Quasilinear theory from first principles: reinstating the ponderomotive forces

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Abstract

The classic quasilinear theory (QLT) of resonant wave-particle interactions [1] fails to conserve the action of nonresonant waves and mostly misses (oversimplifies) the adiabatic ponderomotive effects caused by the slow evolution of the wave parameters in time and space. Postulating action conservation instead of the usual amplitude equation for the waves' electric-field amplitudes undermines QLT's exact energy-momentum conservation, which is a significant part of the classic QLT's appeal. The 'oscillation-center' QLT [2] reinstates both action and energy-momentum conservation ad hoc, but a general first-principle QLT has been lacking. Here, we report a rigorous formulation of QLT based on the Weyl symbol calculus [3, 4]. This formulation captures both adiabatic and nonadiabatic dynamics and leads to an exactly conservative model for any Hamiltonian wave-plasma interactions. Effects of plasma inhomogeneity and Balescu–Lenard collisions are also accommodated. The known results for electrostatic, relativistic electromagnetic, and gravitational interactions are reproduced as special cases.

[1] T. H. Stix, Waves in Plasmas (Springer, 1992), 2nd edition.

[2] R. L. Dewar, Oscillation center quasilinear theory, Phys. Fluids 16, 1102 (1973).

[3] I. Y. Dodin, Quasilinear theory: the lost ponderomotive effects and why they matter, Rev. Mod. Plasma Phys. 8, 35 (2024).

[4] I. Y. Dodin, Quasilinear theory for inhomogeneous plasma, J. Plasma Phys. 88, 905880407 (2022), arxiv:2201.08562.

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Outline

- What's wrong with the textbook theory?

  Fails to conserve the wave action, misses inhomogeneities and collisions.

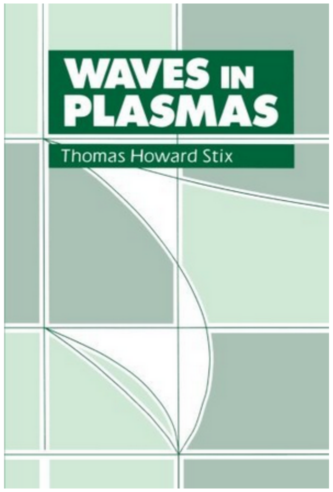
- How do we fix this?

  Use the Weyl calculus and keep the derivation general.

- Examples: grand unification

  Electrostatic turbulence, electromagnetic turbulence, relativistic gravity...

Quasilinear theory: the textbook version



- Homogeneous 1-D Langmuir turbulence, $\widetilde{E} = \sum_n \widetilde{E}_k e^{ikx}$:

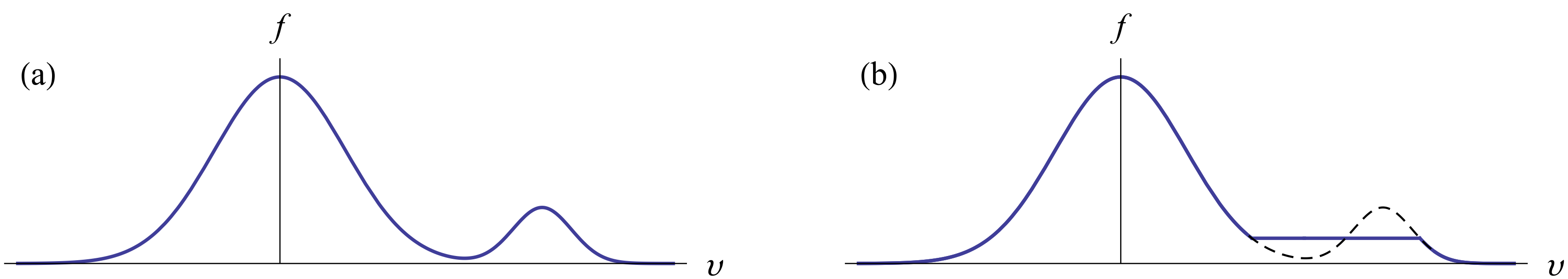
$$\partial_t f + v\partial_x f + ({}^{e}\!/\!_{m})\,\widetilde{E}\,\partial_v f = 0, \quad f = \overline{f} + \widetilde{f}, \quad \widetilde{f} \ll \overline{f}$$

- Assume a linear solution for $\widetilde{f}$ but keep $\partial_t \overline{f}$ nonlinear:

$$\widetilde{f}_k = -\frac{i({}^{e}\!/\!_{m})\widetilde{E}_k}{\omega_k - kv}\frac{\partial \overline{f}}{\partial v}, \qquad \frac{\partial \overline{f}}{\partial t} + \left\langle \frac{e}{m}\widetilde{E}\frac{\partial \widetilde{f}}{\partial v} \right\rangle = 0$$

- Then, the average distribution satisfies a diffusion equation:

$$\frac{\partial \overline{f}}{\partial t} = \frac{\partial}{\partial v}\left(\mathsf{D}(t,v)\,\frac{\partial \overline{f}(t,v)}{\partial v}\right), \qquad \mathsf{D} = \frac{e^2}{m^2}\int_{\mathsf{L}} \frac{\mathrm{d}k}{2\pi L}\frac{|\widetilde{E}_k(t)|^2}{i\,[kv - \omega_k(t)]}$$



The textbook quasilinear theory is conservative only by accident.

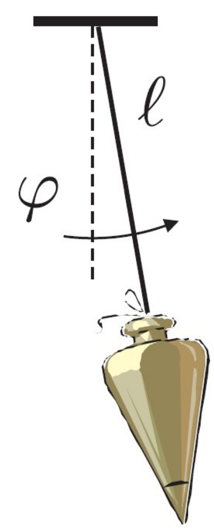
- This model is praised for its exact conservation of momentum and energy with

$$\frac{\mathrm{d}|\widetilde{E}_k|^2}{\mathrm{d}t} = 2\underbrace{\gamma_k}_{\operatorname{im}\omega_k}|\widetilde{E}_k|^2, \qquad 1 - \frac{4\pi e^2}{mk^2}\int_{\mathsf{L}} \mathrm{d}v\, \frac{\partial_v \overline{f}(t,v)}{v - \omega_k(t)/k} = 0$$

- But this field equation violates the action conservation (except in the cold limit):

$$\frac{\mathrm{d}\mathcal{I}_k}{\mathrm{d}t} = 2\gamma_k \mathcal{I}_k, \qquad \mathcal{I}_k = |\widetilde{E}_k|^2 \left.\frac{\partial_\omega(\omega^2 \epsilon_{\mathsf{H}})}{16\pi\omega^2}\right|_{(\omega_{\boldsymbol{k}},\boldsymbol{k})}$$

