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Platonism's Pyrrhic Victory[1]

During my acquaintance with Frederic Fitch, I have learned much from him, both through his writings and in discussion about semantic paradoxes and about modal, deontic, and combinatory logic. But I am most grateful to him for the challenge of his ontological position, with which I am in total temperamental disagreement. Temperament is no substitute for arguments, of course, and it rues me to think how much less articulate or well founded than his my ontological position is. In this paper I shall do my best to spell out why I do not feel a need to believe in the existence of abstract entities.

1. The Worlds of Oz and Id

My dialectic strategy will be a bit roundabout. I shall first tell a fable, for the reader to mull over while grappling with the main arguments; in conclusion I shall comment on the fable.

Once upon a time there were two possible worlds, Oz and Id. These worlds were very much alike and, indeed, very much like our world. Specifically, their inhabitants developed exactly the mathematics and mathematical logic we have today. The main differences were two: (a) In Oz, sets really existed, and in Id no abstract entities existed, but (b) in Id, mathematicians and philosophers were almost universally Platonist, while in Oz they refused, almost to a man, to believe that there existed any abstract entities.

They all lived happily ever after.

2. The Strategy of Realism

There are certain phenomena to be taken into account in the philosophy of mathematics. These include the prevalence of language games roughly codified in set theory, classical analysis, and modal logic. The realist gives an account of these phenomena: the axioms of set theory (his favorite set theory) are literally true, he says; there *exist* sets of the sort described. Depending on his ulterior philosophical motives, he may add that certain other abstract entities also exist, or that sets are logical constructs built from other abstract entities (properties or propositions, say). But, on the other hand, he may consider himself moderate and hold that all other abstract entities are built from sets. These sectarian differences we may ignore; they all hold that there exists a family of abstract entities properly called *sets* because they are as described in some set theory sufficiently powerful to serve as a foundation for a sufficiently large body of mathematics and logic.

This is, of course, in its simpler forms the simplest possible hypothesis accounting for the phenomena: the explanation of mathematical theories is that they are true theories in just the way that botany is a true theory. The challenge to anyone not inclined to realism is to provide an alternative account and to subject this account to the criteria of simplicity and explanatory power. In the forties a small band of men valiantly picked up the gauntlet, proclaimed themselves nominalists, set out to "reconstruct" the phenomena nominalistically, and went down to ignominious defeat.

A realist has relatively little work if he so wishes: once he says, say, that Zermelo-Fraenkel set theory is *true*, there already exists a powerful and elegant theory describing his postulated entities, namely, Zermelo-Fraenkel set theory. Any nonrealist is challenged to provide an alternative theory, free of existence-postulates (beyond the acknowledged existence of a finite number of concrete entities) and powerful enough to "reconstruct" this set theory. A Herculean task indeed, I would say, if Hercules' intellect had not lagged so miserably behind his muscles.

Realism wins by default. And that has been the realist's strategy through the ages, to win by default once he has spelled out what his opponent must do to win.

3. *The World of Realism*

At the turn of the century, the axiomatic method had progressed from construction *more geometrico* to formalization. Proceeding axiomatically had at first consisted of choosing some technical or

common terms as primitive and then using these terms plus syncategorematic devices of natural language to formulate one's postulates. Formalization, the *fin de siècle* refinement, eliminated that vestige of reliance on natural language. To intuitionists and their immediate forebears formalization could be at most a means for presenting and communicating mathematical constructions already at hand. But others held the more extreme view that

(a) the formal axiomatic formulation of a mathematical theory captures, in principle, exactly what can be known about the subject matter of that theory;

(b) mathematical concepts are *implicitly defined;* that is, their total significance and meaning is presented through the axiomatic formulation;

(c) an adequate axiomatic theory describes its subject matter *categorically* (that is, uniquely up to isomorphism), and for every mathematical subject there exists, in principle, such a theory.

