Starting Point

- We derived the induced current in the form

$$\mathbf{j}(t,\mathbf{x}) = -\frac{m\omega_p^2}{4\pi}\int d^3p\,\mathbf{v}\int_{t_0}^{t} e^{i\mathbf{k}\cdot\mathbf{x}'-i\omega t'}\,\mathbf{E}\cdot\left[\hat{\mathbf{I}}\left(1-\frac{\mathbf{k}\cdot\mathbf{v}'}{\omega}\right)+\frac{\mathbf{v}'\mathbf{k}}{\omega}\right]\cdot\frac{\partial f_0(\mathbf{p}')}{\partial\mathbf{p}'}\,dt'$$

- We now want to rewrite it in the following form:

$$\mathbf{j}(t,\mathbf{x}) = -\frac{i\omega}{4\pi}\,\hat{\chi}\,\mathbf{E}\,e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}} \equiv \frac{i\omega}{4\pi}\,\hat{\chi}\,\mathbf{E}(t,\mathbf{x})$$

- The coefficient χ will then be the sought susceptibility.

Here **E** is the field *amplitude*, and **E**(t, **x**) is the actual complex field.

Single-particle Motion

$$\frac{d\mathbf{p}'}{dt'} = \frac{q}{m}\mathbf{v}' \times \mathbf{B}_0, \qquad \frac{d\mathbf{r}'}{dt'} = \mathbf{v}', \qquad \mathbf{v}' = \frac{\mathbf{p}'}{\gamma m}$$

$$\mathbf{p}'(t' = t) = \mathbf{p}, \qquad \mathbf{r}'(t' = t) = \mathbf{r}$$

$$v_x = v_\perp \cos\phi, \qquad v_y = v_\perp \sin\phi$$

$$\Omega = qB_0/(mc\gamma)$$

$$\tau \equiv t - t'$$

$$v'_x = v_\perp \cos(\phi + \Omega\tau),$$

$$v'_y = v_\perp \sin(\phi + \Omega\tau),$$

$$v'_z = v_\parallel,$$

$$x' = x - \frac{v_\perp}{\Omega}\left[\sin(\phi + \Omega\tau) - \sin\phi\right],$$

$$y' = y + \frac{v_\perp}{\Omega}\left[\cos(\phi + \Omega\tau) - \cos\phi\right],$$

$$z' = z - v_\parallel \tau$$

Assumption of Azimuthal Isotropy

$$f_0 = f_0\left(\sqrt{p_x^2 + p_y^2},\ p_\parallel\right)$$

$$\frac{\partial f_0}{\partial \mathbf{p}} = \mathbf{z}^0 \frac{\partial f_0}{\partial p_\parallel} + \frac{\mathbf{p}_\perp}{p_\perp}\frac{\partial f_0}{\partial p_\perp}$$

$$k_x = k_\perp \cos\theta, \quad k_y = k_\perp \sin\theta$$

$$U = \frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp}\right),$$

$$V = \frac{k_\perp}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp}\right),$$

$$W = \left(1 - \frac{n\Omega}{\omega}\right)\frac{\partial f_0}{\partial p_\parallel} + \frac{n\Omega p_\parallel}{\omega p_\perp}\frac{\partial f_0}{\partial p_\perp}.$$

$$\mathbf{j}(t,\mathbf{x}) = -e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\frac{m\omega_p^2}{4\pi}\int d^3p\,\mathbf{v}\int_0^{t-t_0} d\tau\, e^{i\beta} \times \Big\{E_x U \cos(\phi + \Omega\tau) + E_y U \sin(\phi + \Omega\tau) + E_z\left[\partial f_0/\partial p_\parallel - V\cos(\phi - \theta + \Omega\tau)\right]\Big\}$$

$$\beta = -\frac{k_\perp v_\perp}{\Omega}\left[\sin(\phi - \theta + \Omega\tau) - \sin(\phi - \theta)\right] + (\omega - k_\parallel v_\parallel)\tau.$$

Order of Integration. Integrating over the Azimuthal Angle First

$$\int d^3p\, d\tau = \int dp_{\parallel} \int p_{\perp}\, dp_{\perp} \int d\tau \int d\phi$$