- QL theory is conservative only by accident: the error in the equation for $\widetilde{E}_k$ is compensated by another error in the equation for $\widetilde{f}$:

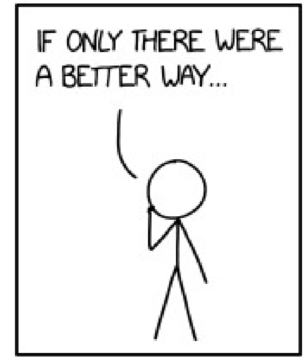


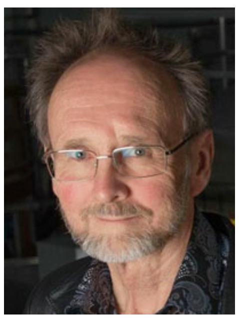
$$\widetilde{f}_k = -\frac{i(e/m)\widetilde{E}_k}{\omega_k - kv}\frac{\partial \overline{f}}{\partial v} + \underbrace{\boldsymbol{\mathcal{O}(\partial_t \overline{f})}}_{\textbf{non-negligible}}$$

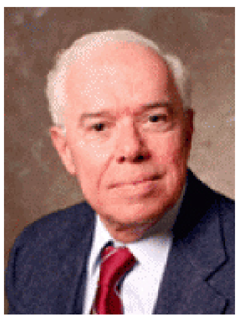
- The ponderomotive forces, caused by $\mathcal{O}(\bar{\partial}_t, \bar{\partial}_x)$, are also missing. Coincidence?..

‘Kind-of-known’ approach: oscillation-center (OC) QL theory

- With $\mathcal{O}(\overline{\partial}_t, \overline{\partial}_x)$ retained, the only known approach is heuristic:
  - ignore $\mathcal{O}(\overline{\partial}_t, \overline{\partial}_x)$ near resonances (separated out arbitrarily)
  - OC coordinate transformation for non-resonant particles
  - *assume* the proper equation for the wave action







$$\frac{\partial F_0}{\partial t} + \frac{\partial \langle K \rangle}{\partial \mathbf{p}} \cdot \frac{\partial F_0}{\partial \mathbf{x}} - \frac{\partial \langle K \rangle}{\partial \mathbf{x}} \cdot \frac{\partial F_0}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathsf{D} \cdot \frac{\partial F_0}{\partial \mathbf{p}} \right)$$

$$\frac{\partial n_k{}^l}{\partial t} + \frac{\partial \omega_k{}^l}{\partial \mathbf{k}} \cdot \frac{\partial n_k{}^l}{\partial \mathbf{x}} - \frac{\partial \omega_k{}^l}{\partial \mathbf{x}} \cdot \frac{\partial n_k{}^l}{\partial \mathbf{k}} = 2\gamma_k{}^l n_k{}^l, \qquad n_k{}^l \equiv \frac{\partial \epsilon_r}{\partial \omega_k{}^l} \frac{k^2 \,|\, \phi_k{}^l \,|^2}{8\pi L^6}$$

$$\mathsf{D} = 4\pi e^2 \sum_{k,l} (|\, \phi_k{}^l \,|^2 / 8\pi L^6) \mathbf{k}\mathbf{k} 2\pi \delta(\omega_k{}^l - \mathbf{k} \cdot \mathbf{p}/m), \qquad \langle K \rangle = H_0 - \sum_{k,l} \frac{4\pi e^2}{m} \frac{k^2 \,|\, \phi_k{}^l \,|^2}{8\pi L^6} \frac{\partial}{\partial \omega_k{}^l} \frac{\Pi(\omega_k{}^l - \mathbf{k} \cdot \mathbf{p}/m)}{\omega_k{}^l - \mathbf{k} \cdot \mathbf{p}/m}$$

- Oscillation-center QL theory is not actually proven, not expressed in terms of measurable quantities, not extendable to off-shell waves and collisional plasmas.

Dewar (1973); McDonald *et al.* (1985)

Image: adapted from xkcd.com

A first-principle approach

- Use $\partial_t f = \{\overline{H} + \widetilde{H}, f\}$, split $f = \overline{f} + \widetilde{f}$, and linearize the equation for fluctuations $\widetilde{f}$:

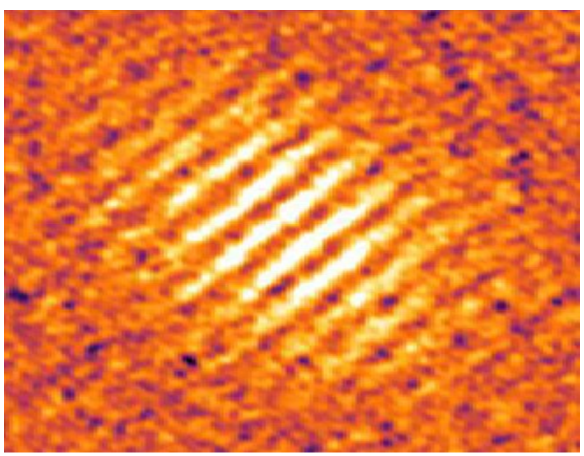
$$\partial_t \widetilde{f} - \{\overline{H}, \widetilde{f}\} = \{\widetilde{H}, \overline{f}\}, \qquad \partial_t \overline{f} - \{\overline{H}, \overline{f}\} = \langle\{\widetilde{H}, \widetilde{f}\}\rangle$$

- Define phase-space velocities in general *canonical* coordinates $\boldsymbol{z}$, with $\boldsymbol{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$:

$$v^\alpha(t, \boldsymbol{z}) \doteq J^{\alpha\beta} \underbrace{\partial_\beta \overline{H}(t, \boldsymbol{z})}_{\mathcal{O}(1)} \qquad u^\alpha(t, \boldsymbol{z}) \doteq J^{\alpha\beta} \underbrace{\partial_\beta \widetilde{H}(t, \boldsymbol{z})}_{\mathcal{O}(\varepsilon)}$$

- In terms of the Green's operator $\widehat{G}$ and unperturbed microscopic fluctuations g:

$$\widetilde{f} = g - \widehat{G}\widehat{u}^\alpha \partial_\alpha \overline{f}, \qquad \widehat{u}^\alpha = u^\alpha(t, \widehat{\boldsymbol{z}}), \qquad \widehat{G} = \lim_{\nu \to 0+} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\nu\tau - \tau(\partial_t + v^\alpha \partial_\alpha)}$$



- Define $\widehat{D}^{\alpha\beta} = \overline{\widehat{u}^\alpha \widehat{G} \widehat{u}^\beta}$ and $\mathfrak{F}^\alpha = \overline{u^\alpha g}/\overline{f}$. Then,

$$\boxed{\partial_t \overline{f} - \{\overline{H}, \overline{f}\} = \partial_\alpha(\widehat{D}^{\alpha\beta} \partial_\beta \overline{f} - \mathfrak{F}^\alpha \overline{f})}$$

It remains to approximate $\widehat{D}^{\alpha\beta}$. To do this, let's introduce some machinery...

