t)\, dt = 1$
- resulting change in $v(t)$ is $v(\epsilon) - v(0) = 1$
- can approximate i as impulse at $t = 0$ with magnitude 1

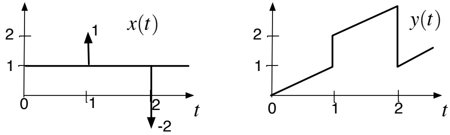
Modeling current as impulse

- obscures details of current signal
- obscures details of voltage change during the rapid charging
- preserves total change in charge, voltage
- is reasonable model for time scales $\gg \epsilon$

Integrals of impulsive functions

Integral of a function with impulses has jump at each impulse, equal to the magnitude of impulse

example: $x(t) = 1 + \delta(t-1) - 2\delta(t-2)$; define $y(t) = \int_0^t x(\tau)\, d\tau$

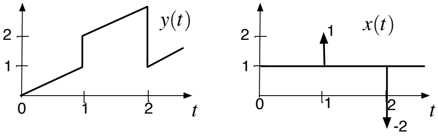


Derivatives of discontinuous functions

Conversely, derivative of function with discontinuities has impulse at each jump in function

- Derivative of unit step function $u(t)$ is $\delta(t)$
- Signal y of previous page

$$y'(t) = 1 + \delta(t-1) - 2\delta(t-2)$$



Derivatives of impulse functions

Integration by parts suggests we define

$$\int_{-\infty}^{\infty} \delta'(t) f(t)\, dt = \delta(t) f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t) f'(t)\, dt = -f'(0)$$

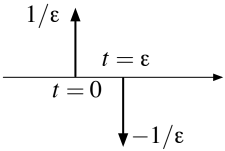
provided f' continuous at $t = 0$

- δ' is called *doublet*
- δ', δ'', etc. are called *higher-order impulses*
- Similar rules for higher-order impulses:

$$\int_{-\infty}^{\infty} \delta^{(k)}(t) f(t)\, dt = (-1)^k f^{(k)}(0)$$

if $f^{(k)}$ continuous at $t = 0$

interpretation of doublet δ': take two impulses with magnitude $\pm 1/\epsilon$, a distance ϵ apart, and let $\epsilon \to 0$



Then

$$\int_{-\infty}^{\infty} f(t)\left(\frac{\delta(t)}{\epsilon} - \frac{\delta(t-\epsilon)}{\epsilon}\right)\, dt = \frac{f(0) - f(\epsilon)}{\epsilon}$$

converges to $-f'(0)$ if $\epsilon \to 0$

Caveat

$\delta(t)$ is not a signal or function in the ordinary sense, it only makes mathematical sense when inside an integral sign

- We manipulate impulsive functions as if they were real functions, which they aren't
- It is safe to use impulsive functions in expressions like

  $$\int_{-\infty}^{\infty} f(t)\delta(t-T)\, dt, \quad \int_{-\infty}^{\infty} f(t)\delta'(t-T)\, dt$$

  provided f (resp, f') is continuous at $t = T$.
- Some innocent looking expressions don't make any sense at all (*e.g.*, $\delta(t)^2$ or $\delta(t^2)$)

Break

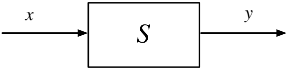
Talk about Office hours and coming to the first lab.

Systems

- A system transforms *input signals* into *output signals*.
- A system is a *function* mapping input signals into output signals.
- We will concentrate on systems with one input and one output *i.e. single-input, single-output* (SISO) systems.
- Notation:
  - $y = Sx$ or $y = S(x)$, meaning the system S acts on an input signal x to produce output signal y.
  - $y = Sx$ does not (in general) mean multiplication!

Block diagrams

Systems often denoted by *block diagram*:



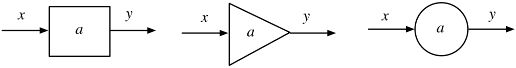
- Lines with arrows denote signals (*not* wires).
- Boxes denote systems; arrows show inputs & outputs.
- Special symbols for some systems.

