



Lecture 09: Multi-period Model *Fixed Income, Futures, Swaps*

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Overview

1. Bond basics
2. Duration
3. Term structure of the real interest rate
4. Forwards and futures
 1. Forwards versus futures prices
 2. Currency futures
 3. Commodity futures: backwardation and contango
5. Repos
6. Swaps



Bond basics

- Example: U.S. Treasury (Table 7.1)
 - Bills (<1 year), no coupons, sell at discount
 - Notes (1-10 years), Bonds (10-30 years), coupons, sell at par
 - STRIPS: claim to a single coupon or principal, zero-coupon
- Notation:
 - $r_t(t_1, t_2)$: Interest rate from time t_1 to t_2 prevailing at time t .
 - $P_{t_0}(t_1, t_2)$: Price of a bond quoted at $t = t_0$ to be purchased at $t = t_1$ maturing at $t = t_2$
 - Yield to maturity: Percentage increase in \$s earned from the bond



Bond basics (cont.)

- Zero-coupon bonds make a single payment at maturity

TABLE 7.1 Five ways to present equivalent information about default-free interest rates. All rates but those in the the last column are effective annual rates.

	(1)	(2)	(3)	(4)	(5)
	Zero-Coupon	Zero-Coupon	One-Year Implied	Par	Continuously
Maturity	Bond Yield	Bond Price	Forward Rate	Coupon	Compounded
					Zero Yield
1	6.00%	0.943396	6.00000%	6.00000%	5.82689%
2	6.50	0.881659	7.00236	6.48423	6.29748
3	7.00	0.816298	8.00705	6.95485	6.76586

- One year zero-coupon bond: $P(0,1)=0.943396$
 - Pay \$0.943396 today to receive \$1 at $t=1$
 - Yield to maturity (YTM) = $1/0.943396 - 1 = 0.06 = 6\% = r(0,1)$
- Two year zero-coupon bond: $P(0,2)=0.881659$
 - YTM = $1/0.881659 - 1 = 0.134225 = (1+r(0,2))^2 - 1 \Rightarrow r(0,2) = 0.065 = 6.5\%$



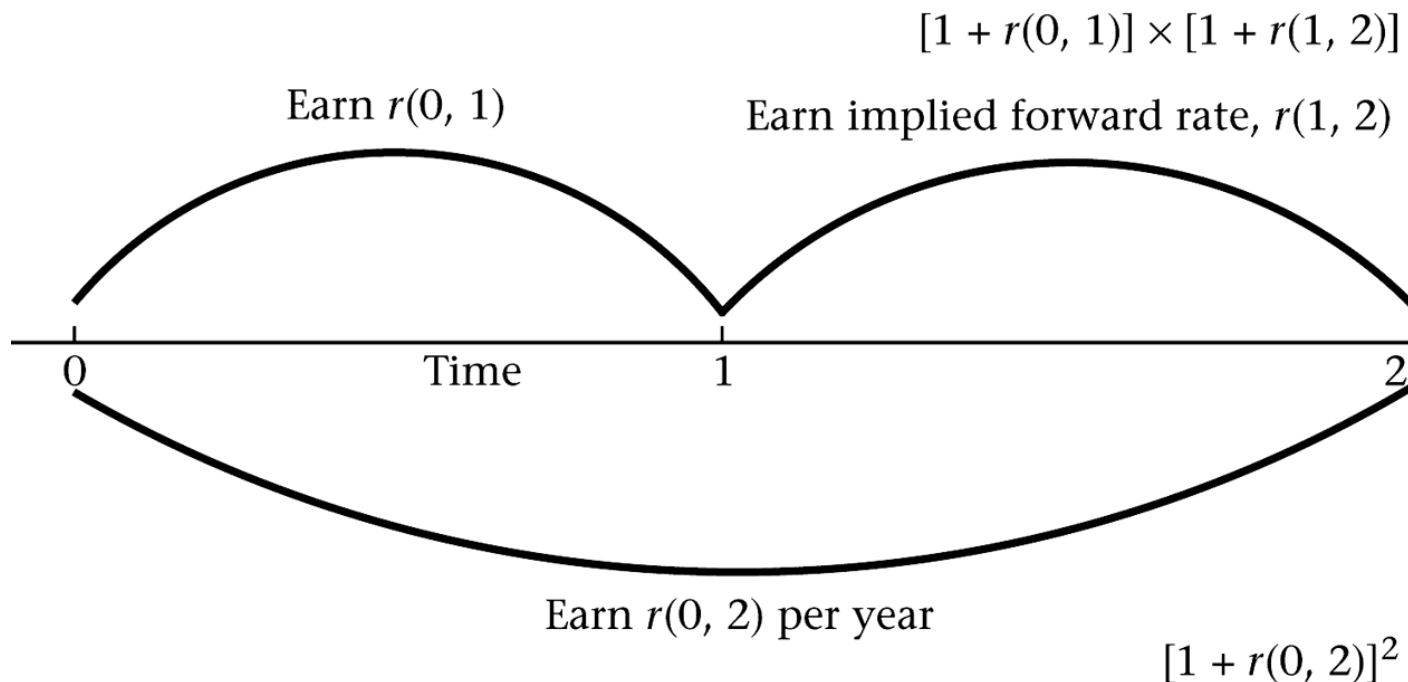
Bond basics (cont.)

