Nonlinear frequency shift of electrostatic waves in general collisionless plasma: unifying theory of fluid and kinetic nonlinearities

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Abstract

The nonlinear frequency shift is derived in a transparent asymptotic form for intense Langmuir waves in general collisionless plasma. The formula describes both fluid and kinetic effects simultaneously. The fluid nonlinearity is expressed, for the first time, through the plasma dielectric function, and the kinetic nonlinearity accounts for both smooth distributions and trapped-particle beams. Various known limiting scalings are reproduced as special cases. The calculation avoids differential equations and can be extended straightforwardly to other nonlinear plasma waves.

$$\left[\frac{\partial \epsilon(\omega_0, k)}{\partial \omega_0}\right] \delta\omega \approx \underbrace{\frac{a_0^2}{96}\left\{\left[\omega_0^2 \frac{\partial^2 \epsilon(\omega_0, k)}{\partial \omega_0^2}\right]^2 - \frac{\omega_0^4}{2}\frac{\partial^4 \epsilon(\omega_0, k)}{\partial \omega_0^4}\right\}}_{\text{fluid, due to } \phi_2} + \underbrace{\eta \sqrt{a_0}\, f_0''(u_0)\, \frac{\omega_0 \omega_p^2}{k^3}}_{\text{kinetic, due to } \phi_1} + \underbrace{\frac{\vartheta}{a_0^r}}_{\text{beam}}$$

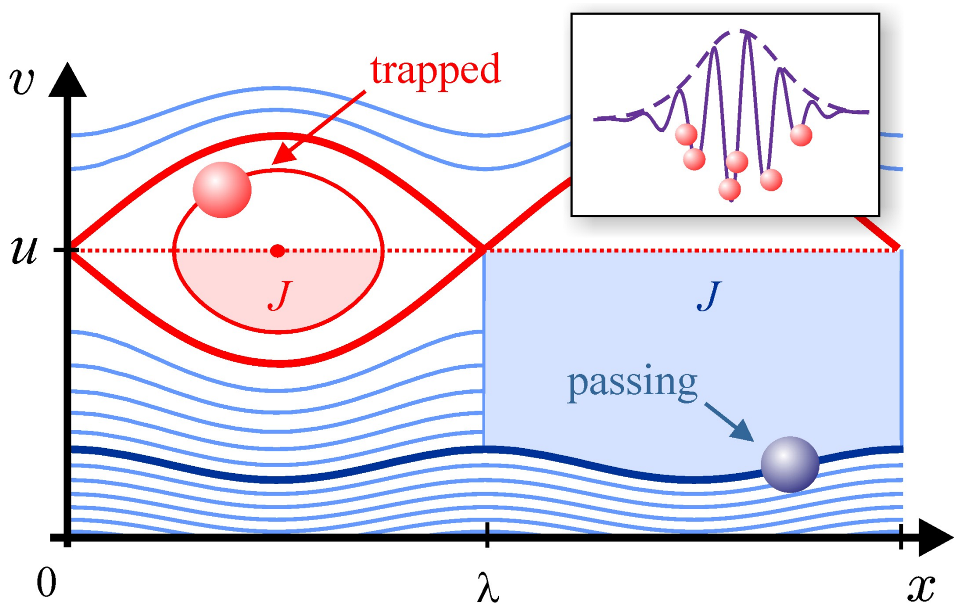
$$\phi = \phi_1 \cos(\xi) + \phi_2 \cos(2\xi + \chi), \qquad a_0 = e k^2 \phi_1 / m \omega_0^2$$

Variational principle

- The dispersion (and dynamics) of an adiabatic wave can be derived from the least action principle $\delta\Lambda = 0$, where $\Lambda = \int \mathfrak{L}\, d^3x\, dt$, and $\mathfrak{L}$ is the Lagrangian density:

$$\mathfrak{L} = \frac{\langle(\partial_x \phi)^2\rangle}{8\pi} - \sum_{\mathsf{s}} \bar{n}_{\mathsf{s}} \langle\!\langle \mathcal{H}_{\mathsf{s}} \rangle\!\rangle, \qquad \underbrace{\mathcal{H} = \dot{\zeta}\,\partial_{\dot{\zeta}}\langle L\rangle - \langle L\rangle}_{\text{oscillation-center Hamiltonian}}$$