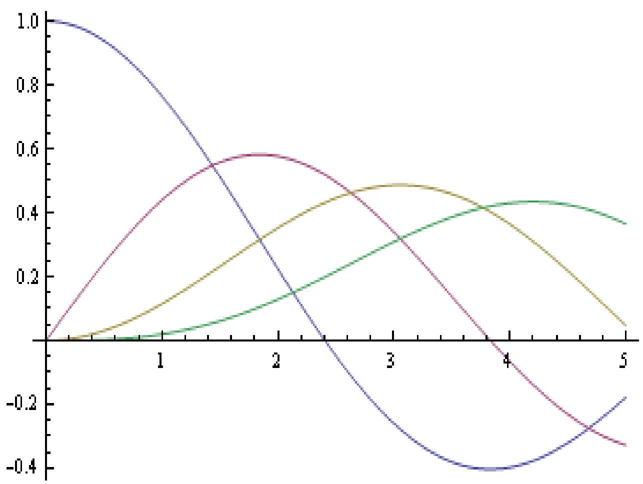
$$\int_0^{2\pi} d\phi\, e^{-iz[\sin(\phi+\Omega\tau)-\sin\phi]} \begin{pmatrix} \sin\phi\sin(\phi+\Omega\tau) \\ \sin\phi\cos(\phi+\Omega\tau) \\ \cos\phi\sin(\phi+\Omega\tau) \\ \cos\phi\cos(\phi+\Omega\tau) \\ 1 \\ \sin\phi \\ \cos\phi \\ \sin(\phi+\Omega\tau) \\ \cos(\phi+\Omega\tau) \end{pmatrix} = 2\pi \sum_{n=-\infty}^{\infty} e^{-in\Omega\tau} \begin{pmatrix} (J_n')^2 \\ -\frac{in}{z} J_n J_n' \\ \frac{in}{z} J_n J_n' \\ \frac{n^2}{z^2} J_n^2 \\ J_n^2 \\ -iJ_n J_n' \\ \frac{n}{z} J_n^2 \\ iJ_n J_n' \\ \frac{n}{z} J_n^2 \end{pmatrix}$$

$J_n(z)$, z denotes $k_{\perp} v_{\perp} / \Omega$

Bessel Functions

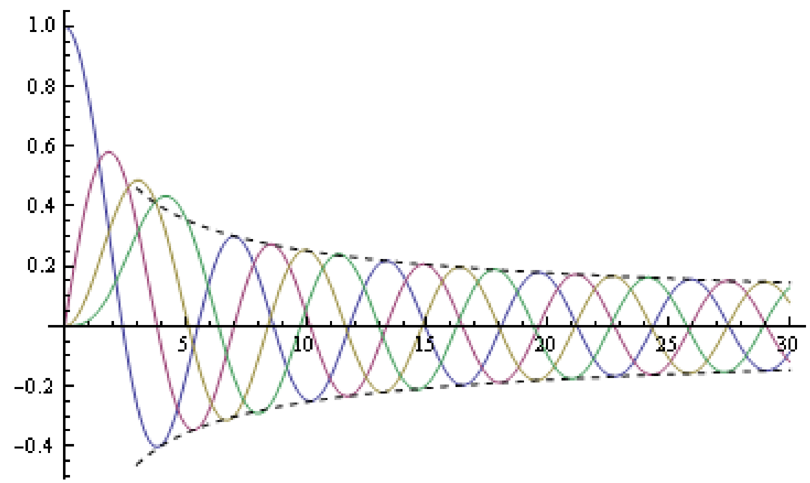
$$y'' + y'/x + (1 - n^2/x^2)y = 0$$

$$x \to 0$$



$$J_n(x) \sim \frac{1}{n!}\left(\frac{x}{2}\right)^n$$

$$x \to \infty$$



$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(\frac{\pi}{4} + \frac{n\pi}{2} - x\right)$$

Integrating over Time

$$\int d^3p\, d\tau = \int dp_\parallel \int p_\perp\, dp_\perp \int d\tau \int d\phi$$

