Metaplectic Geometrical Optics

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Fusionopolis, A*STAR

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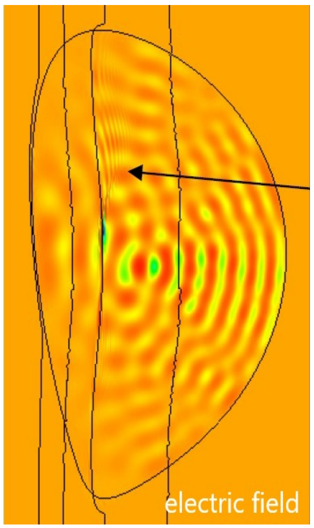
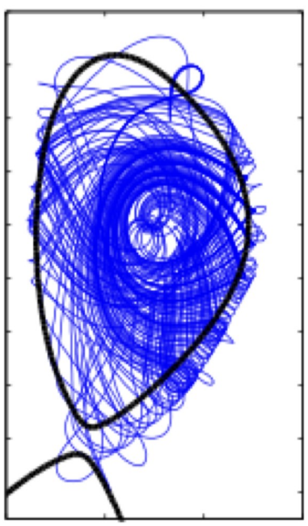
Introduction

- Modeling RF waves is fusion is important, amounts to solving linear Maxwell's eqs:





$$\nabla \times \nabla \times \boldsymbol{E} - (\omega^2/c^2)(\hat{\boldsymbol{\epsilon}}\boldsymbol{E}) = \text{source}$$

- **'Full-wave'** approach: solve as a boundary-value problem on a grid, $A_{ab}\psi_b = S_a$. Expensive, mostly used for the IC range.
- **'Ray tracing'** (beam tracing, quasioptics...) for EC & LH waves:
  - For $\boldsymbol{E} = \mathrm{e}^{i\theta}\boldsymbol{\Psi}$ with large $\boldsymbol{k} \doteq \nabla\theta$, or small $\lambda \doteq 2\pi/|\boldsymbol{k}|$:

$$\underbrace{\boldsymbol{k} \times \boldsymbol{k} \times \boldsymbol{E} + (\omega^2/c^2)\,\boldsymbol{\epsilon}_{\mathrm{H}}(\omega, \boldsymbol{k}; \boldsymbol{x})\boldsymbol{E} = 0}_{\boldsymbol{D}\boldsymbol{E} = 0, \quad D \doteq \det \boldsymbol{D}}$$

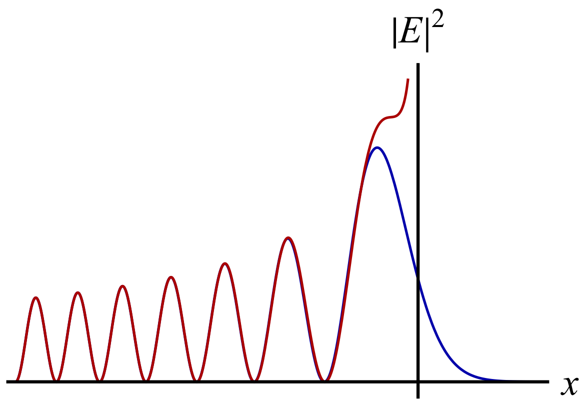
$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = -\frac{\partial D/\partial k_a}{\partial D/\partial \omega}, \qquad \frac{\mathrm{d}k_a}{\mathrm{d}t} = \frac{\partial D/\partial x^a}{\partial D/\partial \omega}$$

  - Initial-value problem: calculate the amplitude on the rays,

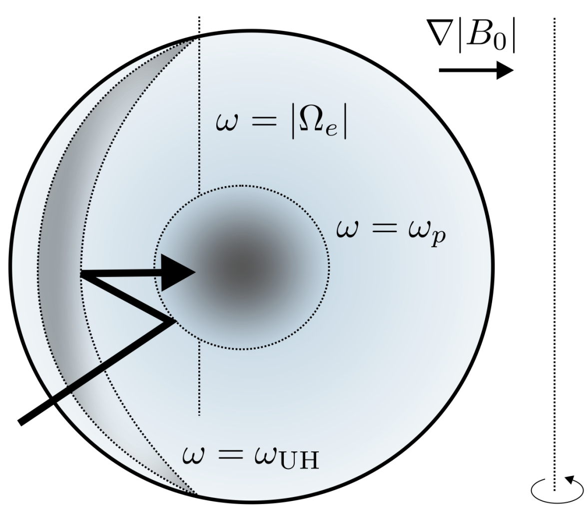
$$\dot{W} = -(\nabla \cdot \boldsymbol{v}_{\mathrm{g}} + 2\gamma)W, \qquad P_{\mathrm{abs}} = 2\gamma W$$

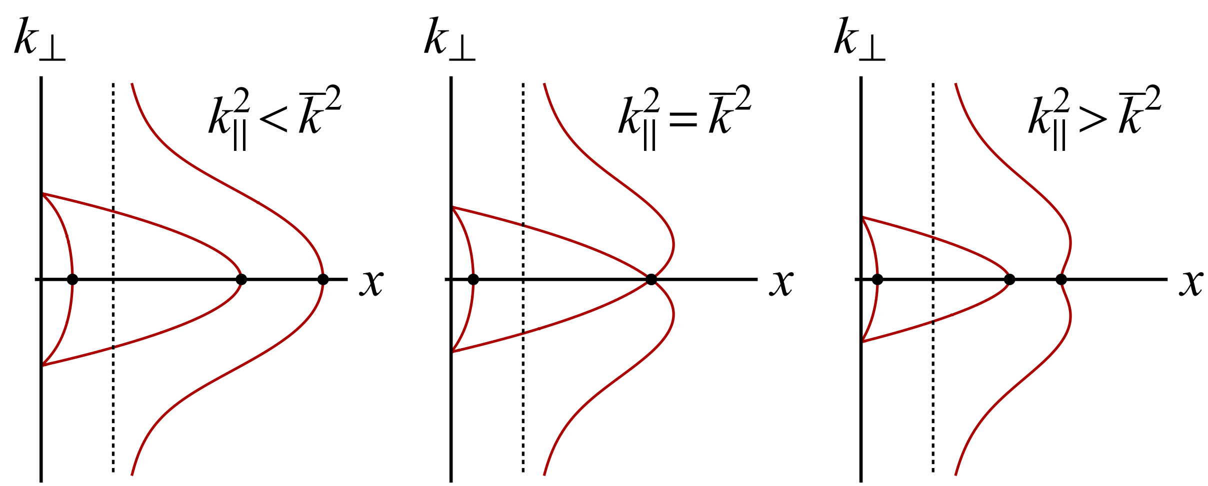
Images: Bonoli *et al.* (2007); Poli et al. (2017)

A more general formulation of geometrical optics is long overdue.



- Geometrical-optics ordering breaks down near reflection points, where $\lambda/L \to \infty$. Applications:
  - Dense plasmas could use EBW heating → need O–X conversion near $\omega_p = \omega$.
  - Parametric instabilities with trapped modes.





Need a different formulation of GO, with a different small parameter.

(How are we even ok with modeling reflection with tracing without it?!)

Preinhaelter and Kopecky (1973); Hansen *et al.* (1985); Hansen *et al.* (2019); Gusakov and Popov (2020); Clod *et al.* (2024)

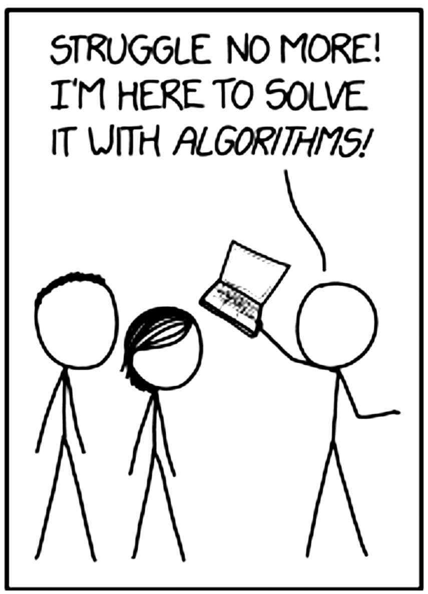
Let's introduce some machinery...