Weyl symbol: a phase-space representation of an operator

- Any operator $\widehat{A}\psi(\mathsf{x}) = \int \mathsf{A}(\mathsf{x},\mathsf{x}')\,\psi(\mathsf{x}')\,\mathrm{d}\mathsf{x}'$ on space/spacetime* x can be expressed through its *symbol* using $\widehat{\mathsf{x}} = \mathsf{x}$, $\widehat{\mathsf{k}} = -i\nabla$:

$$A(\mathsf{x},\mathsf{k}) = \int \mathsf{A}(\mathsf{x} + \mathsf{s}/2, \mathsf{x} - \mathsf{s}/2)\,\mathrm{e}^{-i\mathsf{k}\cdot\mathsf{s}}\,\mathrm{d}\mathsf{s}$$

$$\widehat{A} = \frac{1}{(2\pi)^{2\mathsf{n}}} \int A(\mathsf{x}',\mathsf{k}')\,\mathrm{e}^{i\mathsf{k}''\cdot(\mathsf{x}'-\widehat{\mathsf{x}}) - i\mathsf{x}''\cdot(\mathsf{k}'-\widehat{\mathsf{k}})}\,\mathrm{d}\mathsf{x}'\,\mathrm{d}\mathsf{k}'\,\mathrm{d}\mathsf{x}''\,\mathrm{d}\mathsf{k}''$$

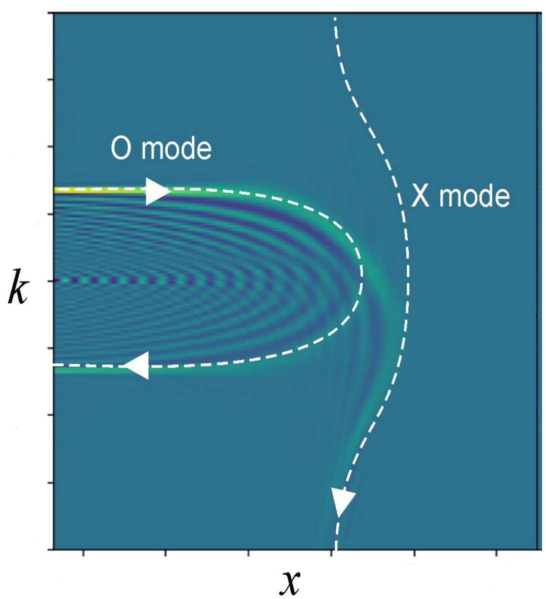
$\widehat{1} \Leftrightarrow 1$

$\widehat{\mathsf{x}} \Leftrightarrow \mathsf{x}$

$\widehat{\mathsf{k}} \Leftrightarrow \mathsf{k}$

$\widehat{\mathsf{A}}^\dagger \Leftrightarrow \mathsf{A}^\dagger$

$\widehat{\mathsf{A}}\widehat{\mathsf{B}} \Leftrightarrow \mathsf{A} \star \mathsf{B}$



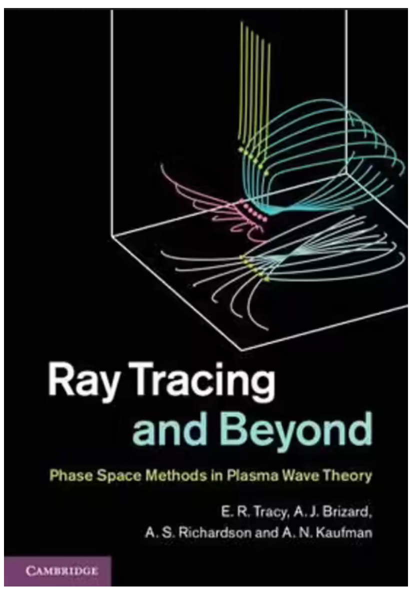
- **Example 1:** The dielectric tensor $\boldsymbol{\epsilon}(t,\boldsymbol{x},\omega,\boldsymbol{k})$ is actually the Weyl symbol of $\widehat{\boldsymbol{\epsilon}}$ (up to $\mathcal{O}(1/\omega\tau, 1/kL)$).

- **Example 2:** Spectrum of the 2-point correlation function of $\boldsymbol{E}$ is the symbol of $|E_a\rangle\langle E_b|$, a.k.a. Wigner matrix:

$$\overline{W}_{ab}(t,\boldsymbol{x},\omega,\boldsymbol{k}) = (2\pi)^{-4}\int \mathrm{d}\tau\,\mathrm{d}\boldsymbol{s}\,\mathrm{e}^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}} \times \langle E_a(t+\tau/2, \boldsymbol{x}+\boldsymbol{s}/2)\,E_b^*(t-\tau/2,\boldsymbol{x}-\boldsymbol{s}/2)\rangle$$

* $(\widehat{\mathsf{x}},\widehat{\mathsf{k}}) \equiv (t, \boldsymbol{x}, -i\partial_t, -i\nabla)$

To approximate an operator, approximate its symbol: $\widehat{A} \to A \approx A_{\text{appr}} \to \widehat{A}_{\text{appr}}$



$$\mathsf{k}F(\mathsf{x}) \Leftrightarrow {}^1\!/_2\left(\widehat{\mathsf{k}}F(\widehat{\mathsf{x}}) + F(\widehat{\mathsf{x}})\widehat{\mathsf{k}}\right) = \widehat{\mathsf{k}}F(\widehat{\mathsf{x}}) + {}^1\!/_2 \underbrace{\left[\widehat{\mathsf{k}}, F(\widehat{\mathsf{x}})\right]}_{-i\nabla\cdot F}$$

- For operators acting on slow functions:

$$A(\mathsf{x},\mathsf{k}) = A(\mathsf{x},0) + \mathsf{k}\cdot V_0 + \dots, \quad V_0 = \partial_{\mathsf{k}} A(\mathsf{x},0)$$

$$\widehat{A}\Psi = \left[A(\mathsf{x},0) - iV_0\cdot\nabla - {}^i\!/_2\,(\nabla\cdot V_0) + \dots\right]\Psi$$

- For operators acting on quasimonochromatic functions:

$$\widehat{A}(\mathrm{e}^{i\theta(\mathsf{x})}\Psi) = \mathrm{e}^{i\theta(\mathsf{x})}\left[A(\mathsf{x},\bar{\mathsf{k}}) - iV\cdot\nabla - {}^i\!/_2\,(\nabla\cdot V) + \dots\right]\Psi$$

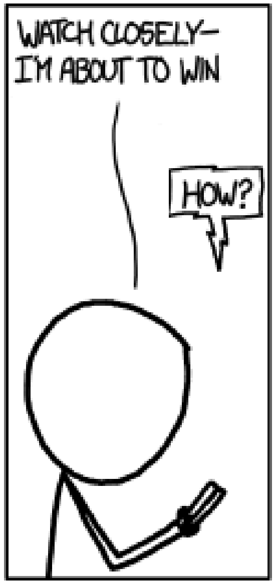
$$\bar{\mathsf{k}} = \nabla\theta(\mathsf{x}), \quad V = \partial_{\mathsf{k}} A(\mathsf{x},\bar{\mathsf{k}}(\mathsf{x}))$$

- Applications to linear waves = modern geometrical optics. But we can also use this in quasilinear theory...