$$\int_0^{2\pi} d\phi\, e^{-iz[\sin(\phi+\Omega\tau)-\sin\phi]} \begin{pmatrix} \sin\phi\sin(\phi+\Omega\tau) \\ \sin\phi\cos(\phi+\Omega\tau) \\ \cos\phi\sin(\phi+\Omega\tau) \\ \cos\phi\cos(\phi+\Omega\tau) \\ 1 \\ \sin\phi \\ \cos\phi \\ \sin(\phi+\Omega\tau) \\ \cos(\phi+\Omega\tau) \end{pmatrix} = 2\pi \sum_{n=-\infty}^{\infty} e^{-in\Omega\tau} \begin{pmatrix} (J_n')^2 \\ -\frac{in}{z} J_n J_n' \\ \frac{in}{z} J_n J_n' \\ \frac{n^2}{z^2} J_n^2 \\ J_n^2 \\ -iJ_n J_n' \\ \frac{n}{z} J_n^2 \\ iJ_n J_n' \\ \frac{n}{z} J_n^2 \end{pmatrix}$$

$$\beta = -\frac{k_\perp v_\perp}{\Omega}[\sin(\phi-\theta+\Omega\tau) - \sin(\phi-\theta)] + (\omega - k_\parallel v_\parallel)\tau.$$

$\mathrm{Im}\,\omega > 0$

$$\int_0^{t-t_0} e^{i(\omega-k_\parallel v_\parallel-n\Omega)\tau}\, d\tau = \frac{e^{i(\omega-k_\parallel v_\parallel-n\Omega)(t-t_0)}-1}{i(\omega-k_\parallel v_\parallel-n\Omega)} \rightarrow -\frac{1}{i(\omega-k_\parallel v_\parallel-n\Omega)}$$

Susceptibility Tensor For an Arbitrary Azimuthally-isotropic f_0

$$\epsilon(\omega,\mathbf{k}) = \mathbf{1} + \sum_s \chi_s(\omega,\mathbf{k})$$

$$\chi_s = \frac{\omega_{p0,s}^2}{\omega\Omega_{0,s}} \sum_{n=-\infty}^{\infty} \int_0^{\infty} 2\pi p_\perp \, dp_\perp \int_{-\infty}^{\infty} dp_\parallel \left(\frac{\Omega}{\omega - k_\parallel v_\parallel - n\Omega} \mathbf{S}_n \right)_s$$

$$\mathbf{S}_n = \begin{pmatrix} \frac{n^2 J_n^2}{z^2} p_\perp U & \frac{inJ_nJ_n'}{z} p_\perp U & \frac{nJ_n^2}{z} p_\perp W \\ -\frac{inJ_nJ_n'}{z} p_\perp U & (J_n')^2 p_\perp U & -iJ_nJ_n' p_\perp W \\ \frac{nJ_n^2}{z} p_\parallel U & iJ_nJ_n' p_\parallel U & J_n^2 p_\parallel W \end{pmatrix}$$

$$U = \frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp} \right),$$

$$V = \frac{k_\perp}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp} \right),$$

$$W = \left(1 - \frac{n\Omega}{\omega}\right) \frac{\partial f_0}{\partial p_\parallel} + \frac{n\Omega p_\parallel}{\omega p_\perp} \frac{\partial f_0}{\partial p_\perp}.$$

Symmetrized Form

One can also put the tensor in a symmetrized form:

$$\chi_s = \frac{\omega_{p0,s}^2}{\omega\Omega_{0,s}} \int_0^\infty 2\pi p_\perp \, dp_\perp \int_{-\infty}^{\infty} dp_\parallel \left[\hat{\mathbf{e}}_\parallel \hat{\mathbf{e}}_\parallel \frac{\Omega}{\omega} \left(\frac{1}{p_\parallel} \frac{\partial f_0}{\partial p_\parallel} - \frac{1}{p_\perp} \frac{\partial f_0}{\partial p_\perp} \right) p_\parallel^2 + \sum_{n=-\infty}^{\infty} \frac{\Omega p_\perp U}{\omega - k_\parallel v_\parallel - n\Omega} \mathbf{T}_n \right]_s$$

$$\mathbf{T}_n = \begin{pmatrix} \dfrac{n^2 J_n^2}{z^2} & \dfrac{inJ_nJ_n'}{z} & \dfrac{nJ_n^2 p_\parallel}{zp_\perp} \\ -\dfrac{inJ_nJ_n'}{z} & (J_n')^2 & -\dfrac{iJ_nJ_n' p_\parallel}{p_\perp} \\ \dfrac{nJ_n^2 p_\parallel}{zp_\perp} & \dfrac{iJ_nJ_n' p_\parallel}{p_\perp} & \dfrac{J_n^2 p_\parallel^2}{p_\perp^2} \end{pmatrix}$$

Like before, this holds for any f_0.

