Lecture 8
ELE 301: Signals and Systems

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Properties of the Fourier Transform

- Properties of the Fourier Transform
  - Linearity
  - Time-shift
  - Time Scaling
  - Conjugation
  - Duality
  - Parseval
- Convolution and Modulation
- Periodic Signals
- Constant-Coefficient Differential Equations

Linearity

Linear combination of two signals $x_1(t)$ and $x_2(t)$ is a signal of the form $ax_1(t) + bx_2(t)$.

Linearity Theorem: The Fourier transform is linear; that is, given two signals $x_1(t)$ and $x_2(t)$ and two complex numbers a and b, then

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega).$$

This follows from linearity of integrals:

$$\begin{aligned}
&\int_{-\infty}^{\infty} (ax_1(t) + bx_2(t))e^{-j2\pi ft}\, dt \\
&= a\int_{-\infty}^{\infty} x_1(t)e^{-j2\pi ft}\, dt + b\int_{-\infty}^{\infty} x_2(t)e^{-j2\pi ft}\, dt \\
&= aX_1(f) + bX_2(f)
\end{aligned}$$

Finite Sums

This easily extends to finite combinations. Given signals $x_k(t)$ with Fourier transforms $X_k(f)$ and complex constants a_k, $k = 1, 2, \ldots K$, then

$$\sum_{k=1}^{K} a_k x_k(t) \Leftrightarrow \sum_{k=1}^{K} a_k X_k(f).$$

If you consider a system which has a signal $x(t)$ as its input and the Fourier transform $X(f)$ as its output, the system is linear!

Linearity Example

Find the Fourier transform of the signal

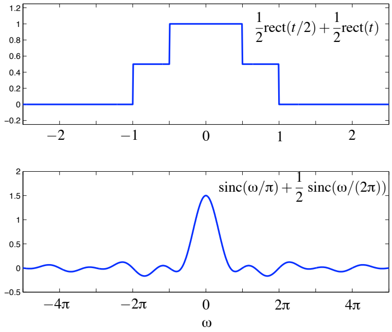
$$x(t) = \begin{cases} \frac{1}{2} & \frac{1}{2} \le |t| < 1 \\ 1 & |t| \le \frac{1}{2} \end{cases}$$

This signal can be recognized as

$$x(t) = \frac{1}{2}\,\mathrm{rect}\left(\frac{t}{2}\right) + \frac{1}{2}\,\mathrm{rect}\,(t)$$

and hence from linearity we have

$$X(f) = \left(\frac{1}{2}\right) 2\,\mathrm{sinc}(2f) + \frac{1}{2}\,\mathrm{sinc}(f) = \mathrm{sinc}(2f) + \frac{1}{2}\,\mathrm{sinc}(f)$$



Scaling Theorem

Stretch (Scaling) Theorem: Given a transform pair $x(t) \Leftrightarrow X(f)$, and a real-valued nonzero constant a,

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Proof: Here consider only $a > 0$. (negative a left as an exercise) Change variables $\tau = at$

$$\int_{-\infty}^{\infty} x(at) e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau/a} \frac{d\tau}{a} = \frac{1}{a} X\left(\frac{f}{a}\right).$$

If $a = -1 \Rightarrow$ "time reversal theorem:"

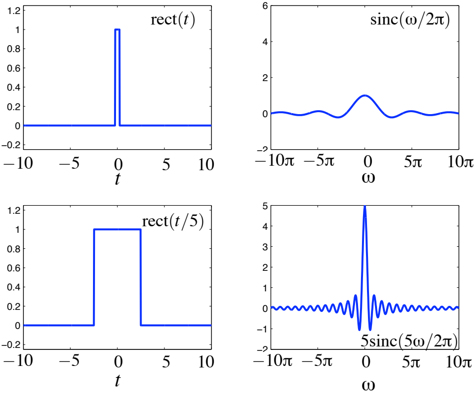
$$X(-t) \Leftrightarrow X(-f)$$

Scaling Examples

We have already seen that $\mathrm{rect}(t/T) \Leftrightarrow T\,\mathrm{sinc}(Tf)$ by brute force integration. The scaling theorem provides a shortcut proof given the simpler result $\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(f)$.