- Zero-coupon bond price that pays C_t at t :
- **Yield** curve: Graph of annualized bond yields against time
$$P(0,t) = \frac{C_t}{[1 + r(0,t)]^t}$$
- Implied **forward rates**
 - Suppose current one-year rate $r(0,1)$ and two-year rate $r(0,2)$
 - Current forward rate from year 1 to year 2, $r_0(1,2)$, must satisfy:

$$[1 + r_0(0,1)] [1 + r_0(1,2)] = [1 + r_0(0,2)]^2$$



Bond basics (cont.)





Bond basics (cont.)

- In general: $[1 + r_0(t_1, t_2)]^{t_2 - t_1} = \frac{[1 + r_0(0, t_2)]^{t_2}}{[1 + r_0(0, t_1)]^{t_1}} = \frac{P(0, t_1)}{P(0, t_2)}$
- Example 7.1:
 - What are the implied forward rate $r_0(2, 3)$ and forward zero-coupon bond price $P_0(2, 3)$ from year 2 to year 3? (use Table 7.1)

$$r_0(2, 3) = \frac{P(0, 2)}{P(0, 3)} - 1 = \frac{0.881659}{0.816298} - 1 = 0.0800705$$

$$P_0(2, 3) = \frac{P(0, 3)}{P(0, 2)} = \frac{0.816298}{0.881659} = 0.925865$$



Coupon bonds

- The price at time of issue of t of a bond maturing at time T that pays n coupons of size c and maturity payment of \$1:

$$B_t(t, T, c, n) = \sum_{i=1}^n c P_t(t, t_i) + P_t(t, T)$$

where $t_i = t + i(T - t)/n$

- For the bond to sell at par, i.e. $B(t, T, c, n) = 1$ the coupon size must be:
$$c = \frac{1 - P_t(t, T)}{\sum_{i=1}^n P_t(t, t_i)}$$



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Duration

- Duration is a measure of sensitivity of a bond's price to changes in interest rates

- Duration

\$ Change in price for
a unit change in yield

divide by 100 (10,000) for change in price given a 1% (1 basis point) change in yield

$$\frac{\text{Change in bond price}}{\text{Unit change in yield}} = -\frac{1}{1+y} \sum_{i=1}^n i \frac{C_i}{(1+y)^i}$$

- Modified Duration

% Change in price for
a unit change in yield

$$-\text{Duration} \times \frac{1}{B(y)} = \frac{1}{1+y} \sum_{i=1}^n i \left[\frac{C_i}{(1+y)^i} \frac{1}{B(y)} \right]$$

- Macaulay Duration – Duration $\times \frac{1+y}{B(y)} = \sum_{i=1}^n i \left[\frac{C_i}{(1+y)^i} \frac{1}{B(y)} \right]$

Size-weighted average
of time until payments

- y : yield per period;

to annualize divide by # of payments per year

- $B(y)$: bond price as a function of yield y



Duration (Examples)