Distribution Maxwellian in the perpendicular velocity

$$\chi_s = \frac{\omega_{p0,s}^2}{\omega\Omega_{0,s}} \int_0^\infty 2\pi p_\perp \, dp_\perp \int_{-\infty}^{\infty} dp_\parallel \Big[\hat{\mathbf{e}}_\parallel \hat{\mathbf{e}}_\parallel \frac{\Omega}{\omega} \left(\frac{1}{p_\parallel} \frac{\partial f_0}{\partial p_\parallel} - \frac{1}{p_\perp} \frac{\partial f_0}{\partial p_\perp} \right) p_\parallel^2 + \sum_{n=-\infty}^{\infty} \frac{\Omega p_\perp U}{\omega - k_\parallel v_\parallel - n\Omega} \mathbf{T}_n \Big]_s$$

$$\mathbf{T}_n = \begin{pmatrix} \frac{n^2 J_n^2}{z^2} & \frac{inJ_n J_n'}{z} & \frac{nJ_n^2 p_\parallel}{z p_\perp} \\ -\frac{inJ_n J_n'}{z} & (J_n')^2 & -\frac{iJ_n J_n' p_\parallel}{p_\perp} \\ \frac{nJ_n^2 p_\parallel}{z p_\perp} & \frac{iJ_n J_n' p_\parallel}{p_\perp} & \frac{J_n^2 p_\parallel^2}{p_\perp^2} \end{pmatrix}$$

$J_n(z)$, z denotes $k_\perp v_\perp / \Omega$

$$f_0(v_\perp, v_\parallel) = h(v_\parallel) \frac{1}{\pi w_\perp^2} \exp\left(-\frac{v_\perp^2}{w_\perp^2} \right)$$

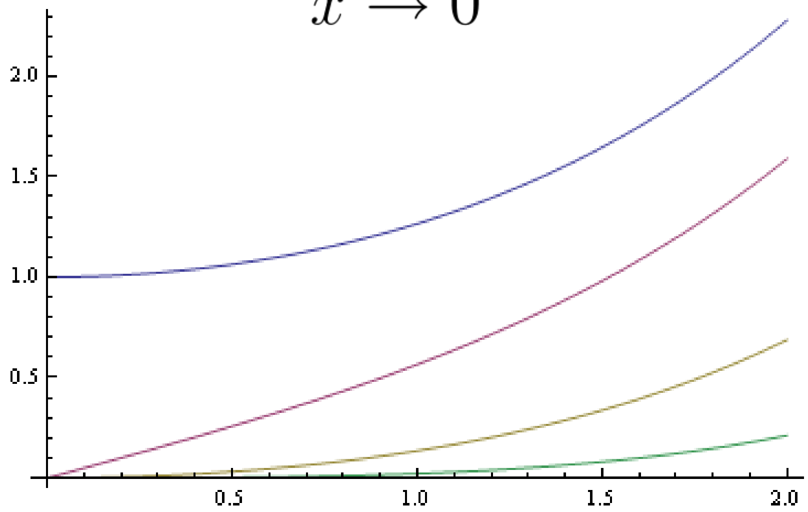
$$\int_0^\infty t \, dt \, J_\nu(at) J_\nu(bt) e^{-p^2 t^2} = \frac{1}{2p^2} \exp\left(-\frac{a^2 + b^2}{4p^2} \right) I_\nu\left(\frac{ab}{2p^2} \right)$$