Tracy *et al.* (2014); McDonald (1988); Dodin *et al.* (2019). . .

Images: xkcd.com

A first-principle approach: avoid solving for $\tilde{f}$ explicitly



- $\overline{f}$ lives in a 7D space $(t, \boldsymbol{x}, \boldsymbol{p}) \equiv \boldsymbol{X} \to$ 14D phase space $(\boldsymbol{X}, \boldsymbol{K})$.
- One can show that the symbol of $\widehat{\boldsymbol{D}}$ can be expressed as

$$\boldsymbol{D}(\boldsymbol{X}, \boldsymbol{K}) = \int \mathrm{d}\boldsymbol{K}' \; \underbrace{\overline{\boldsymbol{W}}_{\boldsymbol{u}}(\boldsymbol{X}, \boldsymbol{K}')}_{\text{Wigner matrix of } \boldsymbol{u}} \; G(\boldsymbol{X}, \boldsymbol{K} - \boldsymbol{K}')$$

$$\boldsymbol{D}(\boldsymbol{X}, \boldsymbol{K}) \approx \underbrace{\boldsymbol{D}(\boldsymbol{X}, \boldsymbol{0})}_{\text{usual QLT}} + \underbrace{(\boldsymbol{K} \cdot \partial_{\boldsymbol{K}})\boldsymbol{D}(\boldsymbol{X}, \boldsymbol{0})}_{\text{new term}}$$

- The Wigner matrix of the quiver phase-space velocity $\boldsymbol{u}$ is the symbol of $|\boldsymbol{u}\rangle\langle\boldsymbol{u}|$.
- Using $|\boldsymbol{u}\rangle = i\boldsymbol{J}\widehat{\boldsymbol{q}}\,|\widetilde{H}\rangle$, where $\widehat{\boldsymbol{q}} = -i\partial_{\boldsymbol{z}}$, one can express $\overline{\boldsymbol{W}}_{\boldsymbol{u}}$ through the scalar Wigner function of $\widetilde{H}$, which is the symbol of $|\widetilde{H}\rangle\langle\widetilde{H}|$:

$$\overline{W}_{\widetilde{H}} = \int \frac{\mathrm{d}\tau}{2\pi} \frac{\mathrm{d}\boldsymbol{s}}{(2\pi)^n} \mathrm{e}^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}} \, \overline{\widetilde{H}(t + \tau/2, \boldsymbol{x} + \boldsymbol{s}/2, \boldsymbol{p}) \, \widetilde{H}(t - \tau/2, \boldsymbol{x} - \boldsymbol{s}/2, \boldsymbol{p})}$$

Image: adapted from xkcd.com

Dodin (2022); Dodin (2024)

Equation for the dressed, or "oscillation-center", distribution $\boldsymbol{F}$

- Equation for the "oscillation-center" distribution $F \doteq \overline{f} + \partial_{\boldsymbol{p}} \cdot (\Theta\, \partial_{\boldsymbol{p}} \overline{f})$ captures both QL diffusion and ponderomotive forces:

$$\frac{\partial F}{\partial t} = \{\overline{H} + \Phi, F\} + \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(\mathsf{D}\,\frac{\partial F}{\partial \boldsymbol{p}}\right)$$

$$\boldsymbol{\Theta} = \frac{\partial}{\partial \vartheta} \fint \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}\boldsymbol{k}^\dagger\, \overline{\boldsymbol{W}}_{\widetilde{\boldsymbol{H}}}}{2(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \vartheta)}\bigg|_{\vartheta=0}$$

$$\Phi = \frac{\partial}{\partial \boldsymbol{p}} \cdot \fint \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}\overline{\boldsymbol{W}}_{\widetilde{\boldsymbol{H}}}}{2(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}$$

$$\mathsf{D} = \pi \int \mathrm{d}\boldsymbol{k}\; \boldsymbol{k}\boldsymbol{k}^\dagger\, \overline{\boldsymbol{W}}_{\widetilde{\boldsymbol{H}}}(t, \boldsymbol{x}, \boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k}; \boldsymbol{p})$$

No coordinate transformations → no singularities!

- When averaging over phase-space volume $\Delta x\, \Delta k \gtrsim 1$, the function $\overline{W}_{\widetilde{H}}$ is nonnegative → D is positive-semidefinite → H-theorem.

Dodin (2022); Dodin (2024)

Image: adapted from xkcd.com

Let's make the fields self-consistent (but not necessarily on-shell).

- Use a generic linear-wave action for vacuum and a generic particle Hamiltonian. Then $\widetilde{\mathbf{\Psi}}$ satisfies a linear equation with initial conditions g_s as sources.

$$S_0 = \frac{1}{2}\int \widetilde{\mathbf{\Psi}}^\dagger \widehat{\mathbf{\Xi}}_0 \widetilde{\mathbf{\Psi}}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}, \qquad H_s \approx H_{0s} + \widehat{\boldsymbol{\alpha}}_s^\dagger \widetilde{\mathbf{\Psi}} + \frac{1}{2}(\widehat{\boldsymbol{L}}_s \widetilde{\mathbf{\Psi}})^\dagger (\widehat{\boldsymbol{R}}_s \widetilde{\mathbf{\Psi}}) \quad \to \quad \widehat{\mathbf{\Xi}}\widetilde{\mathbf{\Psi}} = \sum_s \int \mathrm{d}\boldsymbol{p}\,\widehat{\boldsymbol{\alpha}}_s g_s$$

$$\mathbf{\Xi}(\omega,\boldsymbol{k}) \approx \mathbf{\Xi}_0(\omega,\boldsymbol{k}) - \sum_s \int \mathrm{d}\boldsymbol{p}\,(\boldsymbol{L}_s^\dagger \boldsymbol{R}_s)_{\mathsf{H}}(\omega,\boldsymbol{k};\boldsymbol{p}) F_s(\boldsymbol{p}) + \sum_s ⨍ \mathrm{d}\boldsymbol{p}\,\frac{\boldsymbol{\alpha}_s(\omega,\boldsymbol{k};\boldsymbol{p})\boldsymbol{\alpha}_s^\dagger(\omega,\boldsymbol{k};\boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s + i0}\,\boldsymbol{k}\cdot\frac{\partial F_s(\boldsymbol{p})}{\partial \boldsymbol{p}}$$