- OC Hamiltonians $\mathcal{H}$ are found by time-averaging single-particle Lagrangians L



$$\mathcal{H}_{\mathsf{p}} = Pu + \mathcal{E} - mu^2/2$$

$$\mathcal{H}_{\mathsf{t}} = \mathcal{E} - mu^2/2$$

$$P = mu + kJ\,\mathsf{sgn}(v-u)$$

$$\mathcal{E} = m(v-u)^2/2 + e\phi$$

$$J = (2\pi)^{-1}\int m(v-u)\,dx$$

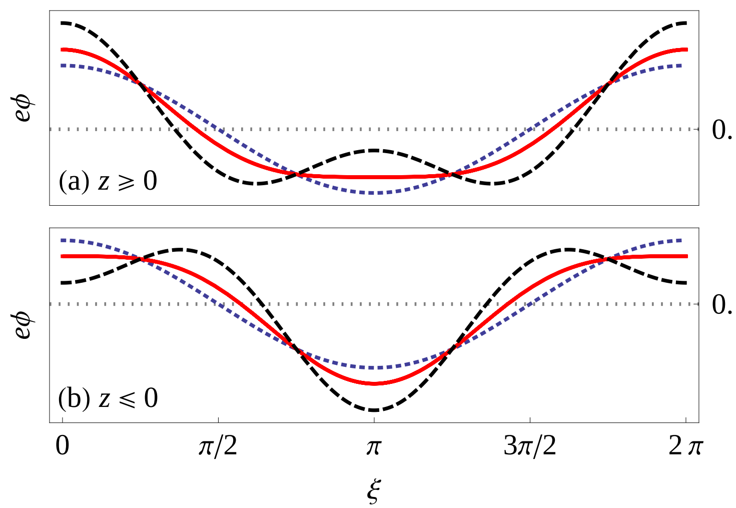
I. Y. Dodin, Fusion Sci. Tech. 65, 54 (2014)

Action distribution and independent variables

- Assume that a wave is excited with homogeneous amplitude and $u = \text{const}$:

$$\text{const} = P = mV_0, \qquad J = |P - mu|/k = |1 - V_0/u|\hat{J}, \qquad \hat{J} = m\omega/k^2$$

$$V_0 \mapsto J: \qquad F(J) = \int \delta\big(J - |1 - {}^{V_0}\!/_{u}|\,\hat{J}\big)\, f_0(V_0)\, dV_0$$



$$F(J) = \frac{k}{m}\left[f_0\left(u + \frac{kJ}{m}\right) + f_0\left(u - \frac{kJ}{m}\right)\right]$$

- Parametrize ϕ with independent $a_{1,2}$ and χ:

$$e\phi/mu^2 = \underbrace{a_1}_{a}\cos(\xi) + \underbrace{a_2}_{2az}\cos(2\xi + \chi)$$

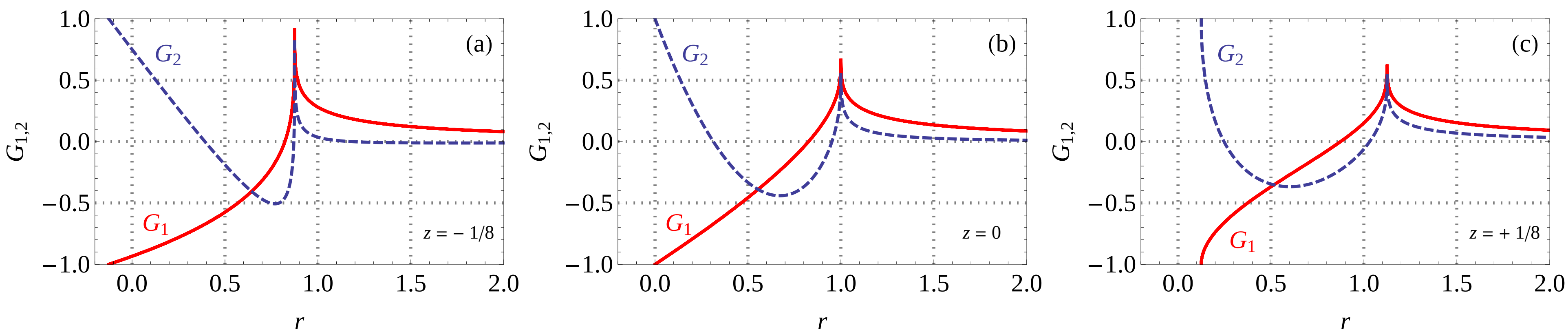
$$\mathfrak{L}(a_1, a_2, \chi) = \bar{n}mu^2\,\frac{\omega^2}{2\omega_p^2}\left(\frac{a_1^2}{2} + 2a_2^2\right) - \bar{n}\langle\langle\mathcal{E}\rangle\rangle - \underbrace{\bar{n}_p\langle\langle P\rangle\rangle u + \frac{\bar{n}mu^2}{2}}_{\text{independent of } a_{1,2} \text{ and } \chi}$$

Euler-Lagrange equations

$$\langle\langle G_1 \rangle\rangle - a\omega^2/2\omega_p^2 = 0, \qquad 8z\langle\langle G_1 \rangle\rangle - \langle\langle G_2 \rangle\rangle = 0, \qquad \chi = 0$$

$$G_{1,2}(r, z) = \frac{\partial}{\partial a_{1,2}} \left[\frac{\mathcal{E}(J, a_1, a_2, \omega, k)}{mu^2} \right], \qquad r = \frac{1}{2}\left(\frac{\mathcal{E}}{mu^2 a} + 1 \right)$$