Examples

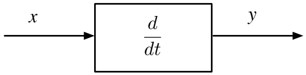
(with input signal x and output signal y)

Scaling system: $y(t) = ax(t)$

- Called an *amplifier* if $|a| > 1$.
- Called an *attenuator* if $|a| < 1$.
- Called *inverting* if $a < 0$.
- a is called the *gain* or *scale factor*.
- Sometimes denoted by triangle or circle in block diagram:

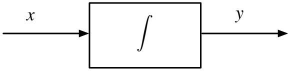


Differentiator: $y(t) = x'(t)$



Integrator: $y(t) = \int_a^t x(\tau)\, d\tau$ (a is often 0 or $-\infty$)

Common notation for integrator:



time shift system: $y(t) = x(t - T)$

- called a *delay system* if $T > 0$
- called a *predictor system* if $T < 0$

convolution system:

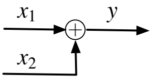
$$y(t) = \int x(t-\tau)h(\tau)\, d\tau,$$

where h is a given function (you'll be hearing much more about this!)

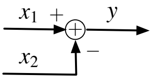
Examples with multiple inputs

Inputs $x_1(t)$, $x_2(t)$, and Output $y(t)$)

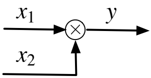
- **summing system:** $y(t) = x_1(t) + x_2(t)$



- **difference system:** $y(t) = x_1(t) - x_2(t)$



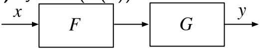
- **multiplier system:** $y(t) = x_1(t)x_2(t)$



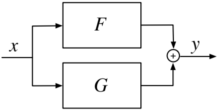
Interconnection of Systems

We can interconnect systems to form new systems,

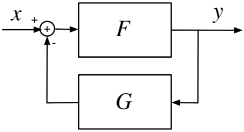
- **cascade (or series):** $y = G(F(x)) = GFx$



  (note that block diagrams and algebra are *reversed*)
- **sum (or parallel):** $y = Fx + Gx$



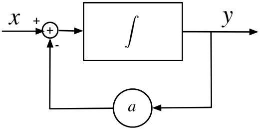
- **feedback:** $y = F(x - Gy)$



In general,

- Block diagrams are a symbolic way to describe a connection of systems.
- We can just as well write out the equations relating the signals.
- We can go back and forth between the system block diagram and the system equations.

Example: Integrator with feedback



Input to integrator is $x - ay$, so

$$\int^{t} (x(\tau) - ay(\tau))\, d\tau = y(t)$$

Another useful method: the *input* to an integrator is the derivative of its output, so we have

$$x - ay = y'$$

(of course, same as above)

Linearity

A system F is **linear** if the following two properties hold:

1. **homogeneity:** if x is any signal and a is any scalar,

$$F(ax) = aF(x)$$

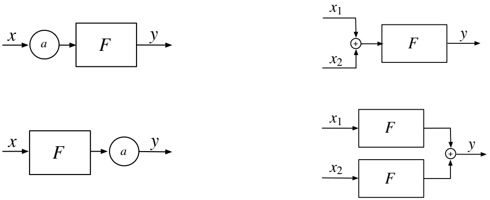
2. **superposition:** if x and $\tilde{x}$ are any two signals,

$$F(x + \tilde{x}) = F(x) + F(\tilde{x})$$

In words, linearity means:

- Scaling before or after the system is the same.
- Summing before or after the system is the same.

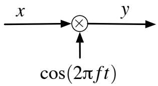
Linearity means the following pairs of block diagrams are equivalent, *i.e.*, have the same output for any input(s)



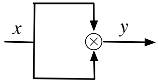
Examples of linear systems: scaling system, differentiator, integrator, running average, time shift, convolution, modulator, sampler.