- Approximating $\hat{\boldsymbol{D}}\boldsymbol{E} = 0$ means approximating $\hat{\boldsymbol{D}}$.
- Any operator $\hat{\boldsymbol{D}}\boldsymbol{E}(\boldsymbol{x}) = \int \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{x}')\, \boldsymbol{E}(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}'$ on space $\boldsymbol{x}$ can be expressed through its *Weyl symbol* using $\hat{\boldsymbol{x}} = \boldsymbol{x}$, $\hat{\boldsymbol{k}} = -i\nabla$:

$$\boxed{\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{k}) = \int \boldsymbol{d}(\boldsymbol{x} + {}^{\boldsymbol{s}}\!/2, \boldsymbol{x} - {}^{\boldsymbol{s}}\!/2)\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{s}}\, \mathrm{d}\boldsymbol{s}}$$

$$\hat{\boldsymbol{D}} = \frac{1}{(2\pi)^{2n}} \int \boldsymbol{D}(\boldsymbol{x}', \boldsymbol{k}')\, \mathrm{e}^{i\boldsymbol{k}''\cdot(\boldsymbol{x}' - \hat{\boldsymbol{x}}) - i\boldsymbol{x}''\cdot(\boldsymbol{k}' - \hat{\boldsymbol{k}})}\, \mathrm{d}\boldsymbol{x}'\, \mathrm{d}\boldsymbol{k}'\, \mathrm{d}\boldsymbol{x}''\, \mathrm{d}\boldsymbol{k}''$$

$$\hat{1} \Leftrightarrow 1$$
$$\hat{\mathsf{x}} \Leftrightarrow \mathsf{x}$$
$$\hat{\mathsf{k}} \Leftrightarrow \mathsf{k}$$
$$\hat{\mathsf{A}}^\dagger \Leftrightarrow \mathsf{A}^\dagger$$
$$\hat{\mathsf{A}}\hat{\mathsf{B}} \Leftrightarrow \mathsf{A} \star \mathsf{B}$$

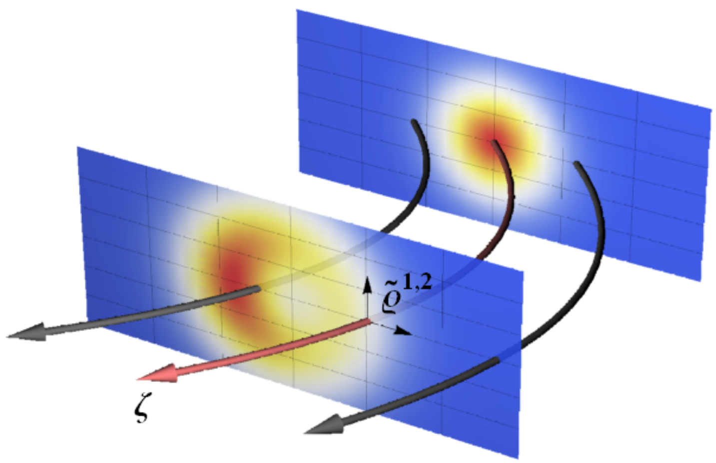


- **Example 1:** The dielectric tensor $\boldsymbol{\epsilon}(t, \boldsymbol{x}, \omega, \boldsymbol{k})$ is actually the Weyl symbol of $\hat{\boldsymbol{\epsilon}}$, at least up to $\mathcal{O}(1/\omega\tau, 1/kL)$.
- **Example 2:** Spectrum of the 2-point correlation function of $\boldsymbol{E}$ is the symbol of $|E_a\rangle\langle E_b|$, a.k.a. Wigner matrix:

$$\overline{W}_{ab}(t, \boldsymbol{x}, \omega, \boldsymbol{k}) = (2\pi)^{-4} \int \mathrm{d}\tau\, \mathrm{d}\boldsymbol{s}\, \mathrm{e}^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}} \times \langle E_a(t + {}^{\tau}\!/2, \boldsymbol{x} + {}^{\boldsymbol{s}}\!/2)\, E_b^*(t - {}^{\tau}\!/2, \boldsymbol{x} - {}^{\boldsymbol{s}}\!/2)\rangle$$

Tracy *et al.* (2014)

Traditional geometrical optics in terms of Weyl symbols



- Consider an eikonal fields $\boldsymbol{E} = \mathrm{e}^{i\theta(\boldsymbol{x})}\boldsymbol{\Psi}(\boldsymbol{x})$ with θ treated as a prescribed field. Invariant form:

$$\hat{\boldsymbol{D}}\,|\boldsymbol{E}\rangle = 0, \quad |\boldsymbol{E}\rangle = \mathrm{e}^{i\theta(\hat{\boldsymbol{x}})}\,|\boldsymbol{\Psi}\rangle, \quad \overline{\boldsymbol{k}} \doteq \boldsymbol{\nabla}\theta(\boldsymbol{x})$$

- Then the envelope $|\boldsymbol{\Psi}\rangle$ is governed by

$$\hat{\boldsymbol{\mathcal{D}}}\,|\boldsymbol{\Psi}\rangle = 0, \qquad \hat{\boldsymbol{\mathcal{D}}} \doteq \mathrm{e}^{-i\theta(\hat{\boldsymbol{x}})}\,\hat{\boldsymbol{D}}\,\mathrm{e}^{i\theta(\hat{\boldsymbol{x}})}$$

- The symbol of the *envelope operator* $\hat{\boldsymbol{\mathcal{D}}}$ is approximately the shifted symbol of $\hat{\boldsymbol{D}}$:

$$\boldsymbol{\mathcal{D}}(\boldsymbol{x},\boldsymbol{k}) \approx \boldsymbol{D}(\boldsymbol{x},\boldsymbol{k}+\overline{\boldsymbol{k}}(\boldsymbol{x})) = \boldsymbol{D}(\boldsymbol{x},\overline{\boldsymbol{k}}(\boldsymbol{x})) + k_\mu \boldsymbol{V}^\mu(\boldsymbol{x}) + {}^1\!/_2\, k_\mu k_\nu \boldsymbol{\Theta}^{\mu\nu}(\boldsymbol{x}) + \ldots$$

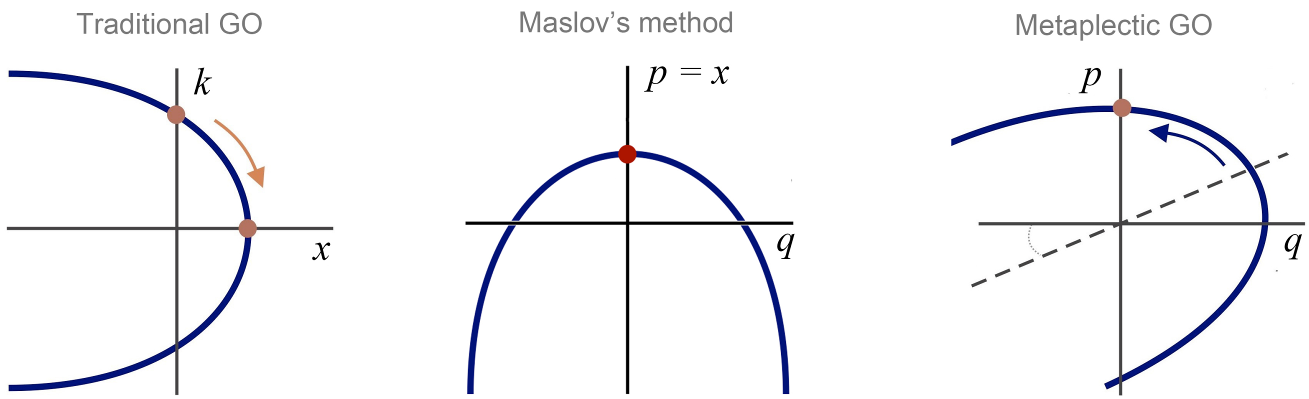
$$\boldsymbol{V}^\mu(\boldsymbol{x}) \doteq (\partial \boldsymbol{D}/\partial k_\mu)(\boldsymbol{x},\bar{\boldsymbol{k}}(\boldsymbol{x})), \qquad \boldsymbol{\Theta}^{\mu\nu}(\boldsymbol{x}) \doteq (\partial^2 \boldsymbol{D}/\partial k_\mu \partial k_\nu)(\boldsymbol{x},\bar{\boldsymbol{k}}(\boldsymbol{x}))$$