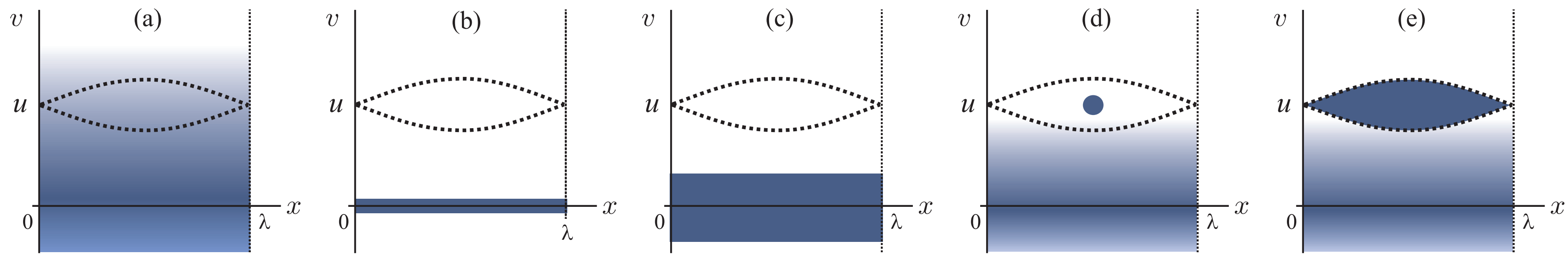
- The dimensionless functions $G_{1,2}$ determine how much a particle with given J contributes to ω and a_2/a_1. Using j, they can be expressed through just (r, z):



$$\frac{J}{\hat{J}\sqrt{a}} \equiv j(r, z) = \frac{2}{\pi} \operatorname{Re} \int_0^\pi \sqrt{r - \sin^2(\theta/2) - z\cos(2\theta)}\, d\theta$$

Approaching the dispersion relation for smooth distributions

- We will consider several types of f_0, starting from smooth distributions



- To calculate $\langle\!\langle G_{1,2} \rangle\!\rangle \equiv \int_0^\infty G_{1,2} F(J)\, dJ$, we need the following asymptotics:

$$j(r,z) = \sqrt{r}\left[2 - \frac{1}{2r} - \frac{3+4z^2}{32r^2} - \frac{5+3z+12z^2}{128r^3} - \frac{5\left(35+48z+144z^2\right)}{8192r^4} \cdots\right]$$

$$r(j,z) = \frac{j^2}{4} + \frac{1}{2} + \frac{1+4z^2}{8j^2} + \frac{3z}{8j^4} + \frac{5(1+32z^2)}{128j^6} + \cdots$$

$$G_1 = \frac{1}{2j^2} + \frac{3z}{2j^4} + \frac{5(1+16z^2)}{16j^6} + \cdots, \qquad G_2 = \frac{z}{j^2} + \frac{3}{8j^4} + \frac{5z}{2j^6} + \cdots$$

Asymptotic expansion of $\langle\!\langle G \rangle\!\rangle$ for smooth $F(J)$. Part 1

$$\langle\!\langle G \rangle\!\rangle = \int_0^\infty G F(J)\, dJ, \qquad G(j) = \frac{c_2}{j^2} + \frac{3c_4}{j^4} + \frac{5c_6}{j^6} + \cdots$$

- To construct an asymptotic expansion of $\langle\!\langle G \rangle\!\rangle$, let us introduce the following:

$$Y(J) = -\int_0^J G\, d\tilde{J} \equiv \hat{J}\sqrt{a}\,\Psi, \qquad \Psi(j) = \frac{c_2}{j} + \frac{c_4}{j^3} + \frac{c_6}{j^5} + \cdots$$

$$F_0^{(q)} \equiv F^{(q)}(0) = 2 f_0^{(q)}(u) \left(\frac{k}{m}\right)^{q+1} \times \begin{cases} 0, & q \text{ is odd} \\ 1, & q \text{ is even} \end{cases}$$

- Assuming that $Y(J)$ is smooth compared to $F(J)$, we can rewrite the integrand as a sum of Q_i, which approximate YF' at small, large, and intermediate J:

$$\langle\!\langle G \rangle\!\rangle = \int_0^\infty Y(J) F'(J)\, dJ \approx \int_0^\infty (Q_1 + Q_2 - Q_3)\, dJ$$

Asymptotic expansion of $\langle\langle G \rangle\rangle$ for smooth $F(J)$. Part 2

- Specifically, the function Q_i are given by the following formulas:

$$Q_1(J) = Y(J)\left(F_0^{(2)}J + F_0^{(4)}\frac{J^3}{3!} + F_0^{(6)}\frac{J^5}{5!} + \cdots\right)$$

$$Q_2(J) = \hat{J}^2 a\left(\frac{c_2}{J} + \frac{c_4}{J^3}\hat{J}^2 a + \frac{c_6}{J^5}\hat{J}^4 a^2 + \cdots\right)F'(J)$$

$$Q_3(J) = \hat{J}^2 a\left(\frac{c_2}{J} + \frac{c_4}{J^3}\hat{J}^2 a + \frac{c_6}{J^5}\hat{J}^4 a^2 + \cdots\right)\left(F_0^{(2)}J + F_0^{(4)}\frac{J^3}{3!} + F_0^{(6)}\frac{J^5}{5!} + \cdots\right)$$

