On a Challenge of Charles Parsons

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Abstract It is suggested how to achieve simultaneously three desiderata proposed by Parsons for the development of arithmetic in set theory.

Charles Parsons (1987), in a survey of the history of attempts to develop of the natural number system in set theory without any axiom of infinity, raises the challenge to develop arithmetic within set theory in such a way that

(A) the set-theoretic axioms used are economical, and in particular do not include infinity;

(B) the construction is general and does not depend on any particular definition of zero 0 and successor S;

(C) the definition of natural number rivals the simplicity of the usual approach depending on an axiom of infinity.

As to (A), what is called *general set theory* (GST) in Boolos (1998, see index) is known to suffice if one does not care about the other desiderata. Its axioms are extensionality, adjunction, and separation, where adjunction says that for any X and y the set $X \cup \{y\}$ exists. Note that separation implies the existence of the null set Ø.

As to (B), the two features needed for zero and successor are these:

(1) $0 \neq \mathrm{S}x$ for any x, and

(2) $Sx \neq Sy$ for any $x \neq y$.

The two definitions most commonly used to obtain this result both let $0 = \emptyset$. Von Neumann's lets $Sx = x \cup \{x\}$. Zermelo's lets $Sx = \{x\}$. Others are possible, but what is wanted is an approach working equally for all.

As to (C), the usual approach, with roots in Frege and Dedekind, defines a set to be inductive if it contains 0 and is closed under S, and assumes as the axiom of infinity that there exists an inductive set. This by separation implies that there is a smallest inductive set, whose elements are those x that belong to every inductive set, and the natural numbers then are defined to be the elements thereof.

As a proposed response to the Parsons challenge — which it would be hard to believe has not been long known as a folk theorem, though no published treatment was found when the author looked for one in response to an inquiry of Oliver Marshall — let us work in GST, claiming thereby to achieve (A), while assuming for 0 and S only (1) and (2), as per (B). The reader will have to judge how far the following definitions achieve (C).

Given a set X call a subset $Y \subseteq X$ is *inductive* in X iff both (a) if $0 \in X$, then $0 \in Y$; and (b) if $y \in Y$ and $Sy \in X$, then $Sy \in Y$. (Intuitively, this implies that Y contains the sequence 0, S0, SS0, … , or at least as long an initial segment thereof as is contained in X.) Define X to be *progressive* iff the only $Y \subseteq X$ that is inductive in X is the whole of X. Note that this implies that if $X \neq \emptyset$, then $0 \in X$. (Intuitively, progressiveness implies that X is an initial segment of the sequence 0, S0, SS0, … , if not the whole.) Finally, define x to be a *natural number* if it is an element of some progressive set.

What remain to be proved are the three remaining Peano postulates beyond (1) and (2), namely the following, of which the last is mathematical induction:

(5) 0 is a natural number;

(6) if x is a natural number, then Sx is a natural number;

(7) any condition that holds for 0 and holds for Sy whenever it holds for y holds for all natural numbers.

As for the proofs, only hints will be given in this note. For (6) it is enough to show that if X is progressive and $x \in X$, then $X \cup \{Sx\}$ is progressive. In (7), by a condition is meant one expressible by a formula Φ of the language of set theory, possibly with parameters; it is enough to show that if X is progressive, then $\{y \in X: \Phi(y)\}$, which exists by separation, is inductive.

Separation is needed *only* for (7): the rest goes through with just extensionality, adjunction, and null-set, which by themselves suffice, according to Tarski-Mostowski-Robinson (1953), who attribute the result to Szmielew and Tarski, to interpret Robinson arithmetic Q, a weak but still essentially undecidable fragment of first-order Peano arithmetic.

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