- Weyl expansion: approximate symbol → approximate operator → $\boldsymbol{x}$ representation

$$0 = \hat{\boldsymbol{\mathcal{D}}}\boldsymbol{\Psi} = \underbrace{\boldsymbol{D}(\boldsymbol{x},\bar{\boldsymbol{k}})\,\boldsymbol{\Psi}}_{\text{dispersion}}\ \underbrace{-i\big(\boldsymbol{V}^\mu\partial_\mu + {}^1\!/_2\,\partial_\mu \boldsymbol{V}^\mu\big)\boldsymbol{\Psi}}_{\text{GO propagation}}\ \underbrace{-{}^1\!/_2\,\partial_\mu\big(\boldsymbol{\Theta}^{\mu\nu}\partial_\nu\boldsymbol{\Psi}\big)}_{\text{diffraction}} + \ldots$$

McDonald (1988); Dodin *et al.* (2019); used in PARADE: Yanagihara *et al.* (2019)a; Yanagihara *et al.* (2019)b...

The basic idea of metaplectic geometrical optics (MGO)



- One can adopt various representations for the kets and the equation $\hat{H}\,|\psi\rangle = 0$:
  - Spatial: use the eigenbasis of the position operator, $\hat{x}\,|\mathfrak{e}_{\hat{x}}(x)\rangle = x\,|\mathfrak{e}_{\hat{x}}(x)\rangle$:

$$\psi_{\hat{x}}(x) = \langle \mathfrak{e}_{\hat{x}}(x)|\psi\rangle\,, \qquad 0 = \langle \mathfrak{e}_{\hat{x}}(x)|\hat{H}|\psi\rangle \equiv (\hat{H}\psi_{\hat{x}})(x)$$

  - One can also use the eigenbasis of a different operator, $\hat{q}\,|\mathfrak{e}_{\hat{q}}(q)\rangle = q\,|\mathfrak{e}_{\hat{q}}(q)\rangle$:

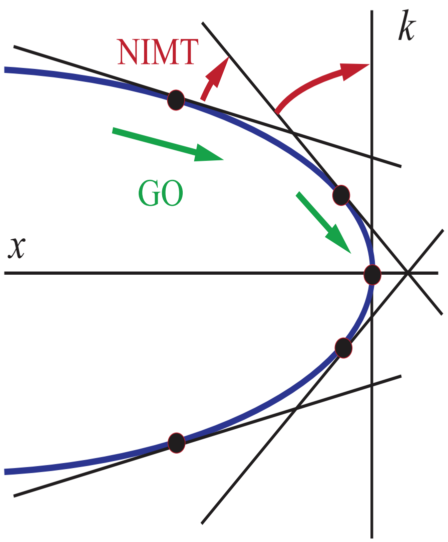
$$\psi_{\hat{q}}(q) = \langle \mathfrak{e}_{\hat{q}}(q)|\psi\rangle\,, \qquad 0 = \langle \mathfrak{e}_{\hat{q}}(q)|\hat{H}|\psi\rangle = (\hat{H}\psi_{\hat{q}})(q)$$

  - Instead of $[\hat{x}, \hat{k}] = i$, the momentum operator is then defined via $[\hat{q}, \hat{p}] = i$.

- The linear transform that connects $\psi_{\hat{q}}$ and $\psi_{\hat{x}}$ is called a metaplectic transform:

$$\psi_{\hat{q}}(q) = \int \langle \mathfrak{e}_{\hat{q}}(q)|\mathfrak{e}_{\hat{x}}(x)\rangle\, \psi_{\hat{x}}(x)\, \mathrm{d}x \equiv (\hat{M}\psi_{\hat{x}})(q)$$

The two approaches to MGO



- Earlier, we did GO on tangent surfaces and used near-identity metaplectic transforms (NIMT) to connect those surfaces:*

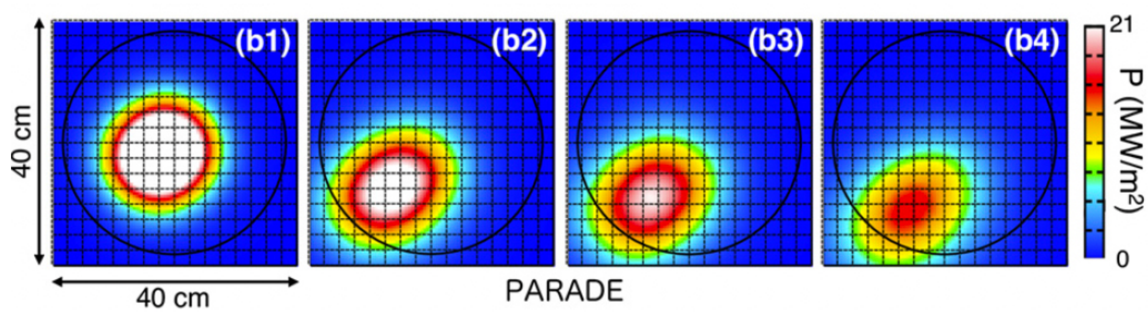
$$\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{k} \end{pmatrix}$$

- Good: linear transformations conserve Weyl symbols.

$$A'(q,p) = A(x(q,p), k(q,p))$$

- Bad: NIMT are hard to compute accurately, and also this formulation does not yield a self-contained PDE.

- Now, we want to develop MGO via *continuous nonlinear transformations*. Goal: symplectically invariant version of PARADE† (diffraction, mode conversion).

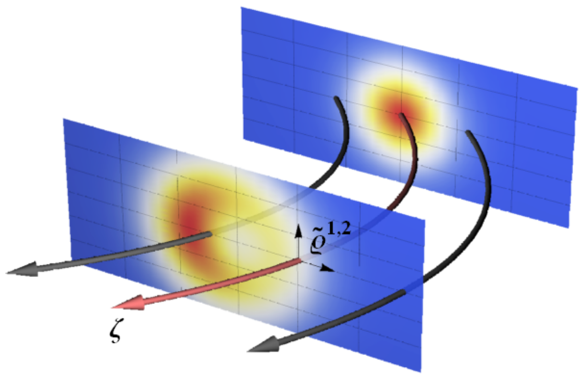


PARADE

*Ph.D. thesis by Lopez (2022); also Littlejohn (1985)

† Yanagihara *et al.* (2019)b; Yanagihara *et al.* (2021) . . .

Deriving the field equations in the new representation is easy, but...