- By rearranging the terms, we can express $\langle\langle G \rangle\rangle$ as a sum of *converging* integrals:

$$\langle\langle G \rangle\rangle = \underbrace{\left[(\hat{J}^2 a)c_2\chi^{(2)} + (\hat{J}^2 a)^2 c_4\chi^{(4)} + (\hat{J}^2 a)^3 c_6\chi^{(6)} + \cdots\right]}_{\text{integer powers of } a}$$

$$+ \underbrace{\left[(\hat{J}^2 a)^{3/2}F_0^{(2)}\eta^{(2)} + (\hat{J}^2 a)^{5/2}F_0^{(4)}\eta^{(4)} + (\hat{J}^2 a)^{7/2}F_0^{(6)}\eta^{(6)} + \cdots\right]}_{\text{half-integer powers of } a}$$

Asymptotic expansion of $\langle\langle G \rangle\rangle$ for smooth $F(J)$. Part 3

- The coefficients $\eta^{(q)}$ are simply *numbers* and can be found numerically:

$$\eta^{(2)} = \int_0^\infty [\Psi(j)j - c_2]\, dj, \qquad \eta^{(4)} = \frac{1}{3!}\int_0^\infty [\Psi(j)j^3 - c_2 j^2 - c_4]\, dj$$

- The coefficients $\chi^{(q)}$ are expressed through the dielectric function:

$$\chi^{(2)} = \int_0^\infty J^{-1} F'(J)\, dJ = \frac{k^4}{m^2\omega_p^2}[1 - \epsilon(\omega, k)]$$

$$\chi^{(4)} = \int_0^\infty J^{-3}[F'(J) - F_0^{(2)} J]\, dJ = -\frac{1}{2}\frac{k^8}{m^4\omega_p^2}\frac{\partial^2\epsilon(\omega, k)}{\partial\omega^2}$$

$$\chi^{(6)} = \int_0^\infty J^{-5}[F'(J) - F_0^{(2)} J - F_0^{(4)} J^3/3!]\, dJ = -\frac{1}{4!}\frac{k^{12}}{m^6\omega_p^2}\frac{\partial^4\epsilon(\omega, k)}{\partial\omega^4}$$

$$\epsilon(\omega, k) \doteq 1 - \frac{\omega_p^2}{k^2}\,\text{P.V.}\int_{-\infty}^{\infty}\frac{f_0'(v)}{v - \omega/k}\, dv$$

- Now let us use these results to actually calculate $\langle\langle G_{1,2} \rangle\rangle$ with appropriate c_q...

Dispersion relation for smooth distributions

- At vanishingly small amplitude, the usual linear dispersion relation is obtained:

$$\epsilon(\omega_0, k) = 0$$

- At nonzero $a_0 \doteq ek^2\phi_1/m\omega_0^2$, the nonlinear frequency shift is found to be

$$\delta\omega \approx \left[\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}\right]^{-1}\left[-\omega_0^2\,\frac{\partial^2\epsilon(\omega_0,k)}{\partial\omega_0^2}\,\frac{za_0}{2} - \omega_0^4\,\frac{\partial^4\epsilon(\omega_0,k)}{\partial\omega_0^4}\,\frac{a_0^2}{192} + 4\eta_1\sqrt{a_0}\,f_0''(u_0)\,\frac{\omega_0\omega_p^2}{k^3}\right], \qquad \eta_1 \approx -0.27$$

- The relative amplitude of the second harmonic, $z \equiv a_2/2a_1$, is given by

$$z \approx -\omega_0^2\,\frac{\partial^2\epsilon(\omega_0,k)}{\partial\omega_0^2}\,\frac{a_0}{48} + \frac{2}{3}\,\eta_2\sqrt{a_0}\,f_0''(u_0)\,\frac{\omega_0\omega_p^2}{k^3}, \qquad \eta_2 \approx 0.11$$

