Lecture 3
ELE 301: Signals and Systems

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Slides courtesy of John Pauly (Stanford)

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Time Domain Analysis of Continuous Time Systems

Today's topics

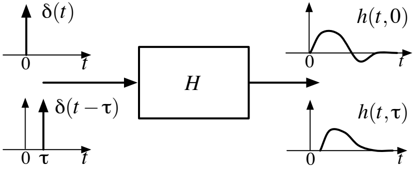
- Impulse response
- Extended linearity
- Response of a linear time-invariant (LTI) system
- Convolution
- Zero-input and zero-state responses of a system

Impulse Response

The *impulse response* of a linear system $h_\tau(t)$ is the output of the system at time t to an impulse at time τ. This can be written as

$$h_\tau = H(\delta_\tau)$$

Care is required in interpreting this expression!



Note: Be aware of potential confusion here:

When you write

$$h_\tau(t) = H(\delta_\tau(t))$$

the variable t serves different roles on each side of the equation.

- t on the left is a specific value for time, the time at which the output is being sampled.
- t on the right is varying over all real numbers, it is not the same t as on the left.
- The output at time specific time t on the left in general depends on the input at all times t on the right (the entire input waveform).

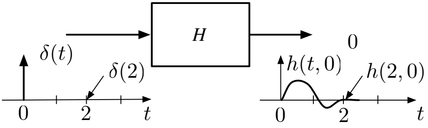
- Assume the input impulse is at $\tau = 0$,

$$h = h_0 = H(\delta_0).$$

We want to know the impulse response at time $t = 2$. It doesn't make any sense to set $t = 2$, and write

$$h(2) = H(\delta(2)) \qquad \Leftarrow \text{No!}$$

First, $\delta(2)$ is something like zero, so $H(0)$ would be zero. Second, the value of $h(2)$ depends on the entire input waveform, not just the value at $t = 2$.

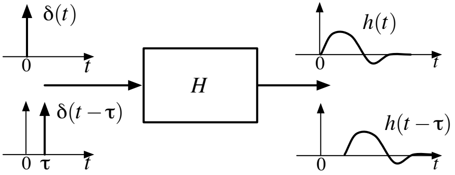


Time-invariance

If H is time invariant, delaying the input and output both by a time τ should produce the same response

$$h_\tau(t) = h(t - \tau).$$

In this case, we don't need to worry about h_τ because it is just h shifted in time.



Linearity and Extended Linearity

Linearity: A system S is linear if it satisfies both

- *Homogeneity:* If $y = Sx$, and a is a constant then
$$ay = S(ax).$$
- *Superposition:* If $y_1 = Sx_1$ and $y_2 = Sx_2$, then
$$y_1 + y_2 = S(x_1 + x_2).$$

Combined Homogeneity and Superposition:
If $y_1 = Sx_1$ and $y_2 = Sx_2$, and a and b are constants,

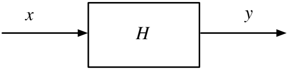
$$ay_1 + by_2 = S(ax_1 + bx_2)$$

Extended Linearity

- *Summation:* If $y_n = Sx_n$ for all n, an integer from $(-\infty < n < \infty)$, and a_n are constants
$$\sum_n a_n y_n = S\left(\sum_n a_n x_n\right)$$
Summation and the system operator commute, and can be interchanged.
- *Integration (Simple Example)* : If $y = Sx$,
$$\int_{-\infty}^{\infty} a(\tau)y(t-\tau)\,d\tau = S\left(\int_{-\infty}^{\infty} a(\tau)x(t-\tau)d\tau\right)$$
Integration and the system operator commute, and can be interchanged.

Output of an LTI System

We would like to determine an expression for the output $y(t)$ of an linear time invariant system, given an input $x(t)$



We can write a signal $x(t)$ as a sample of itself

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta_\tau(t)\, d\tau$$

This means that $x(t)$ can be written as a weighted integral of δ functions.