This is a good point to illustrate a property of transform pairs. Consider this Fourier transform pair for a small T and large T, say $T = 1$ and $T = 5$. The resulting transform pairs are shown below to a common horizontal scale:

Compress in time - Expand in frequency



Scaling Example 2

As another example, find the transform of the time-reversed exponential

$$x(t) = e^{at}u(-t).$$

This is the exponential signal $y(t) = e^{-at}u(t)$ with time scaled by -1, so the Fourier transform is

$$X(f) = Y(-f) = \frac{1}{a - j2\pi f}.$$

Scaling Example 3

As a final example which brings two Fourier theorems into use, find the transform of

$$x(t) = e^{-a|t|}.$$

This signal can be written as $e^{-at}u(t) + e^{at}u(-t)$. Linearity and time-reversal yield

$$\begin{aligned} X(f) &= \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} \\ &= \frac{2a}{a^2 - (j2\pi f)^2} \\ &= \frac{2a}{a^2 + (2\pi f)^2} \end{aligned}$$

Much easier than direct integration!

Complex Conjugation Theorem

Complex Conjugation Theorem: If $x(t) \Leftrightarrow X(f)$, then

$$x^*(t) \Leftrightarrow X^*(-f)$$

Proof: The Fourier transform of $x^*(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi ft}\, dt &= \left(\int_{-\infty}^{\infty} x(t) e^{j2\pi ft}\, dt \right)^* \\ &= \left(\int_{-\infty}^{\infty} x(t) e^{-(-j2\pi f)t}\, dt \right)^* = X^*(-f) \end{aligned}$$

Duality Theorem

We discussed duality in a previous lecture.

Duality Theorem: If $x(t) \Leftrightarrow X(f)$, then $X(t) \Leftrightarrow x(-f)$.

This result effectively gives us two transform pairs for every transform we find.

- *Exercise* What signal $x(t)$ has a Fourier transform $e^{-|f|}$?

Shift Theorem

The Shift Theorem:

$$x(t-\tau) \Leftrightarrow e^{-j2\pi f\tau}X(f)$$

Proof:

Example: square pulse

Consider a causal square pulse $p(t) = 1$ for $t \in [0, T)$ and 0 otherwise. We can write this as

$$p(t) = \mathrm{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

From shift and scaling theorems

$$P(f) = Te^{-j\pi fT}\,\mathrm{sinc}(Tf).$$

The Derivative Theorem

The Derivative Theorem: Given a signal $x(t)$ that is differentiable almost everywhere with Fourier transform $X(f)$,

$$x'(t) \Leftrightarrow j2\pi f X(f)$$

Similarly, if $x(t)$ is n times differentiable, then

$$\frac{d^n x(t)}{dt^n} \Leftrightarrow (j2\pi f)^n X(f)$$

Dual Derivative Formula

There is a dual to the derivative theorem, i.e., a result interchanging the role of t and f. Multiplying a signal by t is related to differentiating the spectrum with respect to f.

$$(-j2\pi t)x(t) \Leftrightarrow X'(f)$$

The Integral Theorem

Recall that we can represent integration by a convolution with a unit step

$$\int_{-\infty}^{t} x(\tau)d\tau = (x * u)(t).$$

Using the Fourier transform of the unit step function we can solve for the Fourier transform of the integral using the convolution theorem,

$$\begin{aligned}\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] &= \mathcal{F}[x(t)]\,\mathcal{F}[u(t)] \\ &= X(f)\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) \\ &= \frac{X(0)}{2}\delta(f) + \frac{X(f)}{j2\pi f}.\end{aligned}$$

Fourier Transform of the Unit Step Function

How do we know the derivative of the unit step function?

The unit step function does not converge under the Fourier transform. But just as we use the delta function to accommodate periodic signals, we can handle the unit step function with some sleight-of-hand.

Use the approximation that $u(t) \approx e^{-at}u(t)$ for small a.