Fluid limit

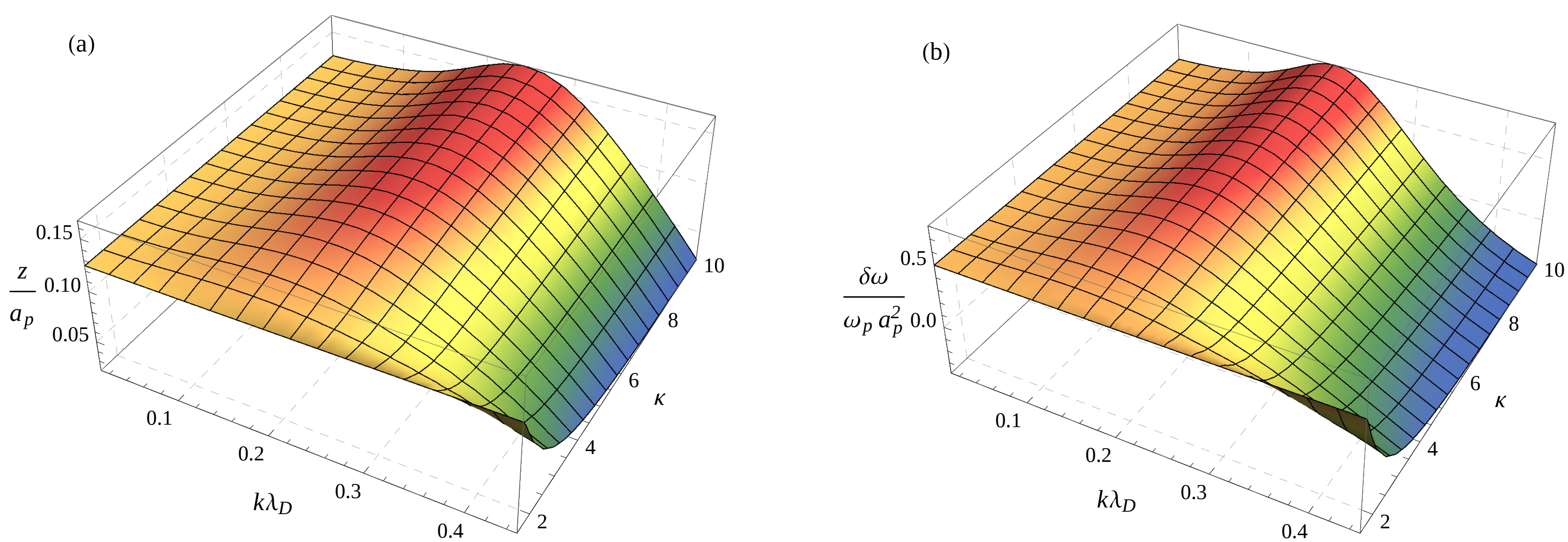
- In cold plasma, one gets a *nonzero* frequency shift:

$$\epsilon(\omega, k) = 1 - \omega_p^2/\omega^2, \qquad z = a_0/8, \qquad \delta\omega = \omega_p a_0^2/2$$

- In the case of waterbag distribution, with $\bar{\alpha} \doteq (k\bar{v}/\omega_0)^2$:

$$\epsilon(\omega, k) = 1 - \frac{\omega_p^2}{\omega^2 - k^2\bar{v}^2}, \qquad z = \frac{a_0(3+\bar{\alpha})}{24(1-\bar{\alpha})^2}, \qquad \delta\omega = \omega_0 a_0^2 \frac{(6 + 9\bar{\alpha} + \bar{\alpha}^2)}{12(1-\bar{\alpha})^3}$$

- In the case of "kappa" distribution, $f_0(v) \propto [1 + v^2/(2\kappa - 3)v_T^2]^{-\kappa}$:



Dewar and Lindl (1972); Winjum *et al* (2007); cf. Sagdeev *et al*, *Nonlinear Physics...* (1988)

Explanation from fluid equations: nonlinear Doppler shift

- Assume a homogeneous amplitude, so $\langle \partial_x(...)\rangle_x \equiv 0$. Then, $\mathcal{P} \doteq mnv$ satisfies

$$\partial_t \mathcal{P} + \partial_x(mnv^2) = neE - \partial_x \Pi$$

$$\partial_t \langle \mathcal{P} \rangle_x = \langle neE \rangle_x \approx \langle \tilde{n} e \tilde{E} \rangle_x$$

$$e\tilde{E} = m\partial_t \tilde{v} + \partial_x \tilde{\Pi}/\bar{n} = m\partial_t \tilde{v} + \bar{T}\partial_x \tilde{n}$$

$$\langle \tilde{n} e \tilde{E} \rangle_x / m = \langle \tilde{n} \partial_t \tilde{v} \rangle_x = \partial_t \langle \tilde{n} \tilde{v} \rangle_x - \langle \tilde{v} \partial_t \tilde{n} \rangle_x = \partial_t \langle \tilde{n} \tilde{v} \rangle_x + \bar{n} \langle \tilde{v} \partial_x \tilde{v} \rangle_x = \partial_t \langle \tilde{n} \tilde{v} \rangle_x$$

$$\langle \Delta \mathcal{P} \rangle_x = \langle m \tilde{n} \tilde{v} \rangle_x = mu \langle \tilde{n}^2 \rangle_x / \bar{n} = m\bar{n} \langle \tilde{v}^2 \rangle_x / u$$

- Now plasma is moving at $\langle\!\langle \Delta V \rangle\!\rangle = \langle \mathcal{P} \rangle / m\bar{n}$, so one gets a Doppler shift $\delta\omega_D$. In cold plasma, $\delta\omega$ consists of this Doppler shift *entirely:*

$$\delta\omega_D = k \langle\!\langle \Delta V \rangle\!\rangle, \qquad \delta\omega_D^{(\text{cold})} = \omega_p a_0^2 / 2$$