- The general solution is $\widetilde{\mathbf{\Psi}} = \widetilde{\mathbf{\Psi}}^{(\mathsf{macro})} + \widetilde{\mathbf{\Psi}}^{(\mathsf{micro})}$:

$$\widehat{\mathbf{\Xi}}\widetilde{\mathbf{\Psi}}^{(\mathsf{macro})} = 0, \qquad \widetilde{\mathbf{\Psi}}^{(\mathsf{micro})} = \textstyle\sum_s \int \mathrm{d}\boldsymbol{p}\,\widehat{\mathbf{\Xi}}^{-1}\widehat{\boldsymbol{\alpha}}_s g_s$$

- The corresponding Wigner tensors are U and $\mathsf{S}/(2\pi)^{n+1}$:

$$\mathsf{U}(\omega,\boldsymbol{k}) = \int \frac{\mathrm{d}\tau}{2\pi}\frac{\mathrm{d}\boldsymbol{s}}{(2\pi)^n}\,\langle \widetilde{\mathbf{\Psi}}^{(\mathsf{macro})}(t+\tau/2,\boldsymbol{x}+\boldsymbol{s}/2)\,\widetilde{\mathbf{\Psi}}^{(\mathsf{macro})\dagger}(t-\tau/2,\boldsymbol{x}-\boldsymbol{s}/2)\rangle\,\mathrm{e}^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}}$$

$$\mathsf{S}(\omega,\boldsymbol{k}) = 2\pi \sum_{s'} \int \mathrm{d}\boldsymbol{p}'\,\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}'_{s'}) F_{s'}(\boldsymbol{p}')\mathbf{\Xi}^{-1}(\omega,\boldsymbol{k})(\boldsymbol{\alpha}_{s'}\boldsymbol{\alpha}^\dagger_{s'})(\omega,\boldsymbol{k};\boldsymbol{p}')\mathbf{\Xi}^{-\dagger}(\omega,\boldsymbol{k})$$

Fluctuation-dissipation theorem: $\mathsf{S}_{\mathsf{eq}} = -{}^{2T}/_{\omega}\,(\mathbf{\Xi}^{-1})_{\mathsf{A}}$

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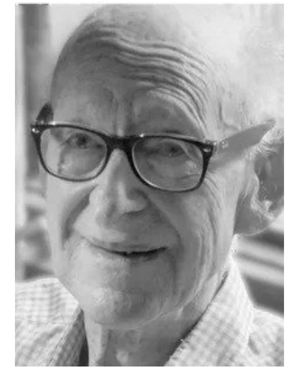
Equation for the OC distribution with self-consistent fields

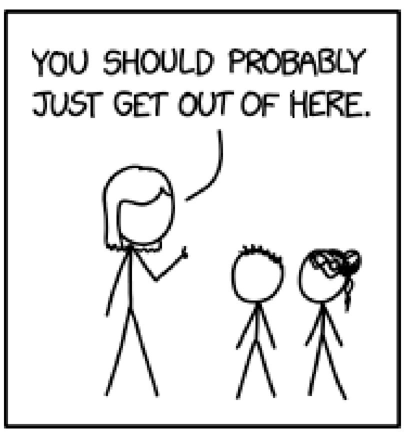
- In a self-consistent field, a collision operator emerges:

$$\frac{\partial F_s}{\partial t} = \{\mathcal{H}_s, F_s\} + \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(\mathsf{D}_s \frac{\partial F_s}{\partial \boldsymbol{p}} \right) + \mathcal{C}_s$$



- $\mathcal{C}_s$ has a Balescu–Lenard form, satisfies ***H*-theorem**, conserves particles and energy–momentum.

$$\mathcal{C}_s = \frac{\partial}{\partial \boldsymbol{p}} \cdot \sum_{s'} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^n} \mathrm{d}\boldsymbol{p}' \, \pi \, \delta(\boldsymbol{k}\cdot\boldsymbol{v}_s - \boldsymbol{k}\cdot\boldsymbol{v}'_{s'}) \, |\boldsymbol{\alpha}_s^\dagger(\boldsymbol{k}\cdot\boldsymbol{v}_s, \boldsymbol{k}; \boldsymbol{p}) \boldsymbol{\Xi}^{-1}(\omega, \boldsymbol{k}) \boldsymbol{\alpha}_{s'}(\boldsymbol{k}\cdot\boldsymbol{v}'_{s'}, \boldsymbol{k}; \boldsymbol{p}')|^2$$
$$\times \, \boldsymbol{k}\boldsymbol{k} \cdot \left(\frac{\partial F_s(\boldsymbol{p})}{\partial \boldsymbol{p}} F_{s'}(\boldsymbol{p}') - F_s(\boldsymbol{p}) \frac{\partial F_{s'}(\boldsymbol{p}')}{\partial \boldsymbol{p}'} \right)$$



- ***K*–χ theorem*** for the ponderomotive energy $\Delta_s \doteq \mathcal{H}_s - H_{0s}$:

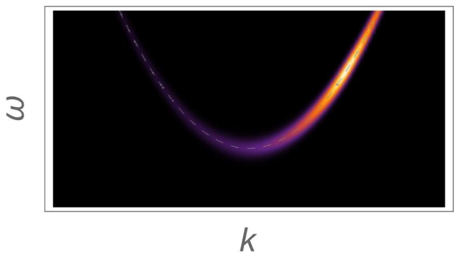
$$\Delta_s = -\frac{1}{2} \frac{\delta}{\delta F_s} \fint \boldsymbol{\Xi}_\mathsf{H} : \mathsf{U} \, \mathrm{d}\omega \, \mathrm{d}\boldsymbol{k}$$

$$\Delta_s = \frac{1}{2} \frac{\partial}{\partial \boldsymbol{p}} \cdot \fint \boldsymbol{k} \frac{(\boldsymbol{\alpha}_s^\dagger \mathsf{U} \boldsymbol{\alpha}_s)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s} \mathrm{d}\omega \, \mathrm{d}\boldsymbol{k} + \frac{1}{2} \int \mathsf{U} : (\boldsymbol{L}_s^\dagger \boldsymbol{R}_s)_\mathsf{H} \, \mathrm{d}\omega \, \mathrm{d}\boldsymbol{k}$$

*e.g. in Kaufman (1987)