Applying the system H to the input $x(t)$,

$$\begin{aligned} y(t) &= H(x(t)) \\ &= H\left(\int_{-\infty}^{\infty} x(\tau)\delta_\tau(t) d\tau\right) \end{aligned}$$

If the system obeys extended linearity we can interchange the order of the system operator and the integration

$$y(t) = \int_{-\infty}^{\infty} x(\tau) H(\delta_\tau(t))\, d\tau.$$

The impulse response is

$$h_\tau(t) = H(\delta_\tau(t)).$$

Substituting for the impulse response gives

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h_\tau(t) d\tau.$$

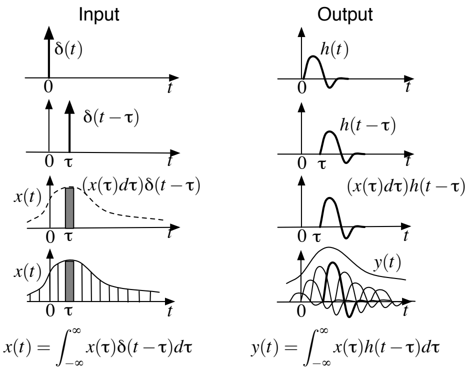
This is a *superposition integral*. The values of $x(\tau)h(t,\tau)d\tau$ are superimposed (added up) for each input time τ.

If H is time invariant, this written more simply as

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h_\tau(t) d\tau.$$

This is in the form of a *convolution integral*, which will be the subject of the next class.

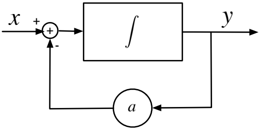
Graphically, this can be represented as:



System Equation

The *System Equation* relates the outputs of a system to its inputs.

Example from last time: the system described by the block diagram



has a system equation

$$y' + ay = x.$$

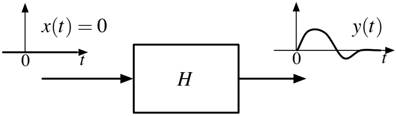
In addition, the initial conditions must be given to uniquely specify a solution.

Solutions for the System Equation

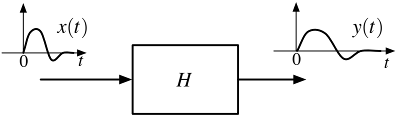
Solving the system equation tells us the output for a given input.

The output consists of two components:

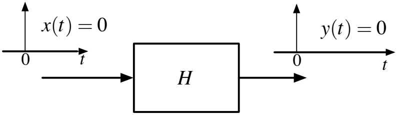
- The *zero-input* response, which is what the system does with no input at all. This is due to initial conditions, such as energy stored in capacitors and inductors.



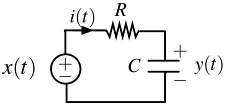
- The *zero-state* response, which is the output of the system with all initial conditions zero.



If H is a linear system, its zero-input response is zero. Homogeneity states if $y = F(ax)$, then $y = aF(x)$. If $a = 0$ then a zero input requires a zero output.



Example: Solve for the voltage across the capacitor $y(t)$ for an arbitrary input voltage $x(t)$, given an initial value $y(0) = Y_0$.



From Kirchhoff's voltage law

$$x(t) = Ri(t) + y(t)$$

Using $i(t) = Cy'(t)$

$$RCy'(t) + y(t) = x(t).$$

This is a first order LCCODE, which is linear with zero initial conditions.

First we solve for the homogeneous solution by setting the right side (the input) to zero

$$RCy'(t) + y(t) = 0.$$

The solution to this is

$$y(t) = Ae^{-t/RC}$$

which can be verified by direct substitution.

To solve for the total response, we let the undetermined coefficient be a function of time

$$y(t) = A(t)e^{-t/RC}.$$

Substituting this into the differential equation

$$RC\left[A'(t)e^{-t/RC} - \frac{1}{RC}A(t)e^{-t/RC}\right] + A(t)e^{-t/RC} = x(t)$$

Simplifying

$$A'(t) = x(t)\left[\frac{1}{RC}e^{t/RC}\right]$$

which can be integrated from $t = 0$ to get

$$A(t) = \int_0^t x(\tau)\left[\frac{1}{RC}e^{\tau/RC}\right]d\tau + A(0)$$

Then

$$\begin{aligned} y(t) &= A(t)e^{-t/RC} \\ &= e^{-t/RC}\int_0^t x(\tau)\left[\frac{1}{RC}e^{\tau/RC}\right]d\tau + A(0)e^{-t/RC} \\ &= \int_0^t x(\tau)\left[\frac{1}{RC}e^{-(t-\tau)/RC}\right]d\tau + A(0)e^{-t/RC} \end{aligned}$$

At $t = 0$, $y(0) = Y_0$, so this gives $A(0) = Y_0$

$$y(t) = \underbrace{\int_0^t x(\tau)\left[\frac{1}{RC}e^{-(t-\tau)/RC}\right]d\tau}_{\text{zero}-\text{state response}} + \underbrace{Y_0 e^{-t/RC}}_{\text{zero}-\text{input response}}.$$