Examples of nonlinear systems: sign detector, multiplier (sometimes), comparator, quantizer, adaptive filter

- Multiplier as a modulator, $y(t) = x(t)\cos(2\pi ft)$, is *linear*.



- Multiplier as a squaring system, $y(t) = x^2(t)$ is *nonlinear*.



System Memory

- A system is *memoryless* if the output depends only on the present input.
  - Ideal amplifier
  - Ideal gear, transmission, or lever in a mechanical system
- A *system with memory* has an output signal that depends on inputs in the past or future.
  - Energy storage circuit elements such as capacitors and inductors
  - Springs or moving masses in mechanical systems
- A *causal* system has an output that depends only on past or present inputs.
  - Any real physical circuit, or mechanical system.

Time-Invariance

- A system is time-invariant if a time shift in the input produces the same time shift in the output.
- For a system F,

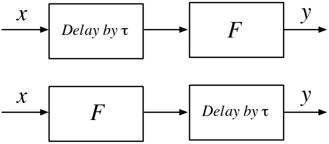
$$y(t) = Fx(t)$$

implies that

$$y(t-\tau) = Fx(t-\tau)$$

for any time shift τ.

- Implies that delay and the system F commute. These block diagrams are equivalent:



- Time invariance is an important system property. It greatly simplifies the analysis of systems.

System Stability

- Stability important for most engineering applications.
- Many definitions
- If a bounded input

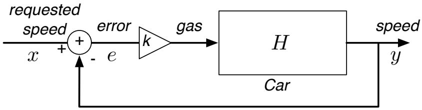
$$|x(t)| \leq M_x < \infty$$

always results in a bounded output

$$|y(t)| \leq M_y < \infty,$$

where M_x and M_y are finite positive numbers, the system is *Bounded Input Bounded Output (BIBO) stable*.

Example: Cruise control, from introduction,



The output y is

$$y = H(k(x - y))$$

We'll see later that this system can become unstable if k is too large (depending on H)

- Positive error adds gas
- Delay car velocity change, speed overshoots
- Negative error cuts gas off
- Delay in velocity change, speed undershoots
- Repeat!

System Invertibility

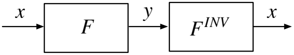
- A system is invertible if the input signal can be recovered from the output signal.
- If F is an invertible system, and

$$y = Fx$$

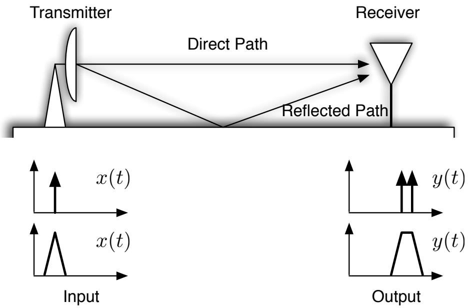
then there is an inverse system F^{INV} such that

$$x = F^{INV} y = F^{INV} F x$$

so $F^{INV} F = I$, the identity operator.



Example: Multipath echo cancelation



Important problem in communications, radar, radio, cell phones.

Generally there will be multiple echoes.

Multipath can be described by a system $y = Fx$

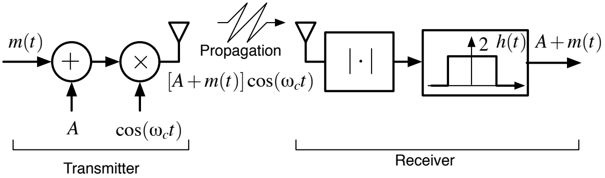
- If we transmit an impulse, we receive multiple delayed impulses.
- One transmitted message gives multiple overlapping messages

We want to find a system F^{INV} that takes the multipath corrupted signal y and recovers x

$$\begin{aligned} F^{INV} y &= F^{INV}(Fx) \\ &= (F^{INV} F)\, x \\ &= x \end{aligned}$$

Often possible if we allow a delay in the output.