Photos: Radu Balescu and Andrew Lenard. Image: xkcd.com

Special case: on-shell waves and the associated conservation laws



- Wave kinetic equation (WKE) for a complexified wave field:*

$$\widehat{\boldsymbol{\Xi}}\,|\widetilde{\boldsymbol{\Psi}}_{\mathsf{c}}\rangle = |\mathbf{0}\rangle \quad\rightarrow\quad \widehat{\boldsymbol{\Xi}}\,|\widetilde{\boldsymbol{\Psi}}_{\mathsf{c}}\rangle\langle\widetilde{\boldsymbol{\Psi}}_{\mathsf{c}}| = \widehat{\mathbf{0}} \quad\rightarrow\quad \mathbf{tr}\int \mathrm{d}\omega\; \boldsymbol{\Xi} \star \mathbf{U}_{\mathsf{c}} = 0$$

- On-shell waves have $\mathbf{U}_{\mathsf{c}} \approx \delta(\Lambda)\, J(t,\boldsymbol{x},\boldsymbol{k})\, \boldsymbol{\eta}\boldsymbol{\eta}^{\dagger}$. In terms of the action density J, the WKE is

$$\boxed{\partial_t J + \boldsymbol{v}_{\mathrm{g}} \cdot \partial_{\boldsymbol{x}} J - \partial_{\boldsymbol{x}}\omega \cdot \partial_{\boldsymbol{k}} J = 2\gamma J}$$

conserves the sign of J

$\int \mathrm{d}\boldsymbol{p}\, \overline{H}_s F_s$	OC energy density
$\int \mathrm{d}\boldsymbol{p}\, \boldsymbol{p} F_s$	OC momentum density
$\int \mathrm{d}\boldsymbol{k}\, \omega J$	wave energy density
$\int \mathrm{d}\boldsymbol{k}\, \boldsymbol{k} J$	wave momentum density

- Combined together, the equations for F_s and J conserve the energy–momentum:

$$\frac{\partial}{\partial t}\left(\sum_s \int \mathrm{d}\boldsymbol{p}\, \overline{H}_s F_s + \int \mathrm{d}\boldsymbol{k}\, \omega J\right) + \frac{\partial}{\partial x^i}\left(\sum_s \int \mathrm{d}\boldsymbol{p}\, (\overline{H}_s + \Delta_s) v_s^i F_s + \int \mathrm{d}\boldsymbol{k}\, \omega v_{\mathrm{g}}^i J\right) = +\sum_s \int \mathrm{d}\boldsymbol{p}\, \frac{\partial \overline{H}_s}{\partial t} F_s$$

$$\frac{\partial}{\partial t}\left(\sum_s \int \mathrm{d}\boldsymbol{p}\, p_l F_s + \int \mathrm{d}\boldsymbol{k}\, k_l J\right) + \frac{\partial}{\partial x^i}\left(\sum_s \int \mathrm{d}\boldsymbol{p}\, (p_l v_s^i + \Delta_s \delta_l^i) F_s + \int \mathrm{d}\boldsymbol{k}\, k_l v_{\mathrm{g}}^i J\right) = -\sum_s \int \mathrm{d}\boldsymbol{p}\, \frac{\partial \overline{H}_s}{\partial x^l} F_s$$

- F_s and J are fundamental objects, the oscillating fields *per se* are not needed.

*cf. McDonald and Kaufman (1985)

So how does one apply all this?

- Need to represent the vacuum-field action and the particle Hamiltonians in the form

$$S_0 = \frac{1}{2}\int \widetilde{\boldsymbol{\Psi}}^\dagger \widehat{\boldsymbol{\Xi}}_0 \widetilde{\boldsymbol{\Psi}}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}, \qquad H_s \approx H_{0s} + \widehat{\boldsymbol{\alpha}}_s^\dagger \widetilde{\boldsymbol{\Psi}} + \frac{1}{2}(\widehat{\boldsymbol{L}}_s \widetilde{\boldsymbol{\Psi}})^\dagger (\widehat{\boldsymbol{R}}_s \widetilde{\boldsymbol{\Psi}})$$

- **Non-relativistic electrostatic interactions:** Dewar's theory, Balescu–Lenard theory, and the formulas for electrostatic fluctuations are subsumed *(see paper)*.

$$S_0 = \int \frac{(\nabla\widetilde{\varphi})^2}{8\pi}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int \widetilde{\varphi}\,\frac{(-\nabla^2)}{4\pi}\,\widetilde{\varphi}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}, \qquad H_s = \frac{p^2}{2m_s} + e_s\overline{\varphi} + e_s\widetilde{\varphi}$$

$$\Xi_0(\omega,\boldsymbol{k}) = {}^{k^2}\!/_{4\pi}, \qquad \alpha_s(\omega,\boldsymbol{k}) = e_s, \qquad L_s(\omega,\boldsymbol{k}) = R_s(\omega,\boldsymbol{k}) = 0$$

- The 'dressing' $F - \overline{f} = \partial_{\boldsymbol{p}} \cdot (\boldsymbol{\Theta}\,\partial_{\boldsymbol{p}}\overline{f})$ carries energy–momentum:

$$\sum_s \int \mathrm{d}\boldsymbol{p}\,H_{0s}F_s + \int \mathrm{d}\boldsymbol{k}\,wJ = \sum_s \int \mathrm{d}\boldsymbol{p}\,H_{0s}\overline{f}_s + \frac{1}{8\pi}\overline{\widetilde{E}^2}$$

$$\sum_s \int \mathrm{d}\boldsymbol{p}\,\boldsymbol{p}F_s + \int \mathrm{d}\boldsymbol{k}\,\boldsymbol{k}J = \sum_s \int \mathrm{d}\boldsymbol{p}\,\boldsymbol{p}\overline{f}_s$$

Relativistic electromagnetic interactions

- Let's adopt $\widetilde{\boldsymbol{E}} = i\widehat{\omega}\widetilde{\boldsymbol{A}}/c$ as the interaction field (Weyl gauge) and $\widetilde{\boldsymbol{B}} = (c\widehat{\boldsymbol{k}}/\widehat{\omega}) \times \widetilde{\boldsymbol{E}}$:

$$S_0 = \int \frac{\widetilde{E}^2 - \widetilde{B}^2}{8\pi}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int \widetilde{\boldsymbol{E}}^\dagger \underbrace{\frac{1}{4\pi}\left[\mathbf{1} + \frac{c^2}{\widehat{\omega}^2}\left(\widehat{\boldsymbol{k}}\widehat{\boldsymbol{k}}^\dagger - \mathbf{1}\widehat{k}^2\right)\right]}_{\widehat{\boldsymbol{\Xi}}_0} \widetilde{\boldsymbol{E}}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x},$$