If today we cannot return to that first fine careless rapture, it must be blamed on a number of metamathematical theorems proved in the half century that followed. I shall state these intuitively, adding precision only where the discussion requires it. The first two are the most famous:

I. (Gödel) A sufficiently strong axiomatic theory is not complete if it is consistent.

II. (Tarski) A sufficiently rich language lacks some means of expression.

If the reader wishes to put these more concretely, he can take "sufficiently strong" and "sufficiently rich" to refer to recursive arithmetic. Hardly less famous is

III. (Löwenheim-Skolem) Any formalized theory has a model which is at most denumerable.

III applies specifically to theories which have axioms saying that there are more than denumerably many things, sets or whatever. Since isomorphism implies equality of cardinality, we have the corollary that no theory about the higher infinities is categorical (if there really are higher infinities). Various philosophers have tried to make it look as if the Löwenheim-Skolem result is innocuous because it involves strained reinterpretations of the primitive terms. But in the proof supplied by Tarski and Vaught it is quite clear that this is not so: any model of a theory will have an at most denumerable submodel of that same theory.[2] This strong form of III was subsequently used by Paul Cohen to derive

IV. (Cohen) The axiom of choice and the continuum hypothesis are independent of the main axioms of set theory.

This theorem is not so general or breathtaking as the first three, but will also appear in my discussion below; a fifth, due to Beth, I shall state later.

My next objective is to consider what reactions to these results are open to the mathematical realist.

I shall divide the realists into the *extreme* (such as Gödel) and the *moderate* (such as Beth). Common to both sects are the following tenets:

(1) The entities purportedly described by mathematical theories exist, and exist independent of any theorizing activity.

(2) These entities are not, and as we now know (see I–III), cannot be adequately (completely or categorically) described by our theories.

(3) Nevertheless, it is possible to refer unambiguously to these mathematical entities, for example, to *the* natural number sequence, or to *the* intended model of classical analysis.

(3) is strongly qualified by (2), in that although we refer to such entities, we do not pretend to have uniquely identifying descriptions for them.

Where extremists and moderates divide is on the adumbration of (3). According to the extremists, the lack of uniquely identifying descriptions does not deprive the word "unambiguously" of any force in (3). We can refer to the number zero or to the natural number sequence and *communicate* to other persons also engaged in mathematics what we are referring to. (There may be gradations in this extremism; perhaps some would say that we can so refer to and communicate about the null set or the class of well-founded sets, though not the number zero. These intrasectarian differences are not important to this discussion.) The moderates, however,

feel that there is no way to guarantee that two mathematicians are referring to the same entities. Since they share the same axioms, this need have no practical effects: mathematician *X* accepts mathematician *Y*'s results because those results hold in *X*'s domain of reference, as he can plainly see. And when *Y* begins to articulate postulates that sound strange to *X*, then *X* soon manages to find a model for *Y*'s assertions within his own interpretation.

Beth described the situation graphically in his discussion of Skolem's results on categoricity:

> Set theory has thus at least two models not isomorphic with each other, which we may denote as M_0 and M_1; a model of set theory I shall call for reasons which will soon become clear, a *milky way system.* Let M_0 be the milky way system which we represent to ourselves when we use set theory, or mean to so represent.
>
> If we take a closer look at the categoricity proof for the Dedekind-Peano axiom system (for natural number arithmetic), we find that what is demonstrated is that any two of its models *which belong to the same milky way system* are isomorphic. "The" natural number sequence belongs to the milky way system M_0; the model constructed by Skolem belongs to a different milky way system, say M_1.[3]

And again, a page later:

> In short, the relativism discovered by Skolem in the theory of sets can be described as follows: there are at least two, and likely more, milky way systems each of which provides a complete realization of the complex of logical and mathematical theories, while neither these milky way systems nor the realizations of theories contained in them, are isomorphic.
>
> And yet we imagine that . . . we possess nevertheless an unambiguous intuitive representation of the *structure* of "the" natural number sequence and of "the" Euclidean space. We imagine that our mathematical reasoning intends [has as subject] a specific milky way system M_0, which is intuitively known to us, although this reasoning is equally valid for other milky way systems.[4]

The resolute use of the first-person plural in these passages, published in 1948, is undermined by Beth's amusing, if somewhat cryptic, paper in the Carnap volume. In this paper, which was I think misunderstood by Carnap, Beth imagines a confrontation between Carnap and a second (hypothetical) logician Carnap* who seems to have an alternative milky way system. Beth concludes:

> The above considerations, which are only variants of the Skolem-Löwenheim paradox, suggests strongly that, if arguments as contained in [*The Logical Syntax of Language*] serve a certain purpose, this can only be the case on account of the fact that they are interpreted by reference to a certain presupposed intuitive model M. Carnap avoids an appeal to such an intuitive model in the discussion of Language II itself, but he could not avoid it in the discussion of its syntax; for the conclusions belonging to its syntax would not be acceptable to Carnap*, though Carnap and Carnap* would, of course, always agree with respect to those conclusions which depend exclusively on formal considerations.[5]

Beth himself proved a major metamathematical result aimed at the pretense that the sense of mathematical concepts is somehow fully conveyed ("implicitly defined") by the axioms of the relevant extant theory. He explicated this view by proposing the principle:

> if F and F′ are implicitly defined by essentially the same axiom sets A and A′, then it must follow from A and A′ together that F = F′

and then proved

V. (Beth) If F is implicitly defined by the theory T, then F is explicitly definable in T.

When F is explicitly definable in T, some expression in which F does not occur does exactly the same job as F there, so that F can be deleted without loss from the primitive terms of the theory. The significance of these results is a main area of contention between realists and antirealists. I can identify two other such areas subordinate to these, which I shall discuss in the next section.

The question most important to me is: Just what advantage is the mathematician supposed to reap from his purported ability to refer unambiguously (in an extreme *or* moderate sense) to "intended models"? This is in part the old question of what *access* the

mathematician has to these independently existing objects. Does he know them by acquaintance or by description? And if the latter, is the description anchored in acquaintance with something, or are we to know those objects *solely* as "that which satisfies the axioms"? While only the most extreme Platonist would appeal to *Wesenschau*, it must be admitted that Husserl correctly identified a task which the mathematical realists have left woefully incomplete: to describe how these nonconcrete entities enter our experience.

IV has lately caused some activity which might provide a clue to mathematicians' purported access to abstract entities. Do mathematicians, faced with the independence of the strong axioms of set theory and hence the logical possibility of adding some of their rivals to the weak axioms, attempt to ferret out what is *true?* Gödel answered in the affirmative: "The mere psychological fact of the existence of an intuition . . . suffices to give meaning to the question of truth or falsity . . . ."[6] But my impression is that mathematicians are so busy exploring which extensions of the weak axioms are, to use Gödel's own terminology, "fruitful" and which "sterile," that the question of truth, or correspondence to a previous intended model, is not considered.

Of course, realists have their account of these phenomena too. Perhaps the situation should be compared to the rise of non-Euclidean geometries. Around 1800, everyone thought of actual space (physical space) as Euclidean, and they continued to do so for a while even after the development of other geometries. But then it became clear that there remained the substantive question of which geometry was the true one, the geometry of actual space. The situation now is similar. There are Cantorian and non-Cantorian set theories, but there is exactly one actual or correct milky way system, and at most one of the rival set theories can describe it correctly.

Perhaps I belabor the obvious if I point out that the question of which is the true geometry is a question that makes sense only if we identify physical correlates of geometric concepts, for example, if we stipulate that paths of light rays shall be geodesics. Is the realist ready to lay down a coordinative definition for ϵ? Or does he imagine that Riemann could have settled the problem by saying that he meant "straight line" in a straight line sense of *straight line?*

4. Subordinate Sound and Fury

Since the realist typically believes that at least the weak axioms of set theory are literally true, he (also typically) believes that there exist more than denumerably many entities. But the Löwenheim-Skolem theorem shows that no matter how he phrases his existence postulates, it will be consistent to say that each of his postulates is true, rightly understood, although there are still only denumerably many things in the world. (The point is simply that "denumerable" is a concept which varies depending on the domain of discourse; what is not denumerable in one domain may be denumerable in another.) So an antirealist, for example, a nominalist, may say that at the least there is no reason to believe that there are more than denumerably many entities in the world.

At exactly this juncture, the realist's gambit of placing the onus of proof on his opponents is most typically realized. I can prove the Löwenheim-Skolem theorem, he says smugly, but can you? Don't come back until you have developed a nominalistic mathematics powerful enough to prove such theorems. And don't try to appropriate by theft what I gained by honest toil.