Example: AM Radio Transmitter and receiver



- Multiple input systems
- Linear and non-linear systems

Systems Described by Differential Equations

Many systems are described by a *linear constant coefficient ordinary differential equation* (LCCODE):

$$a_n y^{(n)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = b_m x^{(m)}(t) + \cdots + b_1 x'(t) + b_0 x(t)$$

with given *initial conditions*

$$y^{(n-1)}(0), \quad \ldots \quad , y'(0), \quad y(0)$$

(which fixes $y(t)$, given $x(t)$)

- n is called the *order* of the system
- $b_0, \ldots, b_m, a_0, \ldots, a_n$ are the *coefficients* of the system

This is important because LCCODE systems are **linear** when initial conditions are all zero.

- Many systems can be described this way
- If we can describe a system this way, we know it is linear

Note that an LCCODE gives an *implicit* description of a system.

- It describes how $x(t)$, $y(t)$, and their derivatives interrelate
- It doesn't give you an explicit solution for $y(t)$ in terms of $x(t)$

Soon we'll be able to *explicitly* express $y(t)$ in terms of $x(t)$

Examples

Simple examples

- scaling system ($a_0 = 1,\ b_0 = a$)

$$y = ax$$

- integrator ($a_1 = 1,\ b_0 = 1$)

$$y' = x$$

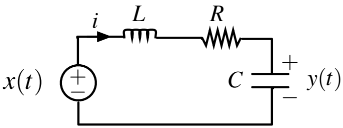
- differentiator ($a_0 = 1,\ b_1 = 1$)

$$y = x'$$

- integrator with feedback (a few slides back, $a_1 = 1, a_0 = a, b_0 = 1$)

$$y' + ay = x$$

2nd Order Circuit Example



By Kirchoff's voltage law

$$x - Li' - Ri - y = 0$$

Using $i = Cy'$,

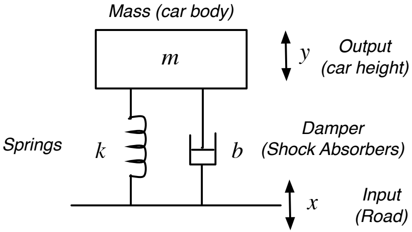
$$x - LCy'' - RCy' - y = 0$$

or

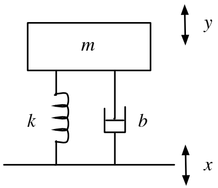
$$LCy'' + RCy' + y = x$$

which is an LCCODE. This is a linear system.

Mechanical System



This can represent suspension system, or building during earthquake, ...



- $x(t)$ is displacement of base; $y(t)$ is displacement of mass
- spring force is $k(x-y)$; damping force is $b(x-y)'$
- Newton's equation is $my'' = b(x-y)' + k(x-y)$

Rewrite as second-order LCCODE

$$my'' + by' + ky = bx' + kx$$

This is a linear system.

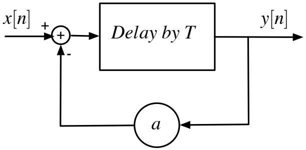
Discrete-Time Systems

- Many of the same block diagram elements
- Scaling and delay blocks common
- The system equations are *difference equations*

$$a_0 y[n] + a_1 y[n-1] + \ldots = b_0 x[n] + b_1 x[n-1] + \ldots$$

where $x[n]$ is the input, and $y[n]$ is the output.

Discrete-Time System Example



- The input into the delay is

$$e[n] = x[n] - ay[n]$$

- The output is $y[n] = e[n-1]$, so

$$y[n] = x[n-1] - ay[n-1].$$

Questions

Are these systems linear? Time invariant?

- $y(t) = \sqrt{x(t)}$
- $y(t) = x(t)z(t)$, where $z(t)$ is a known function
- $y(t) = x(at)$
- $y(t) = 0$
- $y(t) = x(T-t)$

A linear system F has an inverse system F^{inv}. Is F^{inv} linear?