- Relativistic-particle Hamiltonian can be Taylor-expanded and expressed through $\widetilde{\boldsymbol{E}}$:

$$H_s = \sqrt{m_s^2c^4 + (\boldsymbol{p}c - e_s\overline{\boldsymbol{A}} - e_s\widetilde{\boldsymbol{A}})^2} + e_s\overline{\varphi} + e_s\overline{\varphi}$$

$$= H_{0s} + \underbrace{\frac{ie_s}{\widehat{\omega}}\overline{\boldsymbol{v}}_s^\dagger}_{\widehat{\boldsymbol{\alpha}}_s}\widetilde{\boldsymbol{E}} + \frac{1}{2}\Bigg(\underbrace{\frac{e_s^2}{\widehat{\omega}}\widetilde{\boldsymbol{E}}}_{\widehat{L}_s\widetilde{\boldsymbol{E}}}\Bigg)^\dagger\Bigg(\underbrace{\frac{\mathbf{1} - \boldsymbol{v}_s\boldsymbol{v}_s^\dagger/c^2}{m_s\gamma_s}\frac{1}{\widehat{\omega}}\widetilde{\boldsymbol{E}}}_{\widehat{R}_s\widetilde{\boldsymbol{E}}}\Bigg)$$

- Energy–momentum conservation, with $f_s^{(\mathrm{kin})}(\boldsymbol{p}) \doteq f_s(\boldsymbol{p} + e_s\widetilde{\boldsymbol{A}}/c)$:

$$\sum_s \int \mathrm{d}\boldsymbol{p}\, H_{0s}F_s + \int \mathrm{d}\boldsymbol{k}\, wJ = \sum_s \int \mathrm{d}\boldsymbol{p}\, H_{0s}\,\overline{f_s^{(\mathrm{kin})}} + \frac{1}{8\pi}\overline{(\widetilde{\boldsymbol{E}}^\dagger\widetilde{\boldsymbol{E}} + \widetilde{\boldsymbol{B}}^\dagger\widetilde{\boldsymbol{B}})}$$

$$\sum_s \int \mathrm{d}\boldsymbol{p}\, \boldsymbol{p}F_s + \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{k}J = \sum_s \int \mathrm{d}\boldsymbol{p}\, \boldsymbol{p}\,\overline{f_s^{(\mathrm{kin})}} + \frac{\overline{\widetilde{\boldsymbol{E}} \times \widetilde{\boldsymbol{B}}}}{4\pi c}$$

Relativistic electromagnetic interactions

- Relativistic nonlinear potentials (U is the average Wigner tensor of $\widetilde{\boldsymbol{E}}$):

$$\mathsf{D}_s = \pi e_s^2 \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{k}\boldsymbol{k}\, \frac{\boldsymbol{v}_s^\dagger \mathsf{U}(\boldsymbol{k}\cdot\boldsymbol{v}_s, \boldsymbol{k})\boldsymbol{v}_s}{(\boldsymbol{k}\cdot\boldsymbol{v}_s)^2}$$

$$\boldsymbol{\Theta}_s = e_s^2 \frac{\partial}{\partial\vartheta} \fint \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}\boldsymbol{k}}{\omega^2} \frac{(\boldsymbol{v}_s^\dagger \mathsf{U} \boldsymbol{v}_s)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s + \vartheta}\bigg|_{\vartheta=0}$$

$$\Delta_s = \frac{e_s^2}{2}\frac{\partial}{\partial \boldsymbol{p}} \cdot \fint \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}}{\omega^2}\frac{(\boldsymbol{v}_s^\dagger \mathsf{U} \boldsymbol{v}_s)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s} + \frac{e_s^2}{2}\int \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}\, \frac{\mathrm{tr}(\mathsf{U}\boldsymbol{\mu}_s^{-1})}{\omega^2}$$

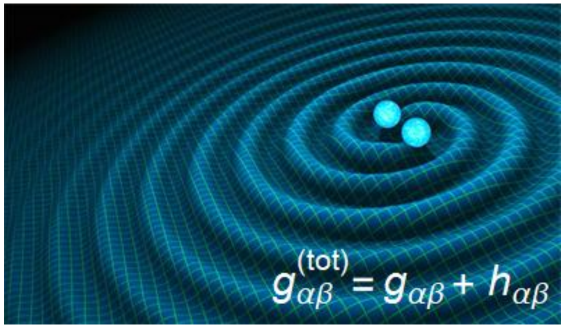
- Fluctuation spectrum and collision operator* ($\boldsymbol{\epsilon}$ is the dielectric tensor):

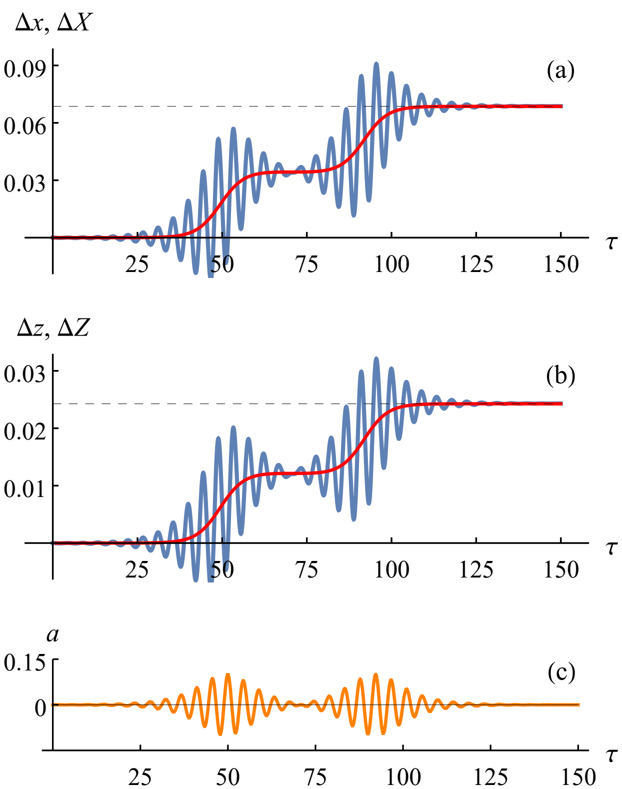
$$\mathsf{S}(\omega, \boldsymbol{k}) = 2\pi \sum_s \left(\frac{4\pi e_s}{\omega}\right)^2 \int \mathrm{d}\boldsymbol{p}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s) F_s(\boldsymbol{p}) \boldsymbol{\epsilon}^{-1}(\omega, \boldsymbol{k})\, \boldsymbol{v}_s \boldsymbol{v}_s^\dagger\, \boldsymbol{\epsilon}^{-\dagger}(\omega, \boldsymbol{k})$$

$$\mathcal{C}_s = \frac{\partial}{\partial \boldsymbol{p}} \cdot \sum_{s'} 2 e_s^2 e_{s'}^2 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \mathrm{d}\boldsymbol{p}'\, \frac{|\boldsymbol{v}_s^\dagger \boldsymbol{\epsilon}^{-1}(\boldsymbol{k}\cdot\boldsymbol{v}_s, \boldsymbol{k})\boldsymbol{v}'_{s'}|^2}{(\boldsymbol{k}\cdot\boldsymbol{v}_s)^4}\, \delta(\boldsymbol{k}\cdot\boldsymbol{v}_s - \boldsymbol{k}\cdot\boldsymbol{v}'_{s'})$$
$$\times\, \boldsymbol{k}\boldsymbol{k} \cdot \left(\frac{\partial F_s(\boldsymbol{p})}{\partial \boldsymbol{p}} F_{s'}(\boldsymbol{p}') - F_s(\boldsymbol{p}) \frac{\partial F_{s'}(\boldsymbol{p}')}{\partial \boldsymbol{p}'}\right)$$