- Narrow beam: one reference ray + diffraction. Having no crossing rays means having no caustic issues.

$$\hat{\boldsymbol{D}}\,|\boldsymbol{E}\rangle = 0, \quad |\boldsymbol{E}\rangle = \mathrm{e}^{i\theta(\hat{\boldsymbol{q}})}\,|\boldsymbol{\Psi}\rangle, \quad \overline{\boldsymbol{p}} \doteq \boldsymbol{\nabla}_{\boldsymbol{q}}\theta$$

- Derive the equation for $\boldsymbol{\Psi}_{\hat{\boldsymbol{q}}}$ as usual:

$$\boldsymbol{D}_{\mathrm{H}}\boldsymbol{\Psi}_{\hat{\boldsymbol{q}}} - i\big(\boldsymbol{V}^{\mu}\partial_{\mu} + {}^{1}\!/_{2}\,\partial_{\mu}\boldsymbol{V}^{\mu} - \boldsymbol{D}_{\mathrm{A}}\big)\boldsymbol{\Psi}_{\hat{\boldsymbol{q}}} - {}^{1}\!/_{2}\,\boldsymbol{\nabla}_{\perp,\sigma}\big(\boldsymbol{\Theta}^{\sigma\sigma'}\boldsymbol{\nabla}_{\perp,\sigma'}\boldsymbol{\Psi}_{\hat{\boldsymbol{q}}}\big) = 0$$

- The actually challenging questions are as follows:

  - What is the small parameter that replaces λ/L?
  - What is the symbol of $\hat{\boldsymbol{D}}$ in the new representation? (The zeroth-order approximation is not enough!)
  - How does one map $\boldsymbol{\Psi}_{\hat{\boldsymbol{q}}}$ to the physical space?

- Need to develop a systematic theory of MT for nonlinear variable transformations.

Image credit: istockphoto.com

Metaplectic transform for nonlinear canonical transformations

- For nonlinear transformations, the existing theory focuses on exactly solvable problems – not useful. Need to develop an asymptotic theory from scratch.
- MT is given by a unitary integral operator that connects two representations:

$$\psi_{\hat{q}}(q) = \int \mathrm{d}x\, M(q,x)\psi_{\hat{x}}(x), \qquad M(q,x) = \bar{M}^*(x,q) = \langle \mathfrak{e}_{\hat{q}}(q) | \mathfrak{e}_{\hat{x}}(x) \rangle$$

- The equations for the kernel of MT, an 'M-wave', are Schrödinger equations:

$\hat{x} = X(\hat{q},\hat{p}), \quad \hat{k} = K(\hat{q},\hat{p})$	$\hat{q} = Q(\hat{x},\hat{k}), \quad \hat{p} = P(\hat{x},\hat{k})$
$\Downarrow$	$\Downarrow$
$X(q,-i\partial_q)M(q,x) = xM(q,x)$	$Q(x,-i\partial_x)\bar{M}(x,q) = q\bar{M}(x,q)$
$K(q,-i\partial_q)M(q,x) = i\partial_x M(q,x)$	$P(x,-i\partial_x)\bar{M}(x,q) = i\partial_q \bar{M}(x,q)$

- Symbols are mapped from $(x,k) \equiv \mathsf{z}$ to $(q,p) \equiv \mathsf{y}$ by the Wigner function of M:

$$A_{\mathsf{y}}(\mathsf{y}) = \int \mathrm{d}\mathsf{z}\, \mu(\mathsf{y},\mathsf{z})\, A_{\mathsf{z}}(\mathsf{z})$$

Littlejohn (1986); Mello and Moshinsky (1975)

Examples of metaplectic transforms

- Phase-space shift: $\hat{q} = \hat{x} + \Delta_q$, $\hat{p} = \hat{k} + \Delta_p$

$$M(q,p) = \mathrm{e}^{i\Delta_p q}\,\delta(q - x - \Delta_q), \qquad \psi_{\hat{q}}(q) = \mathrm{e}^{i\Delta_p q}\psi_{\hat{x}}(q - \Delta_q)$$

- Symplectic rescaling: $\hat{q} = \hat{x}/\alpha$, $\hat{p} = \alpha\hat{k}$, with $\alpha = \text{const}$

$$M(q,x) = |\alpha|^{-1/2}\,\delta(q - x/\alpha), \qquad \psi_{\hat{q}}(q) = |\alpha|^{1/2}\psi_{\hat{x}}(\alpha q)$$

- Linear symplectic transformation: note that $\hat{\boldsymbol{M}}_{\circlearrowleft} = -\mathbf{1}$

$$\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{k} \end{pmatrix}, \qquad M(q,x) = \frac{\mathrm{e}^{\frac{i(Dq^2 - 2xq + Ax^2)}{2B}}}{\sqrt{-2\pi i \boldsymbol{B}}}$$

- Eikonal transform: $\hat{q} = \hat{x}$, $\hat{p} = \hat{k} - \theta'(\hat{x})$

$$M(q,p) = \mathrm{e}^{-i\theta(q)}\,\delta(q - x), \qquad \psi_{\hat{q}}(q) = \mathrm{e}^{-i\theta(x)}\psi_{\hat{x}}(x)$$

Standard GO, $\psi(x) = \Psi(x)\mathrm{e}^{i\theta(x)}$, is just the eikonal metaplectic transform!

Geometrical optics of M-waves

- As mentioned earlier, the equations for M-waves are Schrödinger equations:

$$K(q,-i\partial_q)M(q,x) = i\partial_x M(q,x), \qquad P(x,-i\partial_x)\bar{M}(x,q) = i\partial_q \bar{M}(x,q)$$

- Let us assume an eikonal form, $\bar{M}(x,q) = \mathrm{e}^{iF(x,q)}\mathfrak{M}(x,q)$. One finds that F is the **type-1 generating function** of the canonical transformation $(x,k) \mapsto (q,p)$:

$$k = \partial_x F(x,q), \qquad p = -\partial_q F(x,q)$$

- Functions $K(q,p)$ and $P(x,k)$ serve as ray Hamiltonians, which are conserved, and K_p and P_k serve as group velocities. For example,

$$0 = \mathrm{d}_t K = \partial_x k(x,q) + K_p \partial_q k(x,q) \quad \Rightarrow \quad K_p = -\partial^2_{xx}F/\partial^2_{xq}F.$$

- GO amplitude equation $\partial_x \mathfrak{M}^2 + \partial_q(K_p \mathfrak{M}^2) = 0$ yields $\mathfrak{M}^2 = |\partial^2_{xq}F| \times$ const, so

$$\boxed{\bar{M} \approx \mathrm{e}^{iF}\sqrt{\partial^2_{xq}F/2\pi}}$$

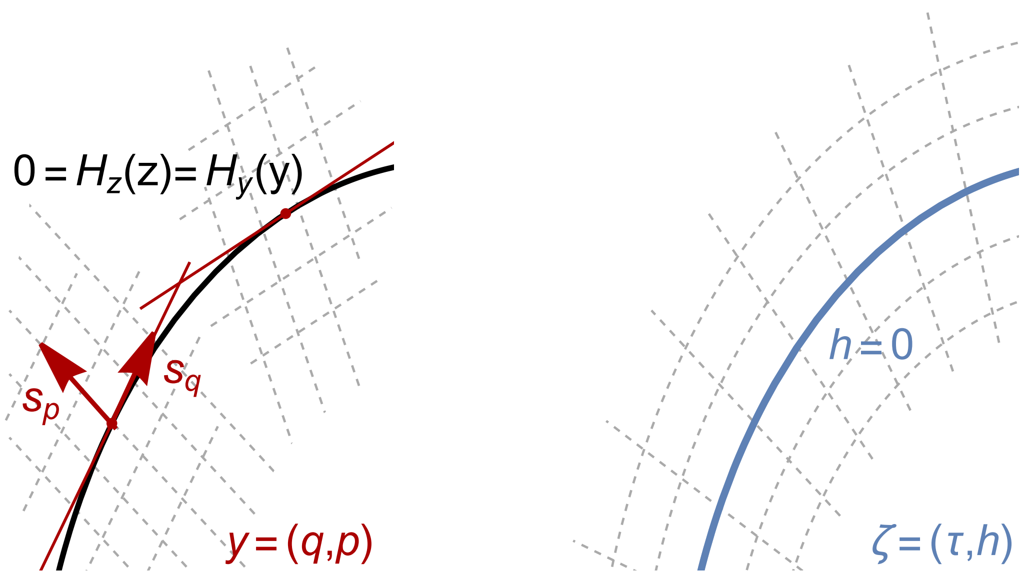
i.e. there *are* ways to approximate M.* But how do we find the new coordinates?