Explanations from the momentum conservation

- Far enough from the resonance, one has the usual ponderomotive Hamiltonian:

$$\mathcal{H}_\mathrm{p} \approx \frac{P^2}{2m} + \Phi, \qquad \Phi = \frac{e^2 k^2 \phi_1^2}{4m(\omega - kP/m)^2}, \qquad P = mV - \frac{\partial \Phi}{\partial V}$$

- If the driver amplitude is homogeneous, $E_d = E_{d0}(t)\,\sin(\omega t - kx)$, then $P = \text{const}$.

$$mV_0 = mV - \partial_V \Phi, \qquad \langle\!\langle \Delta V \rangle\!\rangle = \langle\!\langle \partial_V \Phi \rangle\!\rangle / m$$

- Hence, the plasma is accelerated, and the wave experiences a Doppler shift:

$$k\langle\!\langle \Delta V \rangle\!\rangle = \frac{k\langle\!\langle \partial_V \Phi \rangle\!\rangle}{m} = -\frac{e^2 k^2 \phi_1^2}{4m^2} \frac{\partial}{\partial \omega} \int_{-\infty}^{\infty} \frac{f_0(V_0)}{(V_0 - \omega/k)^2}\, dV_0 = \frac{a_0^2}{4} \frac{\omega^4}{\omega_p^2} \frac{\partial \epsilon(\omega, k)}{\partial \omega}$$

- The associated kinetic momentum is precisely the wave canonical momentum:

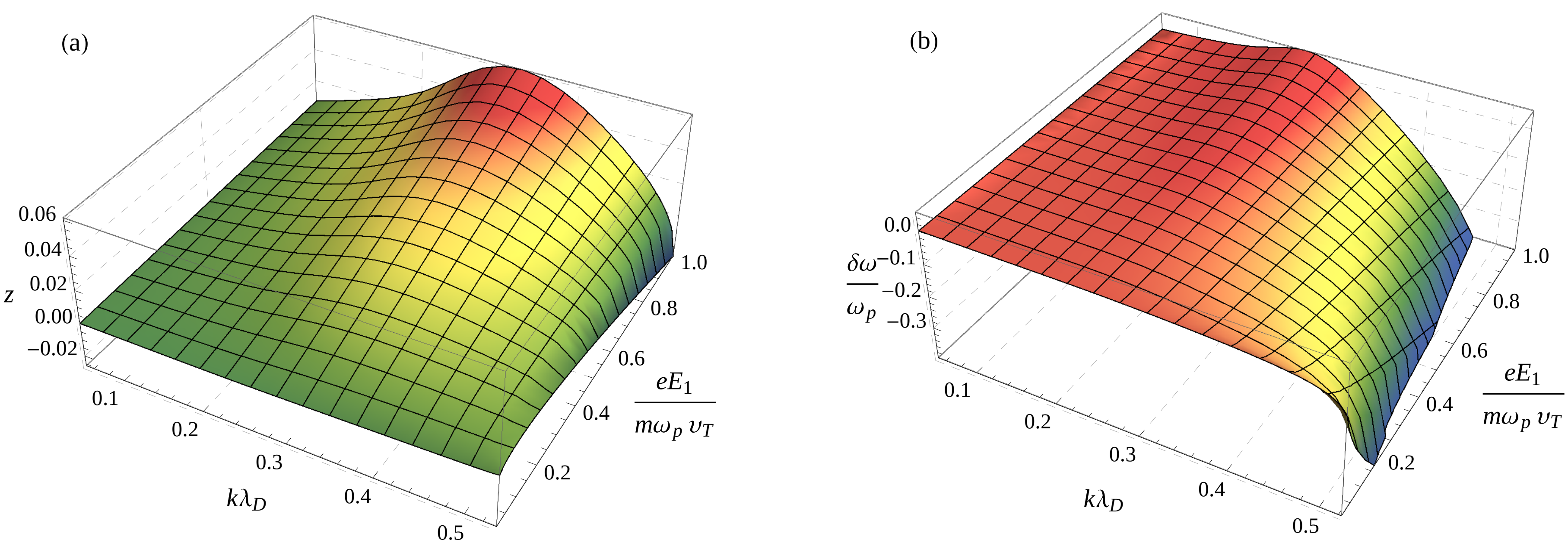
$$m\bar{n}\langle\!\langle \Delta V \rangle\!\rangle = \frac{k\tilde{E}^2}{16\pi} \frac{\partial \epsilon(\omega_0, k)}{\partial \omega_0} = \frac{k}{\omega_0} \times \text{(wave energy)}$$

Kinetic limit

- When kinetic effects dominate over the effect of the second harmonic:

$$z = \frac{2}{3}\,\eta_2\sqrt{a_0}\,f_0''(u_0)\,\frac{\omega_0\omega_p^2}{k^3}, \qquad \delta\omega = 4\eta_1\sqrt{a_0}\,f_0''(u_0)\,\frac{\omega_0\omega_p^2}{k^3}\left[\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}\right]^{-1}$$