Modified Bessel Functions

$$y'' + y'/x - (1 + n^2/x^2)y = 0$$

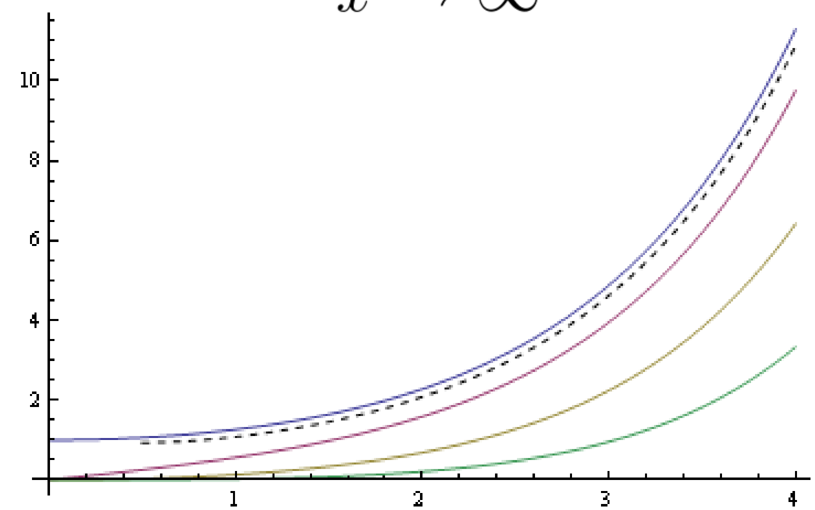
$$I_n(x) = i^{-n} J_n(ix)$$

$$x \longrightarrow 0$$



$$I_n(x) \sim \frac{1}{n!}\left(\frac{x}{2}\right)^n$$

$$x \longrightarrow \infty$$



$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$

After Integrating over the Perpendicular Velocity

$$\chi_s = \left[\hat{\mathbf{e}}_\parallel \hat{\mathbf{e}}_\parallel \frac{2\omega_p^2}{\omega k_\parallel w_\perp^2} \langle v_\parallel \rangle + \frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda} \mathbf{Y}_n(\lambda)\right]_s$$

$$\mathbf{Y}_n(\lambda) = \begin{pmatrix} \frac{n^2 I_n}{\lambda} A_n & -in(I_n - I_n')A_n & \frac{k_\perp}{\Omega}\frac{nI_n}{\lambda} B_n \\ in(I_n - I_n')A_n & \left(\frac{n^2}{\lambda} I_n + 2\lambda I_n - 2\lambda I_n'\right) A_n & \frac{ik_\perp}{\Omega}(I_n - I_n')B_n \\ \frac{k_\perp}{\Omega}\frac{nI_n}{\lambda} B_n & -\frac{ik_\perp}{\Omega}(I_n - I_n')B_n & \frac{2(\omega - n\Omega)}{k_\parallel w_\perp^2} I_n B_n \end{pmatrix}$$

$$\lambda = \frac{k_\perp^2 w_\perp^2}{2\Omega^2}$$

$$A_n = \int_{-\infty}^{\infty} dv_\parallel \frac{H(v_\parallel)}{\omega - k_\parallel v_\parallel - n\Omega} \qquad B_n = \int_{-\infty}^{\infty} dv_\parallel \frac{v_\parallel H(v_\parallel)}{\omega - k_\parallel v_\parallel - n\Omega}$$

$$H(v_\parallel) = -\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right) h(v_\parallel) + \frac{k_\parallel w_\perp^2}{2\omega} h'(v_\parallel)$$

Distribution Maxwellian in the Parallel Velocity

$$h_s(v_\parallel) = \left\{ \frac{1}{\sqrt{\pi} w_\parallel} \exp\left[-\frac{(v_\parallel - V)^2}{w_\parallel^2} \right] \right\}_s$$

$$A_n = \frac{1}{\omega} \frac{T_\perp - T_\parallel}{T_\parallel} + \frac{1}{k_\parallel w_\parallel} \frac{(\omega - k_\parallel V - n\Omega) T_\perp + n\Omega T_\parallel}{\omega T_\parallel} Z_0$$

$$B_n = \frac{1}{k_\parallel} \frac{(\omega - n\Omega) T_\perp - (k_\parallel V - n\Omega) T_\parallel}{\omega T_\parallel}$$

$$+ \frac{1}{k_\parallel} \frac{\omega - n\Omega}{k_\parallel w_\parallel} \frac{(\omega - k_\parallel V - n\Omega) T_\perp + n\Omega T_\parallel}{\omega T_\parallel} Z_0,$$

$$Z_0 = Z_0(\zeta_n), \quad \zeta_n = \frac{\omega - k_\parallel V - n\Omega}{k_\parallel w_\parallel},$$

$$\frac{dZ_0(\zeta_n)}{d\zeta_n} = -2[1 + \zeta_n Z_0(\zeta_n)].$$

Isotropic Maxwellian Distribution

$$\chi_s = \left[\frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda} \mathbf{Y}_n(\lambda) \right]_s$$