*One can also do better than the zeroth-order approximation, see below.

Miller (1974)

So how do we find the desired coordinates?

- For a wave governed by $\hat{H}\,|\psi\rangle = 0$, the ray trajectory is $H_{\mathsf{z}}(\mathsf{z}) = 0$. The new momentum should be $\hat{h} = \hat{H}$, and the position operator $\hat{\tau}$ should satisfy $[\hat{\tau}, \hat{h}] = i$.
- We need them in an *explicit and simple enough form*, so we could solve for M. We can give up the exact equality $\hat{h} = \hat{H}$ to keep the equation for M manageable.
- Let's try $\mathsf{z} \mapsto \mathsf{y} \mapsto \zeta$, where $\mathsf{z} \mapsto \mathsf{y}$ is linear and $\mathsf{y} \mapsto \zeta$ is an asymptotic near-identity.



$$\mathsf{s}_q = \mathsf{v}/v, \quad \mathsf{s}_p = \mathsf{u}/u$$

$$1 = \mathsf{s}_q \wedge \mathsf{s}_p \equiv \mathsf{s}_q^{\mathsf{T}} \mathsf{J} \mathsf{s}_p$$

$$\mathsf{J} \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Tricky part: phase space has no metric, so there are no angles, perpendiculars, etc. $\Rightarrow$ can't define s_q and s_p as orthonormal. Symplecticity requires only $vu = \mathsf{v} \wedge \mathsf{u}$.

Tingjing Xing's project. See also Kamran *et al.* (2009)

Linear shift, rotation, and rescaling

- Near any given z_0 on a ray, where $H_\mathsf{z}(\mathsf{z}_0) = 0$, the symbol of $\hat{H}$ is

$$H_\mathsf{z}(\mathsf{z}) \approx -(\mathsf{z}-\mathsf{z}_0)\cdot \mathsf{J}\dot{\mathsf{z}}_0 + {}^1\!/\!_2\,(\mathsf{z}-\mathsf{z}_0)\cdot \mathsf{g}(\mathsf{z}-\mathsf{z}_0), \qquad \mathsf{g} \doteq (\partial^2_\mathsf{z} H)_0$$

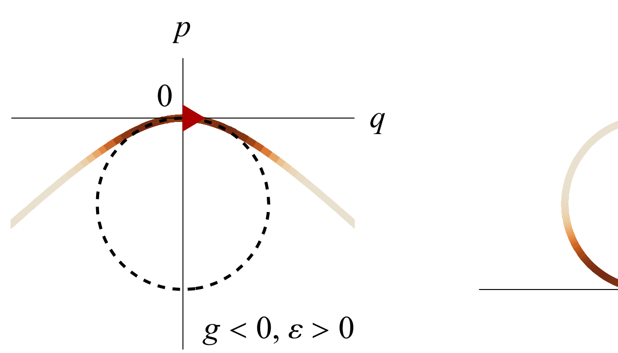
- Assuming the basis $\mathsf{s}_q = \dot{\mathsf{z}}_0/v$ and $\mathsf{s}_p = -v\ddot{\mathsf{z}}_0/\varepsilon$ such that

$$\varepsilon = (\ddot{\mathsf{z}} \wedge \dot{\mathsf{z}})_0, \qquad v = (\varepsilon^2/|g|)^{1/4}, \qquad g \doteq \det \mathsf{g},$$

  the ray is a harmonic oscillator with orbit radius $R = v/\Omega$ and frequency Ω:

$$H_\mathsf{y}(\mathsf{y}) = \frac{\Omega}{2}\left(q^2 + (p+R)^2 - R^2\right), \qquad \Omega = \sqrt{|g|}\,\operatorname{sgn}\varepsilon$$

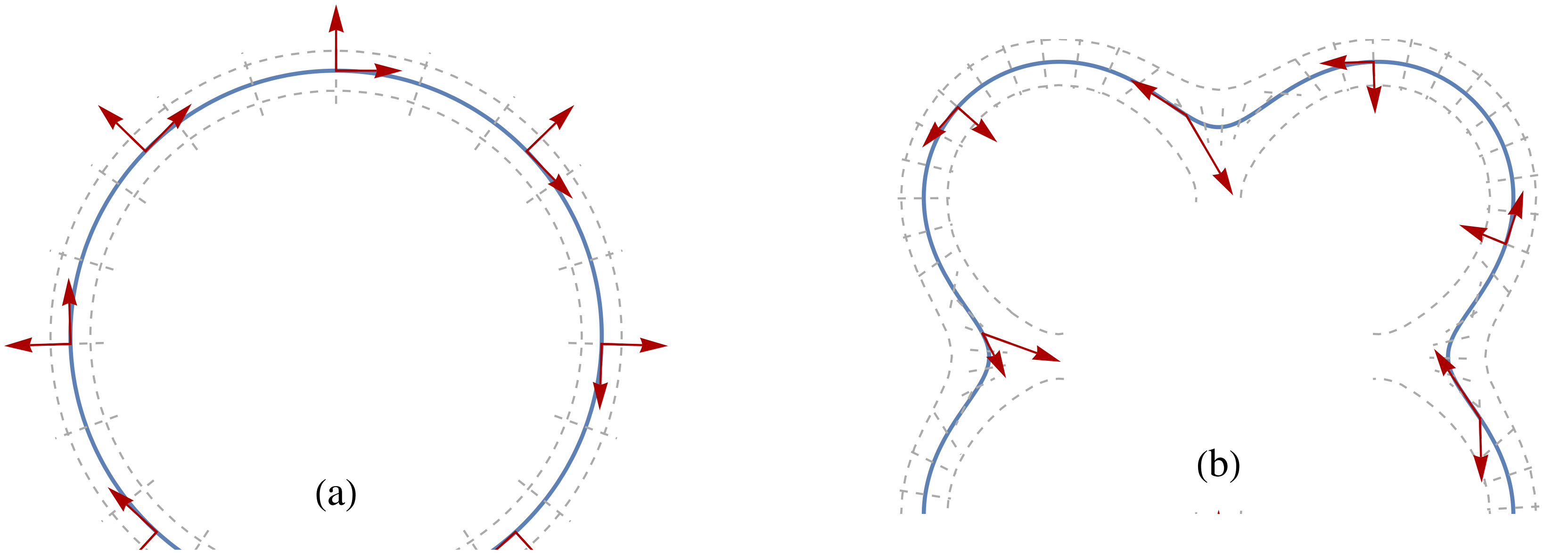
- Then, $\psi_{\hat{q}}$ locally satisfies the equation of a *quantum* harmonic oscillator (QHO).



This is basically a fancified Williamson diagonalization, see Nicacio (2021)

Result #1: The small parameter of MGO

- QHO is well described by GO when its number of quanta, $n = R^2/2$, is large. Thus, the natural small parameter is the symplectic curvature $\boldsymbol{\kappa} \doteq \boldsymbol{R}^{-1}$.