A symmetric construction for approximating $u(t)$

Example: Find the Fourier transform of the signum or sign signal

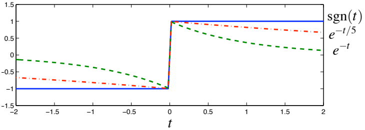
$$f(t) = \mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}.$$

We can approximate $f(t)$ by the signal

$$f_a(t) = e^{-at}u(t) - e^{at}u(-t)$$

as $a \to 0$.

This looks like



As $a \to 0$, $f_a(t) \to \text{sgn}(t)$.

The Fourier transform of $f_a(t)$ is

$$\begin{aligned}
F_a(f) &= \mathcal{F}\left[f_a(t)\right] \\
&= \mathcal{F}\left[e^{-at}u(t) - e^{at}u(-t)\right] \\
&= \mathcal{F}\left[e^{-at}u(t)\right] - \mathcal{F}\left[e^{at}u(-t)\right] \\
&= \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f} \\
&= \frac{-j4\pi f}{a^2 + (2\pi f)^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{a\to 0} F_a(f) &= \lim_{a\to 0} \frac{-j4\pi f}{a^2 + (2\pi f)^2} \\
&= \frac{-j4\pi f}{(2\pi f)^2} \\
&= \frac{1}{j\pi f}.
\end{aligned}$$

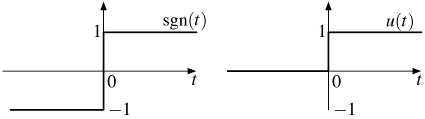
This suggests we *define* the Fourier transform of $\text{sgn}(t)$ as

$$\text{sgn}(t) \Leftrightarrow \begin{cases} \frac{2}{j2\pi f} & f \neq 0 \\ 0 & f = 0 \end{cases}.$$

With this, we can find the Fourier transform of the unit step,

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

as can be seen from the plots

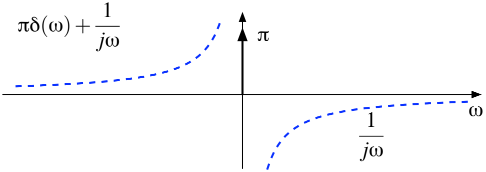


The Fourier transform of the unit step is then

$$\begin{aligned} \mathcal{F}[u(t)] &= \mathcal{F}\left[\frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)\right] \\ &= \frac{1}{2}\delta(f) + \frac{1}{2}\left(\frac{1}{j\pi f}\right). \end{aligned}$$

The transform pair is then

$$u(t) \Leftrightarrow \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}.$$



Parseval's Theorem

(Parseval proved for Fourier series, Rayleigh for Fourier transforms. Also called Plancherel's theorem)

Recall signal *energy* of $x(t)$ is

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Interpretation: energy dissipated in a one ohm resistor if $x(t)$ is a voltage. Can also be viewed as a measure of the size of a signal.

Theorem:

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Example of Parseval's Theorem

Parseval's theorem provides many simple integral evaluations. For example, evaluate

$$\int_{-\infty}^{\infty} \mathrm{sinc}^2(t) \, dt$$

We have seen that $\mathrm{sinc}(t) \Leftrightarrow \mathrm{rect}(f)$.

Parseval's theorem yields

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{sinc}^2(t) \, dt &= \int_{-\infty}^{\infty} \mathrm{rect}^2(f) \, df \\ &= \int_{-1/2}^{1/2} 1 \, df \\ &= 1. \end{aligned}$$

Try to evaluate this integral directly and you will appreciate Parseval's shortcut.

⋆ The Convolution Theorem ⋆

Convolution in the time domain ⇔ multiplication in the frequency domain

This can simplify evaluating convolutions, especially when cascaded.

This is how most simulation programs (e.g., Matlab) compute convolutions, using the FFT.