RC Circuit example

The impulse response of the RC circuit example is

$$h(t) = \frac{1}{RC}e^{-t/RC}$$

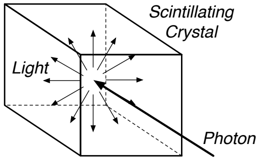
The response of this system to an input $x(t)$ is then

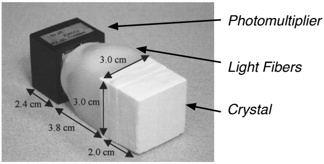
$$\begin{aligned} y(t) &= \int_0^t x(\tau)h\tau(t)d\tau \\ &= \int_0^t x(\tau)\left[\frac{1}{RC}e^{-(t-\tau)/RC}\right]d\tau \end{aligned}$$

which is the zero state solution we found earlier.

Example:

High energy photon detectors can be modeled as having a simple exponential decay impulse response.



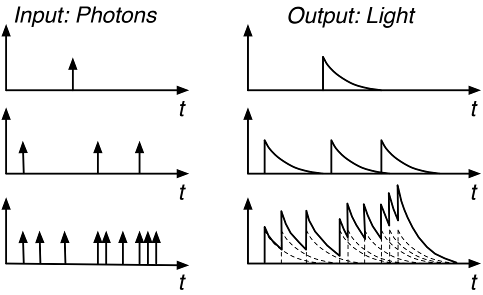


From: Doshi et al, Med Phys. 27(7), p1535 July 2000

These are used in Positiron Emmision Tomography (PET) systems.

Input is a sequence of impulses (photons).

Output is superposition of impulse responses (light).



Summary

- For an input $x(t)$, the output of an linear system is given by the *superposition integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h_\tau(t)\ d\tau$$

- If the system is also time invariant, the result is a *convolution integral*

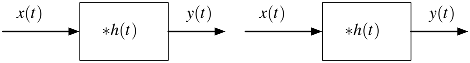
$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\ d\tau$$

- The response of an LTI system is completely characterized by its *impulse response* $h(t)$.

Another expression for the superposition integral can be found by substituting for $\tau = t - \tau_1$. Then $d\tau = -d\tau_1$ and $\tau_1 = t - \tau$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \\ &= \int_{\infty}^{-\infty} x(t-\tau_1) h(t-(t-\tau_1)) d(-\tau_1) \\ &= \int_{-\infty}^{\infty} x(t-\tau_1) h(\tau_1) d\tau_1. \end{aligned}$$

The block diagrams for a system using the impulse response:



Superposition Integral for Causal Systems

For a causal system $h(t) = 0$ for $t < 0$, and

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h^{\tau}(t)\ d\tau.$$

Since $h^{\tau}(t) = 0$ for $t < \tau$, we can replace the upper limit of the integral by t

$$y(t) = \int_{-\infty}^{t} x(\tau) h\tau(t)\ d\tau.$$

Only past and present values of $x(\tau)$ contribute to $y(t)$.

LTI System Response to a Sinusoidal Input

A LTI system has a real impulse response $h(t)$. A sinusoidal input

$$x(t) = A\cos(2\pi f_1 t + \theta)$$

produces an output

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\left[A\cos(2\pi f_1(t-\tau)+\theta)\right] d\tau.$$

Using the identity $\cos(a-b) = \cos a\ \cos b + \sin a\ \sin b$,

$$\begin{aligned} y(t) \quad &= \quad A\cos(2\pi f_1 t + \theta)\int_{-\infty}^{\infty} h(\tau)\cos(2\pi f_1 \tau) d\tau \\ &\quad + A\sin(2\pi f_1 t + \theta)\int_{-\infty}^{\infty} h(\tau)\sin(2\pi f_1 \tau) d\tau. \end{aligned}$$

Since $h(t)$ is real,

$$y(t) = H_c(f_1)A\cos(2\pi f_1 t + \theta) + H_s(f_1)A\sin(2\pi f_1 t + \theta).$$

where

$$\begin{aligned} H_c(f_1) &= \int_{-\infty}^{\infty} h(\tau)\cos(2\pi f_1 \tau)d\tau \\ H_s(f_1) &= \int_{-\infty}^{\infty} h(\tau)\sin(2\pi f_1 \tau)d\tau \end{aligned}$$

are real constants.