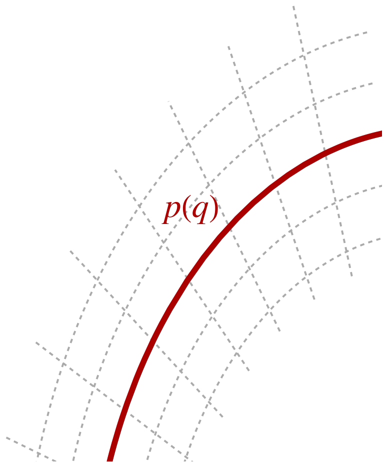


- Tricky part: the coordinate transformation is singular and $\kappa \propto |g|^{3/4}/|\varepsilon|^{1/2}$ is formally infinite near inflection points, where $\varepsilon \to 0$:

$$\varepsilon \equiv (\ddot{\mathsf{z}} \wedge \dot{\mathsf{z}})_0 = -\frac{\mathrm{d}^2 x}{\mathrm{d}k^2}\left(\frac{\partial H_{\mathsf{z}}}{\partial x}\right)^3 = -\frac{\mathrm{d}^2 k}{\mathrm{d}x^2}\left(\frac{\partial H_{\mathsf{z}}}{\partial k}\right)^3$$

- More generally, the symplectic scale is as $R = \sqrt{\Delta x\, \Delta k}$, so $\boldsymbol{\kappa} = (\boldsymbol{\Delta x}\, \boldsymbol{\Delta k})^{-1/2}$.

Result #2: symbol transformation



- Expand the wave Hamiltonian in p near the ray trajectory $p(q)$ as $H(q,p) = V(q)(p - p(q))$. Then,

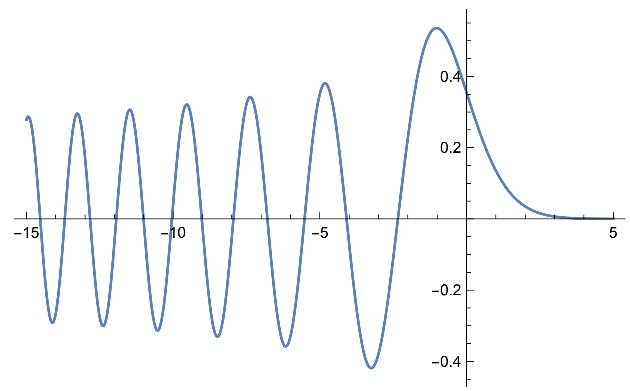
$$\hat{\tau} = \tau(\hat{q}), \qquad \hat{h} = 1/2\,(\hat{V}\hat{p} + \hat{p}\hat{V}) - \hat{V}p(\hat{q}), \qquad [\hat{\tau}, \hat{h}] = i$$

- One can find M explicitly and, with some algebra, show that

$$\mu \approx \mathrm{Ai}_\varepsilon(H_\mathsf{z}(\mathsf{z}) - h)\,\delta(\tau(\mathsf{z}) - \tau)$$

$$\mathrm{Ai}_\varepsilon(X) = \frac{2}{|\varepsilon|^{1/3}}\,\mathrm{Ai}\left(\frac{2X}{|\varepsilon|^{1/3}}\,\mathrm{sgn}\,\varepsilon\right)$$

$$\varepsilon = (\mathsf{J}\partial_\mathsf{z} H_\mathsf{z}) \cdot (\partial_\mathsf{z}^2 H_\mathsf{z})(\mathsf{J}\partial_\mathsf{z} H_\mathsf{z})$$



- The symbols in the two representations are connected by the Airy transform.* For smooth symbols, such as D, **there are no corrections of order $(\boldsymbol{\Delta x\,\Delta k})^{-1}$**.

$$A_\mathsf{z}(\mathsf{z}) \approx \int \mathrm{Ai}_\varepsilon(\tilde{h} - H_\mathsf{z}(\mathsf{z}))\,A_\zeta(\tau(\mathsf{z}), \tilde{h})\,\mathrm{d}\tilde{h}, \qquad D_\mathsf{z}(\mathsf{z}) \approx D_\zeta(\zeta(\mathsf{z}))$$

* Widder (1979)

Result #3: Wigner function $W = \text{symb}(|\psi\rangle\langle\psi|)$

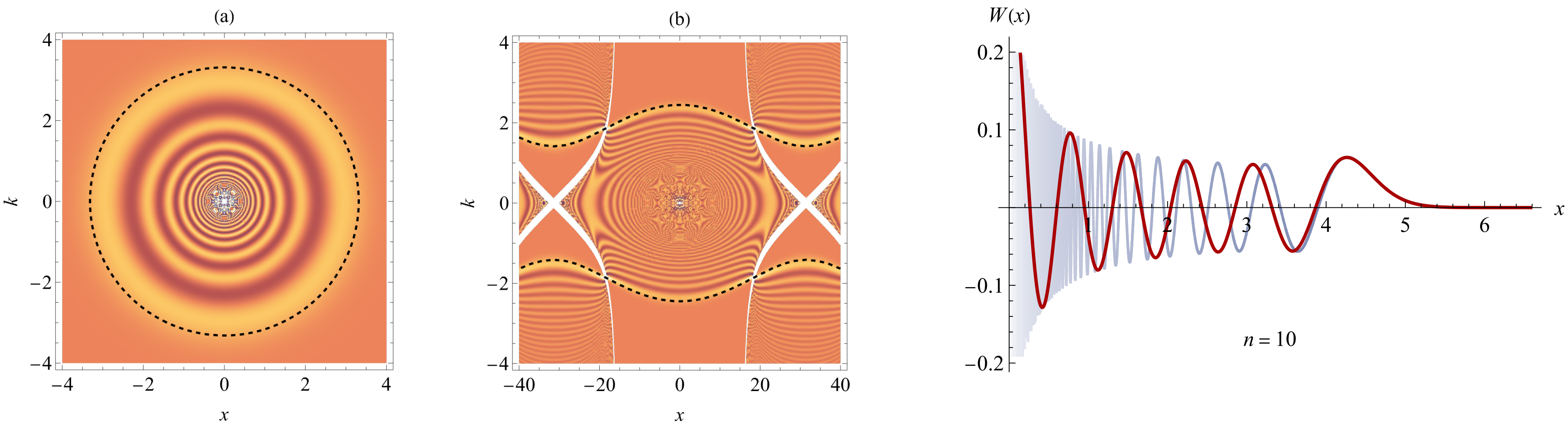
- Everything interesting can be expressed through Wigner functions, so one might not even need a field. As symbols, Wigner functions are easy to remap!
- In the ray-aligned coordinates, waves are stationary (w/o dissipation, diffraction)

$$\hat{H}\,|\psi\rangle = 0 \quad \Rightarrow \quad W_\zeta(\tau, h) = W_0 \delta(h)$$

- In (x, k), MGO waved always have Airy-type Wigner functions with scale ε:

$$W_{\mathsf{z}}(x, k) = \int \mathrm{d}\tau\, \mathrm{d}h\, W_\zeta(\tau, h)\, \mu(x, k, \tau, h) = W_0\, \mathrm{Ai}_\varepsilon(H(x, k)).$$

- These complicated profiles are caused by unfortunate coordinates, not by physics.