But this victory-by-default gambit is simply beside the point. Recall Anselm's venerable strategy of beginning with "Even the fool sayeth in his heart. . . . " On the nominalist position there is no reason to believe that there are nondenumerably many things, since the nominalist denies the existence of abstract entities. On the conceptualist position there is no reason to believe that there are nondenumerably many things. (According to the intuitionist, Cantor's theorem shows only that the concept of the continuum is so rich that its sense is not exhausted by any single humanly followable recipe.) And on the realist position there is no reason to believe that there are nondenumerably many things, since the realist can show via Skolem's proof that his postulates can be satisfied without believing that. So even on the basis of his own position, the realist can find no justification for his belief.

Finally, there is no reason to believe so much. At this point it may be relevant also to mention the substitution interpretation of quantifiers. Not much is to be gained by it, since that still does not show that there is no reason to believe that there are more than finitely many things. But it is one of those subjects where those people who have found the realist's manner of posing the problems inescapable, then found themselves thereby hard pressed. Quine held that one could tell a person's ontic commitments by looking at his syntax and logic; I have found no more precise and tenable way of explaining Quine's criterion of ontic commitment. Arguments showing that Quine's own explications are either confused or inconsistent are familiar to everyone. Henkin (in 1953) and Sellars (in 1960) argued that it is possible so to use the substitution interpretation of quantifiers that one's syntax and logic give no clue to his ontic commitment.[7] Quine's retort that the substitution interpretation validates the Carnap rule '$Fx_1, \ldots, Fx_m, \ldots$ hence $(x)Fx$', where '$x_1, \ldots, x_k, \ldots$' are all the singular terms of the language, is not cogent. It was already forestalled in Henkin's 1953 paper by a point explicitly appreciated by Sellars: in the interpretation of the quantifiers in language L the set of substitution instances need not be defined by the set of singular terms of L itself. And in fact, one's syntax and logic are so far from betraying his ontic commitment that they do not even show whether he accepts the principle of bivalence.

Yet, as Henkin also pointed out, this is not a line of thought by which the nominalist can hope to reconstruct mathematics. If he tries, he will find himself trying, by implication, to provide the finitist proof of the consistency of arithmetic, which Gödel's results ruled out. Putnam suggests that to escape this dilemma the nominalist would be driven to consider *possible* singular terms and *possible* languages, an extremity that would be anathema to the nominalist (though not to a conceptualist).[8]

A second major realist tactic is to claim that the skeptic about abstract entities, having already thereby begun to topple from philosophical common sense, cannot help but fall headlong into skepticism *tout à fait*. His plunge is precipitated, it is argued, because by reason of analogy and because validity is a matter of form, skeptical arguments about abstract entities become skeptical arguments about physical objects. There are two examples of this line of argument.

When Beth comments on the impossibility in principle of developing categorical mathematical theories he says:

> This is at the very least strongly reminiscent of the situation in which we find ourselves with respect to physical objects. Even the most complete physical theory does not exhaust our intui-

> tive knowledge of physical objects. This gives us the conviction that physical objects are not mental entities, somehow created by the physical theory, but that they have an existence independent of any theory, or as we express it more briefly, they are "real".[9]

This is ingenious, but it think it assimilates the problem of not having uniquely identifying descriptions to that of not having complete descriptions (in each case, in principle). We would not be worried by the lack of complete theories if we had some other possible means of finding uniquely identifying descriptions of the supposed objects.

In a more obvious vein, Gödel suggests the argument:

> It seems to me that the assumption of such [mathematical] objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data", i.e. in the latter case the actually occurring sense perceptions.[10]

I wonder what "in the same sense" means here. If Gödel wants an account of mathematics that parallels the account of physics, he is clearly right, given his desiderata. If there is some necessity independent of such a prior choice of desiderata, he has not spelled that out. But let me put the question bluntly: Suppose one realist wishes to know what another uses the term 'the null set' to refer to. Well, how does he know to what the other refers by "the door" or "the Empire State Building"?