We can then write the output as

$$y(t) = |H(f_1)|A\cos\left(2\pi f_1 t + \theta + \angle H(f_1)\right)$$

(using the same trigonometric identity in reverse), where

$$\begin{aligned} |H(f)| &= \sqrt{H_c^2(f_1) + H_s^2(f_1)} \\ \angle H(f_1) &= \tan^{-1}(-H_s(f_1)/H_c(f_1)) \end{aligned}$$

Note that the response to a sinusoidal input is determined by a single complex number $H(f_1)$, which determines the magnitude of the output, and the phase shift.

A sinusoidal input is scaled and delayed by an LTI system, but is otherwise unchanged.

Summary

- The response of an LTI system is completely characterized by its *impulse response* $h(t)$.
- For an input $x(t)$, the output of an linear system is given by the *superposition integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h_\tau(t)\, d\tau$$

- If the system is also time invariant, the result is a *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

- For a sinusoidal input at frequency f,the output is
  - a sinusoid at the same frequency,
  - scaled in amplitude, and
  - phase shifted.

  This can be represented by a single complex number $H(f)$.

Convolution Evaluation and Properties

- Review: response of an LTI system
- Representation of convolution
- Graphical interpretation
- Examples
- Properties of convolution

Convolution Integral

The convolution of an input signal $x(t)$ with and impulse response $h(t)$ is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \\ &= (x * h)(t) \end{aligned}$$

or

$$y = x * h.$$

This is also often written as

$$y(t) = x(t) * h(t)$$

which is potentially confusing, since the t's have different interpretations on the left and right sides of the equation (your book does this).

Convolution Integral for Causal Systems

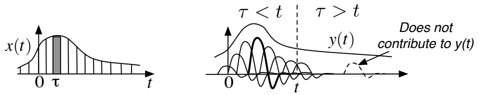
For a causal system $h(t) = 0$ for $t < 0$, and

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau.$$

Since $h(t-\tau) = 0$ for $t < \tau$, the upper limit of the integral is t

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau) d\tau.$$

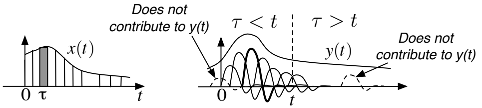
Only past and present values of $x(\tau)$ contribute to $y(t)$.



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

If $x(t)$ is also causal, $x(t) = 0$ for $t < 0$, and the integral further simplifies

$$y(t) = \int_{0}^{t} x(\tau)h(t-\tau)\, d\tau.$$



$$x(t) = \int_{0}^{t} x(\tau)\delta(t-\tau)d\tau \qquad y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

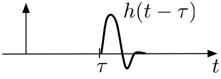
Graphical Interpretation

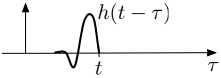
An increment in input $x(\tau)\delta_\tau(t)d\tau$ produces an impulse response $x(\tau)h_\tau(t)d\tau$. The output is the integral of all of these responses

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h\tau(t)\,d\tau$$

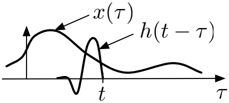
Another perspective is just to look at the integral.

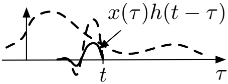
- $h_\tau(t) = h(t-\tau)$ is the impulse response delayed to time τ
- If we consider $h(t-\tau)$ to be a function of τ, then $h(t-\tau)$ is delayed to time t, and *reversed*.





- This is multiplied point by point with the input,



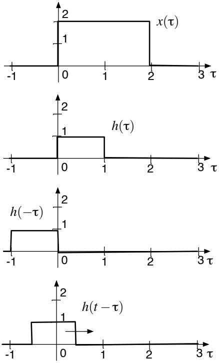


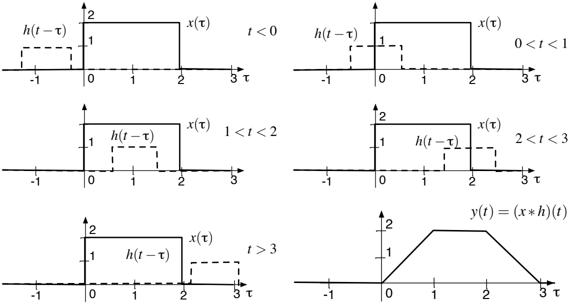
- Then integrate over τ to find $y(t)$ for this t.