*cf. Hizanidis *et al.* (1983); Silin (1961)

Quasilinear theory for gravitational waves





- The same formalism can be readily applied to particle interactions with gravitational waves.

$$\mathsf{H}(\mathsf{x}, \mathsf{p}) = m^2 + g^{\alpha\beta}(\mathsf{x}) p_\alpha p_\beta, \quad g_{\alpha\beta} = \overline{g}_{\alpha\beta} + \widetilde{g}_{\alpha\beta}$$

- The QL coefficients are explicitly found in terms of $\mathfrak{E} \doteq p_\alpha p_\beta p_\gamma p_\delta \mathsf{U}^{\alpha\beta\gamma\delta}$. QL diffusion is gauge-invariant.

$$\mathsf{D} = \frac{\pi}{4P^0} \int \mathrm{d}\mathsf{k}\, \boldsymbol{k}\boldsymbol{k} \mathfrak{E}\, \delta(k^\rho p_\rho)$$

$$\boldsymbol{\Theta} = \frac{1}{4P^0} \frac{\partial}{\partial \vartheta} \fint \mathrm{d}\mathsf{k}\, \frac{\boldsymbol{k}\boldsymbol{k}\mathfrak{E}}{\vartheta P^0 - k^\rho p_\rho} \bigg|_{\vartheta = 0}$$

$$\Delta = \frac{p_\alpha p_\beta}{2P^0} \int \mathrm{d}\mathsf{k}\, \mathsf{U}^{\alpha}{}_{\gamma}{}^{\gamma\beta} - \frac{1}{8P^0} \frac{\partial}{\partial p_\lambda} \fint \mathrm{d}\mathsf{k}\, \frac{k_\lambda \mathfrak{E}}{k^\rho p_\rho}$$

- Vacuum GWs:* effective 'ponderomotive' metric

$$\mathcal{H}' = m^2 + g^{\alpha\beta}_{\mathrm{eff}} p_\alpha p_\beta, \qquad g^{\alpha\beta}_{\mathrm{eff}} \doteq \overline{g}^{\alpha\beta} + \int \mathrm{d}\mathsf{k}\, \mathsf{U}^{\alpha}{}_{\gamma}{}^{\gamma\beta}.$$

Garg and Dodin (2020)

*Lorenz gauge assumed, $\nabla_\alpha \widetilde{g}^{\alpha\beta} = 0$

Local quasilinear theory for magnetized plasma

- The *canonical* $W_{\widetilde{H}}$ cannot be expanded in L^{-1} because $\overline{\boldsymbol{A}}$, unlike $\overline{\boldsymbol{B}}$, depends on $\boldsymbol{x}$ strongly. In *non-canonical* variables, the derivation is too cumbersome.
- **Option 1**: find *global* angle–action coordinates $(\boldsymbol{\phi}, \mathbf{J})$, Fourier-expand in $\boldsymbol{\phi}$, treat each $\widetilde{f}_n$ as a separate $\widetilde{f}$. Since $\widetilde{f}_n$ are $\boldsymbol{\phi}$-independent, there is no problem left.

$$\widetilde{f} = \sum_{\boldsymbol{n}} \widetilde{f}_n(\mathbf{J}) \mathrm{e}^{i\boldsymbol{n}\cdot\boldsymbol{\phi}}$$

- **Option 2**: find *local* canonical coordinates in which the theory works.
  - Homogeneous field: $(Q_3, P_3) = (\theta, \mu)$, $(Q_2, P_2) = (z, p_z)$, $(Q_1, P_1) = (x, y)$.
  - Inhomogeneous field: similar coordinates with $\dot{P}_1 \ll \dot{Q}_1 = v_{\text{drift}}$ and $\dot{P}_2 \ll \dot{Q}_2$.*
  - Fourier-expand in θ but retain weak dependence on the local Q_1 and Q_2:

$$\widetilde{f} = \sum_{l} \widetilde{f}_l(Q_1, Q_2, P_1, P_2; \mu) \mathrm{e}^{il\theta}$$

$$\widetilde{H} = \frac{e}{\omega} \mathrm{re} \sum_{l} \left(\frac{il\Omega}{k_\perp} J_l(k_\perp\rho) \frac{\boldsymbol{k}_\perp}{k_\perp} - \Omega\rho J_l'(k_\perp\rho) \frac{\boldsymbol{b}\times\boldsymbol{k}_\perp}{k_\perp} + \frac{iP_z}{m} J_l(k_\perp\rho)\boldsymbol{b} \right) \cdot \tilde{\boldsymbol{E}}\, \mathrm{e}^{il\theta}$$

  - Reproduced the dielectric tensor and Kennel & Engelmann's (1966) QL diffusion, generalized the known ponderomotive forces (Grebogi et al., 1979). . . *

*in preparation

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Summary

- Quasilinear theory is corrected and derived from first principles as a *local theory*.
  - general Hamiltonian, any interaction field;
  - inhomogeneity, collisions and off-shell waves;
  - H-theorem for inhomogeneous plasma;
  - generalized conservative Balescu–Lenard collision operator;
  - conservation of the action, energy, and momentum for on-shell fields;
  - many known results are subsumed as special cases.
- **Take-home message #1:** $\mathcal{O}(\bar{\partial}_t, \bar{\partial}_x)$ is non-negligible on $t \gg \omega^{-1}$ and $\ell \gg k^{-1}$. Weyl calculus is *the* way to calculate these corrections.

$$\widetilde{f}_k = -\frac{i(e/m)\widetilde{E}_k}{\omega_k - kv}\frac{\partial \overline{f}}{\partial v} + \mathcal{O}(\partial_t \overline{f}, \partial_x \overline{f}), \qquad F - \overline{f} = \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(\boldsymbol{\Theta} \frac{\partial \overline{f}}{\partial \boldsymbol{p}} \right) = \mathcal{O}(\widetilde{E}^2)$$

- **Take-home message #2:** When deriving a reduced theory, transform the distribution, not the coordinates.

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