What is the force of such arguments anyway? Are they not simply another way to put the onus of proof on the opponents of realism? How cogent is it to tell antirealists to get themselves to a nunnery and develop an adequate epistemological account of how persons relate to physical nature before they dare to raise sceptical doubts about entities outside nature?

5. *What the Realist Does Not Know*

Epistemology has always been Platonism's Achilles' heel. The extreme Platonist combines a realist ontology with a theory of

knowledge that implies our having direct acquaintance with abstract entities. We know their existence, he holds, not through abstraction, inference, conception, or confirmation of hypotheses, but as objects of intuition. This is by far the most comfortable epistemology for a realist to have; it suffers only from not having received a respectable elaboration for its own sake in several centuries.

As with most philosophical issues, the opposite extreme is presently espoused, with admirable daring and skill, by Putnam.[11] He proposes to combine a realist ontology with a purely empiricist epistemology. Mathematical entities, he holds, are epistemologically on a par with the theoretical entities of physics. Mathematical practice is, or reasonably can be, hypothetico-deductive, and its hypotheses may be confirmed by the canons of inductive reasoning. On his view, it is reasonable to accept Fermat's last theorem and Goldbach's conjecture, since we have been testing them exhaustively, and they have survived the testing process. This is one consequence of his view, and he draws it unblushingly. A second feature central to his account is that truth does not consist merely in correspondence to the facts, but has pragmatic aspects: the survival of one theory in the struggle of competing theories within the community of mathematicians (or physicists) is a mark of truth. This seems to leave open the possibility that more than one theory might correspond equally well to all the facts, and still exactly one of them be true. (Else, what difference is there?) This is a move one expects of a conceptualist rather than a realist, but the attempt to combine realist ontology with empiricist epistemology is in any case a heroic one.

Anyone who has to defend either such extreme epistemological doctrine is not in an enviable position, but the moderate realist who does not try to make his ontic commitments palatable through epistemological extremism is also in a difficult spot. He does not, for instance, know the abstract entities he purports to refer to by acquaintance. He can find no uniquely identifying descriptions for them, even in principle. Perhaps it looks as if he could for finite or even denumerable sets. The null set, for instance, is the set without members. But one tenable realist position is that sets are logical constructions from properties for which extensionality does not hold, and the theory of such properties is not uniquely determined by the theory of sets. But then if realist *X* holds it, realist *Y* can exhibit *X*'s properties as set-theoretic constructs, and *X*'s sets as constructs of constructs, in which case he says: X's null set is my thingamadiddle. So even if this realist knew his referents, he could not tell us what they are, even in principle. It is hard not to conclude that the realist does not know what he is talking about.

Of course, I only mean this in the purely literal sense that he does not know what he is referring to.

6. *The World of Id*

Could we possibly be living in the world of Id? Certainly realism is rampant; philosophers tend to believe in the existence of almost anything, and mathematicians talk as if they do. Would it have made any difference to the development and usefulness of mathematics if, in addition, there actually were no abstract entities? Is Id really possible? Anyone who says not ought to produce an ontological proof of the existence of the null set, at the very least. ("That than which no emptier can be conceived," anyone?)

Is mathematics possible in Id? Anyone who says not ought to produce a transcendental deduction of the existence of mathematical entities. But suppose someone did. Interpretations of the import of such deductions vary. One interpretation is that it would establish the existence of mathematical entities as a presupposition of the development of mathematics. In that case, would it not suffice to have the aspiring mathematician take an oath to keep presupposing all that? And would the oath, or his presuppositions, be any the less efficacious if this were Id? Would he ever notice anything wrong?

I am not arguing that there are no sets. First, it is philosophically as uninteresting whether there are sets as whether there are unicorns. As a philosopher I am only interested in whether our world is intelligible if we assume there are no sets, and whether it remains equally intelligible if we do not. Personally, I delight in the postulation of occult entities to explain everyday phenomena, I just don't delight in taking it seriously. As a philosopher, however, I look forward to the day when we shall be able to say, "Yes, Virginia, there is a null set," and go on to explain, as the *New York Sun* did of Santa Claus, that of course there isn't one, but still there really is, living in the hearts and minds of men—exactly what a conceptualist by temperament would hope.