Graphically, to find $y(t)$:

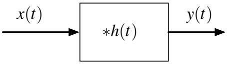
- flip impulse response $h(\tau)$ backwards in time (yields $h(-\tau)$)
- drag to the right over t (yields $h(-(\tau - t))$)
- multiply pointwise by x (yields $x(\tau)h(t-\tau)$)
- integrate over τ to get $y(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

Simple Example

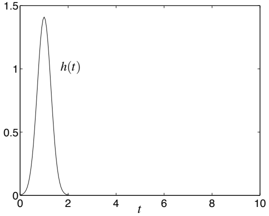




Communication channel, e.g., twisted pair cable

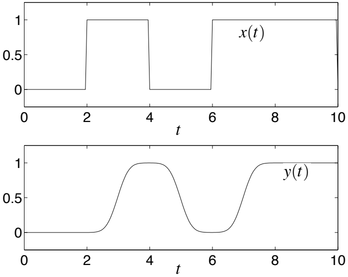


Impulse response:



This is a delay ≈ 1, plus smoothing.

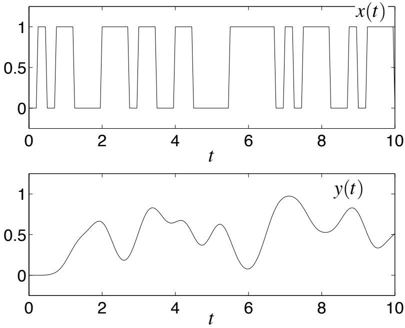
Simple signaling at 0.5 bit/sec; Boolean signal $0, 1, 0, 1, 1, \ldots$



Output is delayed, smoothed version of input.

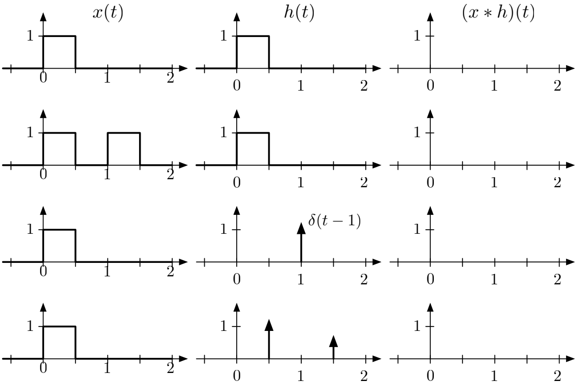
1's & 0's easily distinguished in y

Simple signalling at 4 bit/sec; same Boolean signal



Smoothing makes 1's & 0's very hard to distinguish in y.

Examples: Try these:



Properties of Convolution

For any two functions f and g the convolution is

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau$$

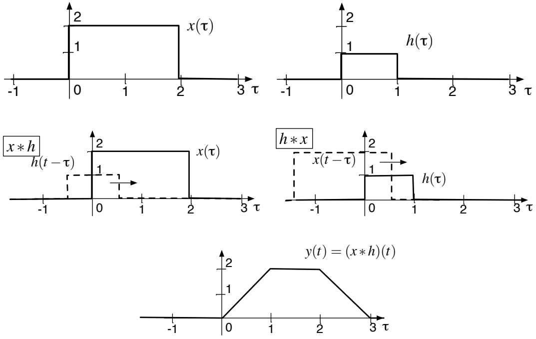
If we make the substitution $\tau_1 = t - \tau$, then $\tau = t - \tau_1$, and $d\tau = -d\tau_1$.

$$\begin{aligned} (f * g)(t) &= \int_{\infty}^{-\infty} f(t-\tau_1)g(\tau_1)\,(-d\tau_1) \\ &= \int_{-\infty}^{\infty} g(\tau)f(t-\tau_1)\, d\tau_1 \\ &= (g * f)(t) \end{aligned}$$

This means that convolution is *commutative*.

Practically, If we have two signals to convolve, we can choose either to be the signal we hold constant and the other to "flip and drag."

Simple Example (x*h)



Convolution is associative

If we convolve three functions f, g, and h

$$(f * (g * h))(t) = ((f * g) * h)(t)$$

which means that convolution is *associative*.

Combining the commutative and associate properties,

$$f * g * h = f * h * g = \cdots = h * g * f$$

We can perform the convolutions in any order.