The Convolution Theorem: Given two signals $x_1(t)$ and $x_2(t)$ with Fourier transforms $X_1(f)$ and $X_2(f)$,

$$(x_1 * x_2)(t) \Leftrightarrow X_1(f)X_2(f)$$

Proof: The Fourier transform of $(x_1 * x_2)(t)$ is

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau \right) e^{-j2\pi ft}\, dt$$
$$= \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j2\pi ft}\, dt \right) d\tau.$$

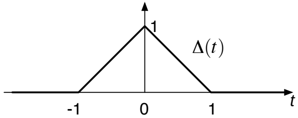
Using the shift theorem, this is

$$= \int_{-\infty}^{\infty} x_1(\tau)\left(e^{-j2\pi f\tau}X_2(f)\right) d\tau$$
$$= X_2(f)\int_{-\infty}^{\infty} x_1(\tau)e^{-j2\pi f\tau}\, d\tau$$
$$= X_2(f)X_1(f).$$

Examples of Convolution Theorem

Unit Triangle Signal $\Delta(t)$

$$\begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



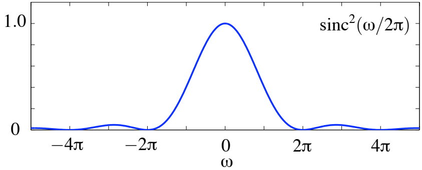
Easy to show $\Delta(t) = \mathrm{rect}(t) * \mathrm{rect}(t)$.

Since

$$\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(f)$$

then

$$\Delta(t) \Leftrightarrow \mathrm{sinc}^2(f)$$



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Multiplication Property

If $x_1(t) \Leftrightarrow X_1(f)$ and $x_2(t) \Leftrightarrow X_2(f)$,

$$x_1(t)x_2(t) \quad \Leftrightarrow \quad (X_1 * X_2)(f).$$

This is the dual property of the convolution property.

Note: If ω is used instead of f, then a $1/2\pi$ term must be included.

Multiplication Example - Bandpass Filter

A bandpass filter can be implemented using a low-pass filter and multiplication by a complex exponential.

Modulation

The Modulation Theorem: Given a signal $x(t)$ with spectrum $x(f)$, then

$$x(t)e^{j2\pi f_0 t} \Leftrightarrow X(f - f_0),$$

$$x(t)\cos(2\pi f_0 t) \Leftrightarrow \frac{1}{2}\left(X(f - f_0) + X(f + f_0)\right),$$

$$x(t)\sin(2\pi f_0 t) \Leftrightarrow \frac{1}{2j}\left(X(f - f_0) - X(f + f_0)\right).$$

Modulating a signal by an exponential shifts the spectrum in the frequency domain. This is a *dual* to the shift theorem. It results from interchanging the roles of t and f.

Modulation by a cosine causes replicas of $X(f)$ to be placed at plus and minus the carrier frequency.

Replicas are called *sidebands*.

Amplitude Modulation (AM)

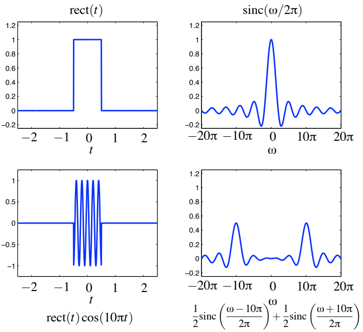
Modulation of complex exponential (carrier) by signal $x(t)$:

$$x_m(t) = x(t)e^{j2\pi f_0 t}$$

Variations:

- $f_c(t) = f(t)\cos(\omega_0 t)$ (DSB-SC)
- $f_s(t) = f(t)\sin(\omega_0 t)$ (DSB-SC)
- $f_a(t) = A[1 + mf(t)]\cos(\omega_0 t)$ (DSB, commercial AM radio)
  - m is the *modulation index*
  - Typically m and $f(t)$ are chosen so that $|mf(t)| < 1$ for all t

Examples of Modulation Theorem



Periodic Signals

Suppose $x(t)$ is periodic with fundamental period T and frequency $f_0 = 1/T$. Then the Fourier *s*eries representation is,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}.$$

Let's substitute in some δ functions using the sifting property:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} \delta(f - kf_0) e^{j2\pi f t} df, \\ &= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_k \delta(f - kf_0) \right) e^{j2\pi f t} df. \end{aligned}$$

This implies the Fourier *t*ransform: $x(t) \Leftrightarrow \sum_{k=-\infty}^{\infty} a_k \delta(f - kf_0)$.

Constant-Coefficient Differential Equations

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

Find the Fourier Transform of the impulse response (the transfer function of the system, $H(f)$) in the frequency domain.