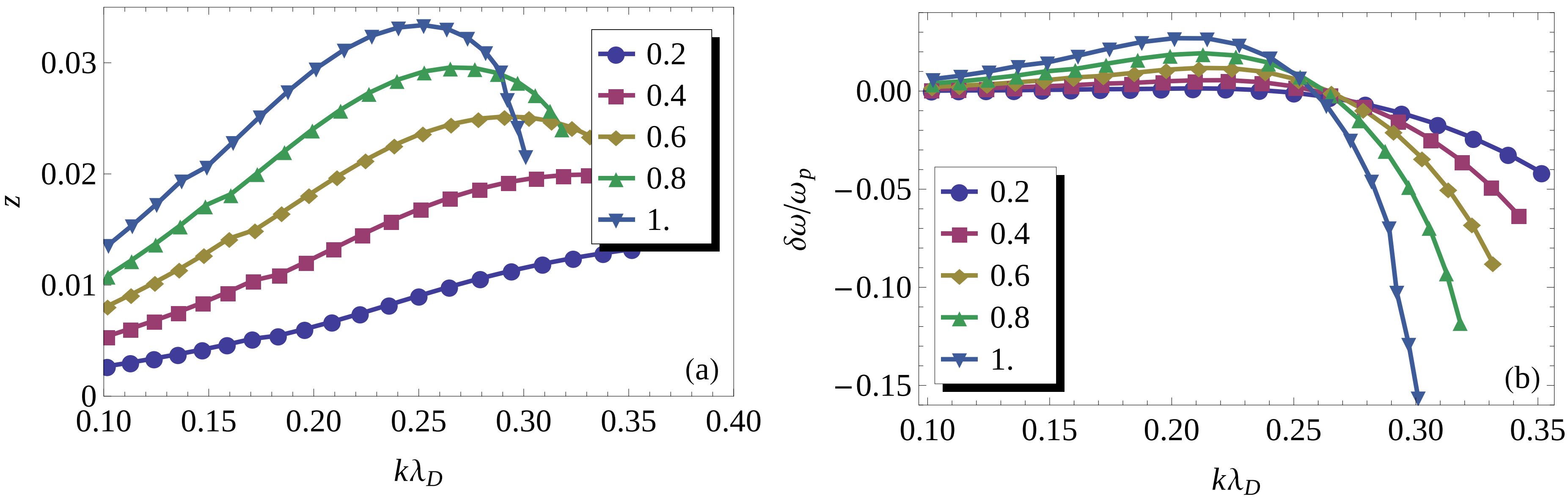
- For Maxwellian plasma, where $\epsilon(\omega,k) = 1-(2k^2\lambda^2)^{-1}\,\mathrm{Re}\,Z'(\omega/kv_T\sqrt{2})$:



Manheimer and Flynn (1971); Dewar (1972); Rose (2001); Berger *et al* (2013)...

Numerical solution for Maxwellian plasma

- We also numerically solved the original Euler-Lagrange equations for a and z assuming Maxwellian plasma. The solution demonstrates a good qualitative agreement with the asymptotic theory.



The different colors correspond to the different values of $ek\phi_1/m\omega_p v_T$ shown in the legend.

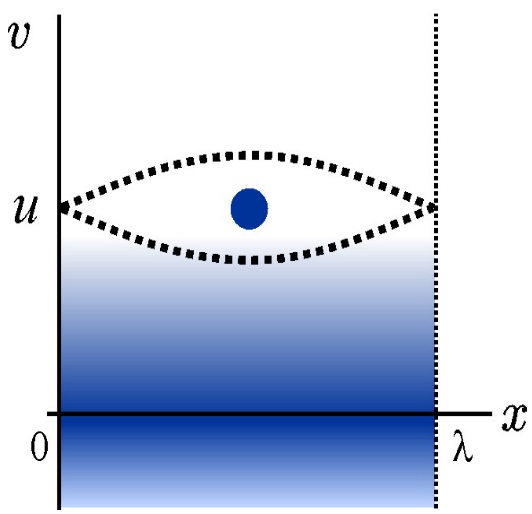
cf. Winjum *et al* (2007)

Beam distributions. Part 1

- If there is an abrupt F_{b} on top of smooth F_0, it causes additional perturbations to $\langle\langle G_{1,2}\rangle\rangle$. The small parameter now is β, not a:

$$\langle\langle \tilde{G}_{1,2}\rangle\rangle = \int G_{1,2} F_{\mathrm{b}}(J)\, dJ, \qquad \beta \doteq \frac{\bar{n}_{\mathrm{b}}}{\bar{n} a_p} \ll 1, \qquad a_p \doteq \frac{e k^2 \phi_1}{m \omega_p^2}$$