$\varepsilon = (\mathsf{J}\partial_{\mathsf{z}} H_{\mathsf{z}}) \cdot (\partial^2_{\mathsf{z}} H_{\mathsf{z}})(\mathsf{J}\partial_{\mathsf{z}} H_{\mathsf{z}})$

$|\boldsymbol{\psi}_{\hat{\boldsymbol{x}}}(\boldsymbol{x})|^2 = \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{W}_{\mathrm{z}}(\boldsymbol{x},\boldsymbol{k}) = \boldsymbol{W}_0 \int \mathrm{d}\boldsymbol{k}\ \mathbf{Ai}_{\boldsymbol{\varepsilon}}(\boldsymbol{H}(\mathrm{z}))$

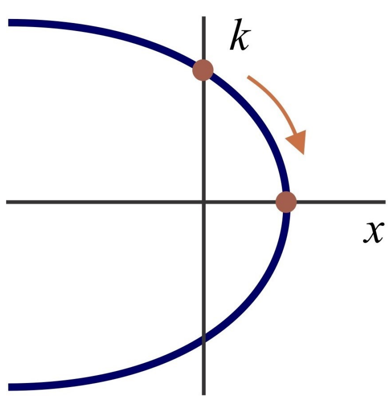
- Using $\mathrm{Ai}_\varepsilon(H(\mathrm{z})) \to \delta(H(\mathrm{z}))$ at $\varepsilon \to 0$, one readily obtains the known WKB result:

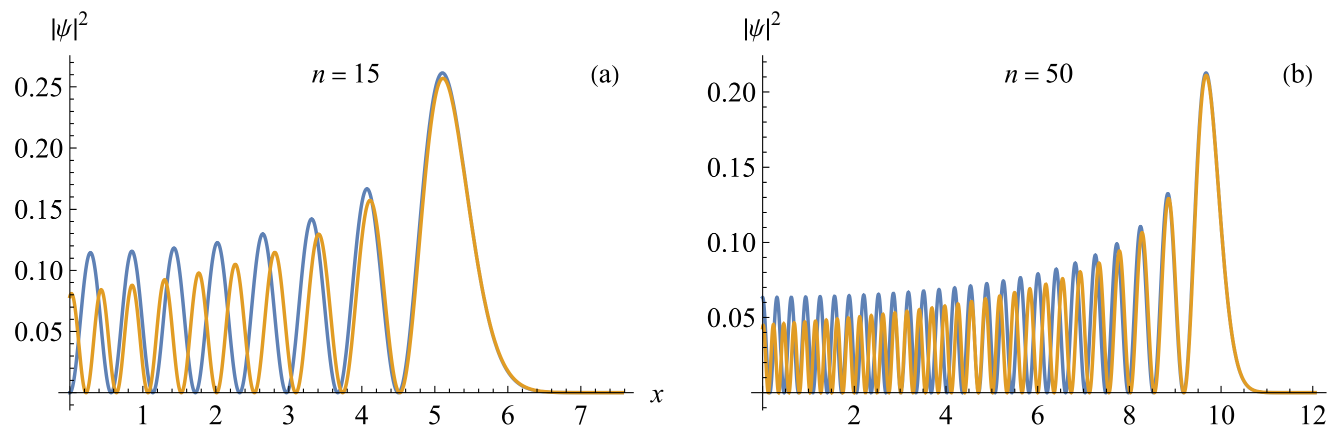
$$|\psi_{\hat{x}}(x)|^2 \approx W_0 \int \mathrm{d}k\, \delta(H_{\mathrm{z}}(x,k)) = \frac{W_0}{|v_{\mathrm{g}}|}, \qquad v_{\mathrm{g}}|\psi_{\hat{x}}(x)|^2 = \text{const}$$

- A more precise result that does not rely on the delta approximation for $\mathrm{Ai}_\varepsilon(H(\mathrm{z}))$:

$$|\psi_{\hat{x}}(x)|^2 = W_0 \frac{\pi v}{2^{1/3}|B|}\, \mathrm{Ai}_\varepsilon^2\left(\frac{v}{2^{2/3}B}\left(x + \frac{D^2}{2B}R\right)\right)$$

- The well-known Airy solution near reflection points is subsumed as a special case.





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So let's return to our 'big questions'...

- What is the new small parameter?
  - $\epsilon = (\Delta x \, \Delta k)^{-1} \ll 1$. The absolute value of k does not matter.
  - Typically, ϵ coincides with the squared symplectic curvature.
- What is the symbol of $\hat{\boldsymbol{D}}$ in the new representation?
  - New symbol = inverse Airy transform of the original one. Corollary: *smooth symbols are preserved* up to errors $\mathcal{O}(\epsilon^2)$, which are negligible in MGO.
  - The Maslov phase is in the metaplectic transform, not in the envelope equation.
- How does one map the field from the ray-aligned coordinates to the x-space?
  - No need to do it during the simulation. Do it only to output the results.
  - If the phase is not needed, calculate the Wigner function, $W_{\mathsf{z}} \approx W_0 \, \mathrm{Ai}_{\varepsilon}(H(\mathsf{z}))$.
  - If one *must* know the phase, there are various asymptotic formulas for the metaplectic-transform kernel. Errors do not propagate, this is just local output.

MGO simulations of O–X conversion (work in progress)

- Assume the usual dispersion function with the cold-plasma dielectric tensor $\boldsymbol{\epsilon}$:

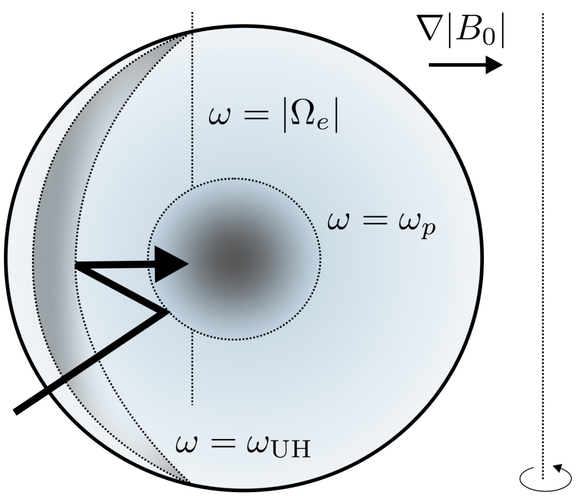
$$D_{ab}(\boldsymbol{x},\boldsymbol{k}) = k_a k_b - \delta_{ab}k^2 + (c^2/\omega^2)\epsilon_{ab}(\boldsymbol{x}), \qquad 0 = \det \boldsymbol{D} = D_{\parallel} D_{\mathrm{O}} D_{\mathrm{X}}$$

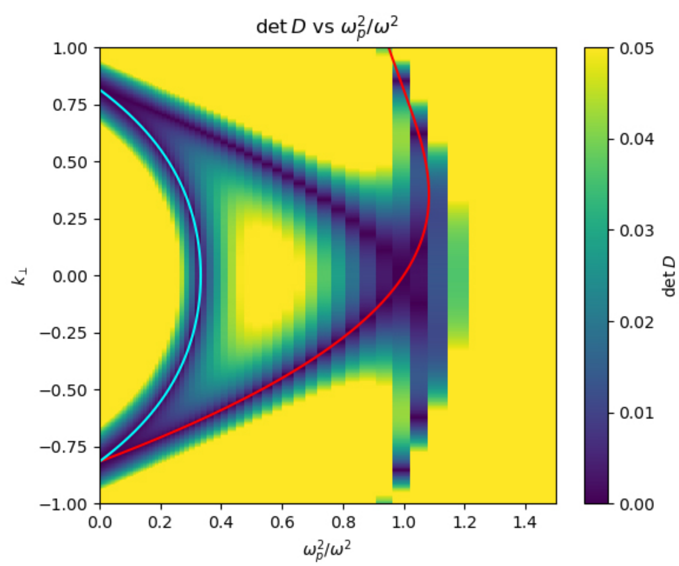
- The eigenvalue D_i corresponds to the ith mode. Propagate a reference ray using

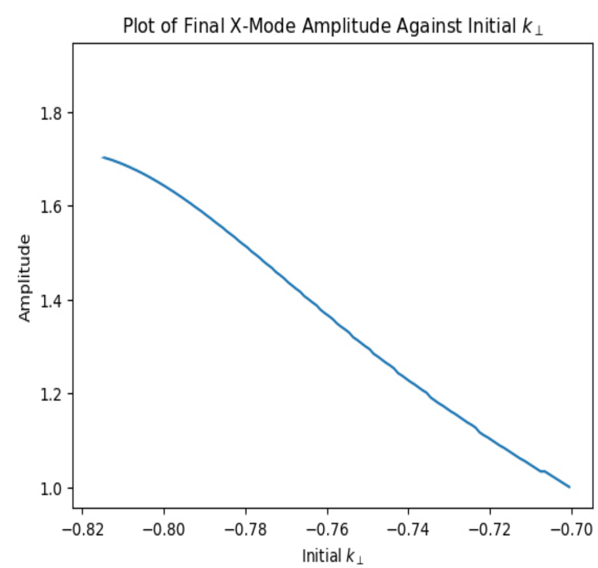
$$H = D_{\mathrm{O}} f + D_{\mathrm{X}}\sqrt{1-f^2}, \qquad f(t\to-\infty)\to 1, \qquad f(t\to+\infty)\to 0$$