Linearity

Convolution is also *distributive*,

$$f * (g + h) = f * g + f * h$$

which is easily shown by writing out the convolution integral,

$$\begin{aligned}
(f * (g + h))(t) &= \int_{-\infty}^{\infty} f(\tau)\left[g(t-\tau) + h(t-\tau)\right]\, d\tau \\
&= \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau + \int_{-\infty}^{\infty} f(\tau) h(t-\tau)\, d\tau \\
&= (f * g)(t) + (f * h)(t)
\end{aligned}$$

Together, the commutative, associative, and distributive properties mean that there is an "algebra of signals" where

- addition is like arithmetic or ordinary algebra, and
- multiplication is replaced by convolution.

Time-invariant

Convolution with a delayed signal gives a delayed output.

$$(f * g_\tau)(t) \ = \ (f_\tau * g)(t) \ = \ (f * g)(t - \tau)$$

Properties of Convolution Systems

The properties of the convolution integral have important consequences for systems described by convolution:

- Convolution systems are **linear**: for all signals x_1, x_2 and all α, $\beta \in \Re$,

$$h * (\alpha x_1 + \beta x_2) = \alpha(h * x_1) + \beta(h * x_2)$$

- Convolution systems are **time-invariant**: if we shift the input signal x by T, *i.e.*, apply the input

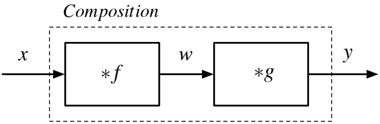
$$x_1(t) = x(t - T)$$

to the system, the output is

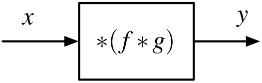
$$y_1(t) = y(t - T).$$

In other words: convolution systems commute with delay.

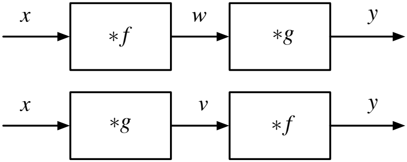
- **Composition** of convolution systems corresponds to convolution of impulse responses.
  The cascade connection of two convolution systems $y = (x * f) * g$



is the same as a single system with an impulse response $h = f * g$



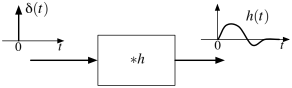
Since convolution is commutative, the convolution systems are also commutative. These two cascade connections have the same response



Many operations can be written as convolutions, and these all commute (integration, differentiation, delay, ...)

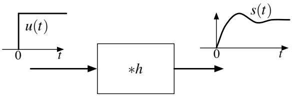
Example: Measuring the impulse response of an LTI system.

We would like to measure the impulse response of an LTI system, described by the impulse response $h(t)$

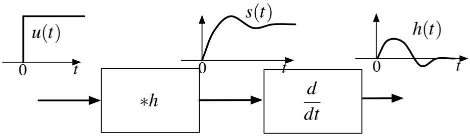


This can be practically difficult because input amplitude is often limited. A very short pulse then has very little energy.

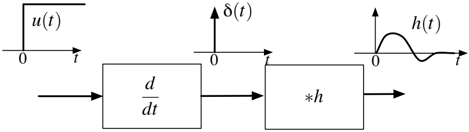
A common alternative is to measure the *step response* $s(t)$, the response to a unit step input $u(t)$



The impulse response is determined by differentiating the step response,



To show this, commute the convolution system and the differentiator to produce a system with the same overall impulse response



Convolution Systems with Complex Exponential Inputs

- If we have a convolution system with an impulse response $h(t)$, and and input e^{st} where $s = \sigma + j\omega$

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} \, d\tau \\
&= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, d\tau
\end{aligned}
$$

- We get the complex exponential back, with a complex constant multiplier

$$
\begin{aligned}
H(s) &= \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, d\tau \\
y(t) &= e^{st} H(s)
\end{aligned}
$$

provided the integral converges.

- $H(s)$ is the *transfer function* of the system.
- If the input is a complex sinusoid $e^{j\omega t}$,

$$
\begin{aligned}
H(j\omega) &= \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} \, d\tau \\
y(t) &= e^{j\omega t} H(j\omega)
\end{aligned}
$$

Summary

- LTI systems can be represented as a the convolution of the input with an impulse response.
- Convolution has many useful properties (associative, commutative, etc).
- These carry over to LTI systems
  - Composition of system blocks
  - Order of system blocks

  Useful both practically, and for understanding.
- While convolution is conceptually simple, it can be practically difficult. It can be tedious to convolve your way through a complex system.
- There has to be a better way ...