- Such beam nonlinearities produce the following terms in the dispersion relation:



$$z \approx \frac{\langle\langle \tilde{G}_2\rangle\rangle}{3 a_p}, \qquad \delta\omega \approx \frac{2\langle\langle \tilde{G}_1\rangle\rangle}{a_p} \left[\frac{\partial \epsilon(\omega_0, k)}{\partial \omega_0}\right]^{-1}$$

- At deep trapping, i.e., $F_{\mathrm{b}}(J) = (\bar{n}_{\mathrm{b}}/\bar{n})\,\delta(J)$:

$$z \approx \frac{\beta}{3}, \qquad \frac{\delta\omega}{\omega_p} \approx -\frac{2\beta}{\omega_p} \left[\frac{\partial \epsilon(\omega_0, k)}{\partial \omega_0}\right]^{-1}$$

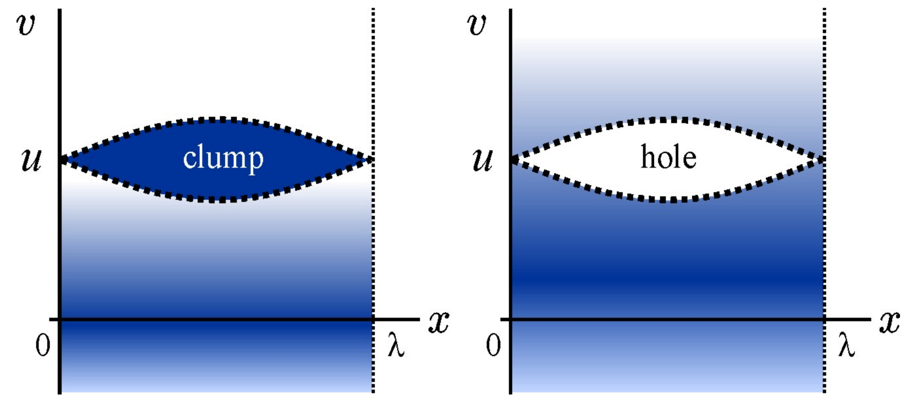
cf. Goldman and Berk (1971)

Beam distributions. Part 2

- For F_{b} flat inside the trapping island, we get $z \approx -0.028\mathcal{N}$ and

$$\frac{\omega}{\omega_p} = \left(1 + \frac{8\mathcal{N}}{3\pi}\right)^{-1/2}$$

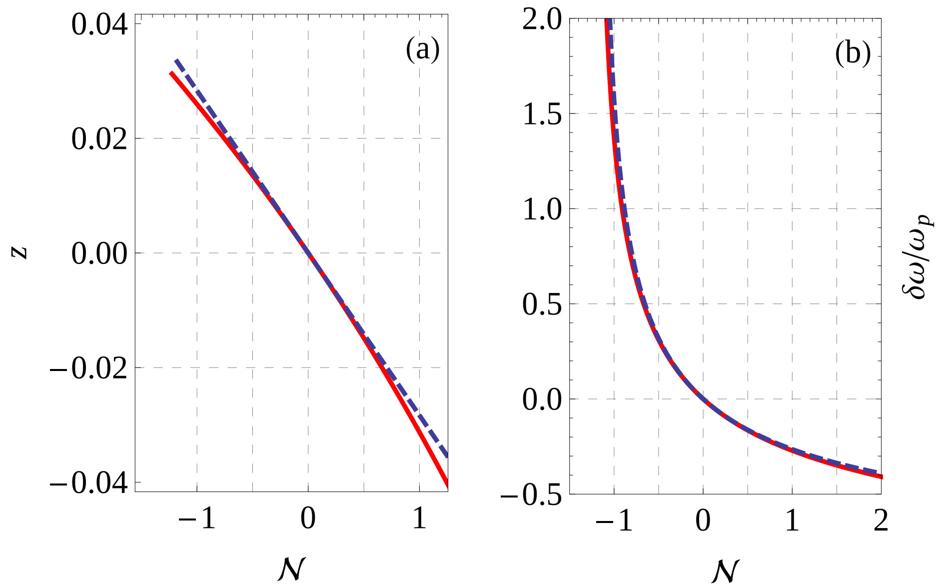
$$\mathcal{N} \doteq \frac{m\omega_p}{k} \frac{F_{\mathsf{b}}}{\sqrt{a_p}} \sim \beta$$



- This agrees with the exact theory:

$$z = \frac{1 - 4\alpha^2}{32 - 8\alpha^2}, \qquad \alpha \doteq \frac{\omega_p}{2\omega}$$

$$\mathcal{N}^{-2} = \frac{2\tan(\pi\alpha)}{3\pi\alpha}(1 - 5\alpha^2 + 4\alpha^4)^{-1}$$



- Captured instability at $\mathcal{N}_c < -\mathcal{N}_c$, where $\mathcal{N}_c^{(\mathrm{our})} \approx 1.22$, and $\mathcal{N}_c^{(\mathrm{exact})} \approx 1.18$.

cf. Eremin and Berk (2004); Krasovsky (2007); Breizman (2010)

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