- In the mode-conversion region, solve the amplitude equation on the reference ray:

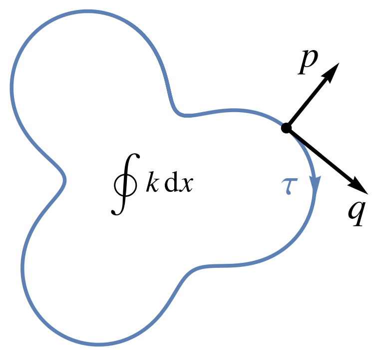
$$\boldsymbol{D}(\boldsymbol{x},\bar{\boldsymbol{k}}(\boldsymbol{x}))\boldsymbol{\Psi}_{\hat{\boldsymbol{q}}} - i\left(\boldsymbol{V}^{\mu}\partial_{\mu} + 1/2\,\partial_{\mu}\boldsymbol{V}^{\mu}\right)\boldsymbol{\Psi}_{\hat{\boldsymbol{q}}} \; - 1/2\,\boldsymbol{\nabla}_{\perp,\sigma}\left(\boldsymbol{\Theta}^{\sigma\sigma'}\boldsymbol{\nabla}_{\perp,\sigma'}\boldsymbol{\Psi}_{\hat{\boldsymbol{q}}}\right) = 0$$







*Standing waves in x-space are propagating waves in τ-space.



$$\psi_{\hat{q}}(q) = \int M_{\hat{q}\leftarrow\hat{\tau}}(q,\tau)\,\psi_{\hat{\tau}}(\tau)\,\mathrm{d}\tau$$

- In the GO limit, one can use

$$M_{\hat{q}\leftarrow\hat{\tau}} = \mathrm{e}^{i\Theta}\sqrt{\Theta_{q\tau}/2\pi}, \quad \mathrm{d}\Theta = p\,\mathrm{d}q - h\,\mathrm{d}\tau$$

- On one hand, $M_{\hat{q}\leftarrow\hat{\tau}}(0,\tau) \approx M_0\delta(\tau)$, $M_0 \doteq [\tau'(0)]^{-1/2}$. Then, $\psi_{\hat{q}}(0) \approx M_0\psi_{\hat{\tau}}(0)$.
- On the other hand, one can also use $M_{\hat{q}\leftarrow\hat{\tau}}$ after a whole rotation:*

$$M_{\hat{q}\leftarrow\hat{\tau}}(0,\tau) \approx -M_0\delta(\tau)\exp(i\Theta_{\circlearrowright}), \qquad \psi_{\hat{q}}(0) \approx -M_0\psi_{\hat{\tau}}(T)\exp(i\Theta_{\circlearrowright})$$

- Since $\psi_{\hat{\tau}}(T) = \psi_{\hat{\tau}}(0)$ and

$$h = 0 \quad \Rightarrow \quad \Theta_{\circlearrowright} = \oint p\,\mathrm{d}q = \oint \mathrm{d}\mathsf{z}\wedge\mathsf{z} = \oint k\,\mathrm{d}x,$$

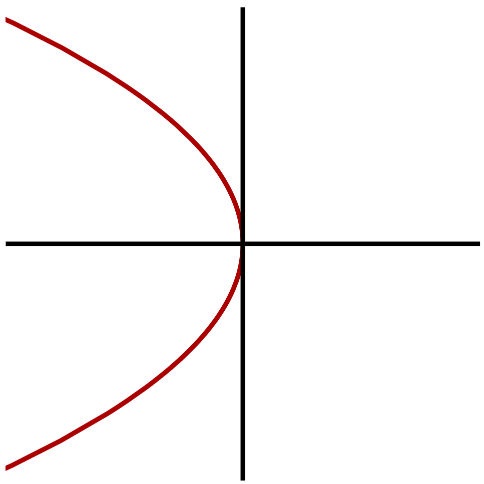
one arrives at the Einstein–Brillouin–Keller quantization:

$$\oint k\,\mathrm{d}x = 2\pi(n + 1/2)$$

*The minus is due to the Maslov phase, which emerges from $\sqrt{\Theta_{q\tau}}$.

*Metaplectic resonances in wave–particle interactions

- The standard condition for resonant interaction with particles is $\boldsymbol{\omega = kv}$, i.e. the phase velocity must equal particle's group velocity*. Not symplectically invariant!



- MGO: the phase of $\psi_{\hat{\tau}}(t,\tau) = \Psi(t,\tau)\mathrm{e}^{i\theta(t,\tau)}$ must satisfy $\partial_\tau \theta = \omega$. Within the OHO model,

$$\partial\theta/\partial\phi = \omega/\Omega, \quad \Omega \doteq \sqrt{|\det(\partial_z^2 H_\mathrm{p})|}$$

i.e. *the Cherenkov and Fermi mechanisms are the same.*

- E.g., for bounded orbits w/harmonic bouncing: $\omega = m\Omega$

- Dissipation power per phase-space volume V, via the Wigner matrix of $\tilde{\boldsymbol{E}}$:

$$\mathrm{d}\mathfrak{E}/\mathrm{dV} = \mathrm{tr}(\boldsymbol{\sigma}_\mathrm{H}\langle \boldsymbol{W}_E\rangle), \qquad \mathrm{dV} \equiv \mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}$$

- In GO, this reduces to the standard formula for the dissipation power density:

$$\mathfrak{P} \doteq \int \frac{\mathrm{d}\mathfrak{E}}{\mathrm{dV}}\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k} = \frac{\bar{\omega}}{8\pi}\,\tilde{\boldsymbol{E}}^\dagger \boldsymbol{\varepsilon}_\mathrm{A}(t,\boldsymbol{x},\bar{\omega},\bar{\boldsymbol{k}})\tilde{\boldsymbol{E}}$$

See more in Ruiz and Dodin (2017); Dodin and Fisch (2014)

Summary

- Developed basic theory of MGO in continuous curved phase-space coordinates:
  - The GO parameter $k\Delta x$ does not matter, what matters is $\boldsymbol{\Delta x\, \Delta k \equiv R^2}$.
  - Theory of *asymptotic* MTs is developed for $\boldsymbol{\epsilon \doteq R^{-2} \ll 1}$ and applied to MGO.
  - A simple alternative is proposed to calculating MTs numerically: locally, all one needs is the Wigner function, which is a symplectic invariant by definition.

$$W_{\mathsf{z}}(\mathsf{z}) \approx W_0\, \mathrm{Ai}_\varepsilon(H(\mathsf{z}))$$

  - The Airy-type solutions for fields in x are extended beyond reflection regions.
  - The Cherenkov condition is generalized to a symplectically invariant form. Basically, in MGO, the Cherenkov and Fermi mechanisms are the same.

$$\partial\theta/\partial\phi = \omega/\Omega, \qquad \Omega \doteq \sqrt{|\det(\partial_{\mathsf{z}}^2 H_{\mathrm{p}})|}$$

  - Symplectically invariant formula for dissipation power per phase-space volume:

$$\mathrm{d}\mathfrak{E}/\mathrm{d}\Gamma = \mathrm{tr}(\boldsymbol{\sigma}_{\mathrm{H}}\langle \boldsymbol{W_E}\rangle)$$

- Next: include transverse diffraction (copy from PARADE) and mode conversion.

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