$$\mathbf{Y}_n(\lambda) = \begin{pmatrix} \frac{n^2 I_n}{\lambda} A_n & -in(I_n - I_n')A_n & \frac{k_\perp}{\Omega}\frac{nI_n}{\lambda} B_n \\ in(I_n - I_n')A_n & \left(\frac{n^2}{\lambda} I_n + 2\lambda I_n - 2\lambda I_n'\right) A_n & \frac{ik_\perp}{\Omega}(I_n - I_n')B_n \\ \frac{k_\perp}{\Omega}\frac{nI_n}{\lambda} B_n & -\frac{ik_\perp}{\Omega}(I_n - I_n')B_n & \frac{2(\omega - n\Omega)}{k_\parallel w_\perp^2} I_n B_n \end{pmatrix}$$

$$A_n = \frac{1}{k_\parallel w} Z_0(\zeta_n) \qquad B_n = -\frac{1}{2k_\parallel} \frac{dZ_0(\zeta_n)}{d\zeta_n} \qquad \lambda = \frac{k_\perp^2 v_T^2}{\Omega^2} = k_\perp^2 r_g^2$$

Summary

Integration over characteristics $\rightarrow f_1 \rightarrow \boldsymbol{j} \rightarrow \hat{\epsilon}$ in the general form

$$\downarrow$$

$$\boxed{\int d^3p\, d\tau = \int dp_{\parallel} \int p_{\perp}\, dp_{\perp} \int d\tau \int d\phi}$$

$$\downarrow$$

$$\boxed{\hat{\epsilon} \text{ for any } f_0} \leftarrow \int d\tau \leftarrow J_n \text{ appear} \leftarrow \int d\phi$$

$$\downarrow$$

$$f_0 = \text{Maxwellian}(v_{\perp}) \times h(v_{\parallel}) \rightarrow I_n, A_n, B_n$$

$$\downarrow$$

$$\text{isotropic Maxwellian} \leftarrow Z(\xi) \leftarrow h = \text{Maxwellian}(v_{\parallel})$$

$$\sum_{n=-\infty}^{\infty} \frac{J_n(z) J_{n-m}(z)}{a-n} = \frac{(-1)^m \pi}{\sin \pi a} J_{m-a}(z) J_a(z) \qquad m \geq 0$$

$$\sum_{n=-\infty}^{\infty} \frac{n^2 J_n^2(z)}{a-n} = \frac{\pi a^2}{\sin \pi a} J_a(z) J_{-a}(z) - a$$

$$\sum_{n=-\infty}^{\infty} \frac{[J_n'(z)]^2}{a-n} = \frac{\pi}{\sin \pi a} J_a'(z) J_{-a}'(z) + \frac{a}{z^2}$$

$$\sum_{n=-\infty}^{\infty} \frac{n J_n(z) J_n'(z)}{a-n} = \frac{\pi a}{\sin \pi a} J_a(z) J_{-a}'(z) + \frac{a}{z}$$

$$\sum_{n=-\infty}^{\infty} \frac{J_n(z) J_n'(z)}{a-n} = \frac{\pi}{\sin \pi a} J_a(z) J_{-a}'(z) + \frac{1}{z}$$

$$\sum_{n=-\infty}^{\infty} \frac{n J_n^2(z)}{a-n} = \frac{\pi a}{\sin \pi a} J_a(z) J_{-a}(z) - 1$$

$$\sum_{n=-\infty}^{\infty} \frac{J_n^2(z)}{a-n} = \frac{\pi}{\sin \pi a} J_a(z) J_{-a}(z).$$

$$\mathfrak{I}_{xx} = \frac{a}{z^2}\left[\frac{\pi a}{\sin(\pi a)} J_{-a}(z) J_a(z) - 1\right]$$

$$\mathfrak{I}_{xx} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} a^2 \sqrt{\pi}\, \Gamma(3/2+m) z^{2m}}{\Gamma(2+m)\Gamma(2-a+m)\Gamma(2+a+m)\sin(\pi a)}$$
$$= \sum_{m=0}^{\infty} \frac{m+1/2}{m+1} \frac{(2m)!}{4^m (m!)^2} \frac{a z^{2m}}{[a^2-(m+1)^2]\dots(a^2-1)}$$

$$\mathfrak{I}_{xx} = \frac{a}{2(a^2-1)} + \frac{3az^2}{8(a^2-1)(a^2-2^2)} + \frac{5az^4}{16(a^2-1)(a^2-2^2)(a^2-3^2)} + \dots$$

Swanson, *Plasma Waves* (2003), 2nd edition, Problem 4.3.4.