• Example 7.4 & 7.5

➤ 3-year zero-coupon bond with maturity value of \$100

- Bond price at YTM of 7.00%: $\$100/(1.0700^3) = \$81.62979$
 - Bond price at YTM of 7.01%: $\$100/(1.0701^3) = \$81.60691$
 - Duration: $-\frac{1}{1.07} \times 3 \times \frac{\$100}{1.07^3} = -\$228.87$
 - For a basis point (0.01%) change: $-\$228.87/10,000 = -\$0.02289$
 - Macaulay duration: $-(-\$228.87) \times \frac{1.07}{\$81.62979} = 3.000$
- $\Delta = -\$0.02288$
-

• Example 7.6

➤ 3-year annual coupon (6.95485%) par bond

- Macaulay Duration:

$$\left(1 \times \frac{0.0695485}{1.0695485}\right) + \left(2 \times \frac{0.0695485}{1.0695485^2}\right) + \left(3 \times \frac{1.0695485}{1.0695485^3}\right) = 2.80915$$



Duration (cont.)

- What is the new bond price $B(y+\varepsilon)$ given a small change ε in yield?
 - Rewrite the Macaulay duration:
$$D_{\text{Mac}} = - \frac{[B(y + \varepsilon) - B(y)]}{\varepsilon} \frac{1 + y}{B(y)}$$
 - And rearrange:
$$B(y + \varepsilon) = B(y) - [D_{\text{Mac}} \times \frac{B(y)\varepsilon}{1 + y}]$$
- Example 7.7
 - Consider the 3-year zero-coupon bond with price \$81.63 and yield 7%
 - What will be the price of the bond if the yield were to increase to 7.25%?
 - $B(7.25\%) = \$81.63 - (3 \times \$81.63 \times 0.0025 / 1.07) = \$81.058$
 - Using ordinary bond pricing: $B(7.25\%) = \$100 / (1.0725)^3 = \$81.060$
- The formula is only approximate due to the bond's convexity



Duration matching

- Suppose we own a bond with time to maturity t_1 , price B_1 , and Macaulay duration D_1
- How many (N) of another bond with time to maturity t_2 , price B_2 , and Macaulay duration D_2 do we need to short to eliminate sensitivity to interest rate changes? The hedge ratio:

$$N = -\frac{D_1 B_1 (y_1) / (1 + y_1)}{D_2 B_2 (y_2) / (1 + y_2)}$$

- Using $B(y + \varepsilon) = B(y) - [D_{\text{Mac}} \times \frac{B(y)\varepsilon}{1 + y}]$ for portfolio $B_1 + N B_2$
- The value of the resulting portfolio with duration zero is $B_1 + N B_2$
- Example 7.8
 - We own a 7-year 6% annual coupon bond yielding 7%
 - Want to match its duration by shorting a 10-year, 8% bond yielding 7.5%
 - You can verify that $B_1 = \$94.611$, $B_2 = \$103.432$, $D_1 = 5.882$, and $D_2 = 7.297$

$$N = -\frac{5.882 \times 94.611 / 1.07}{7.297 \times 103.432 / 1.075} = -0.7409$$



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Term structure of real interest rates

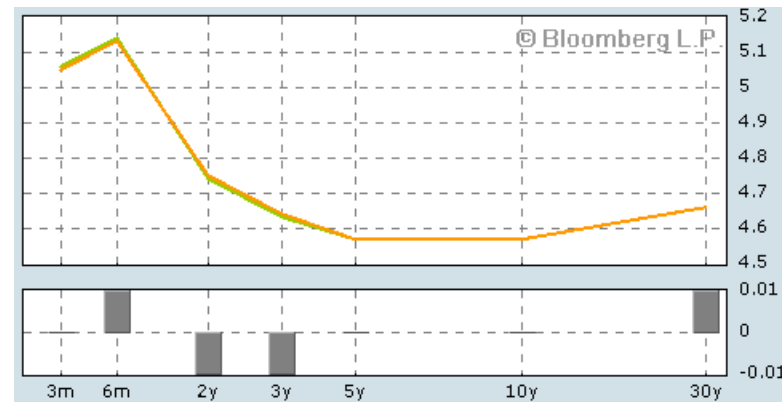
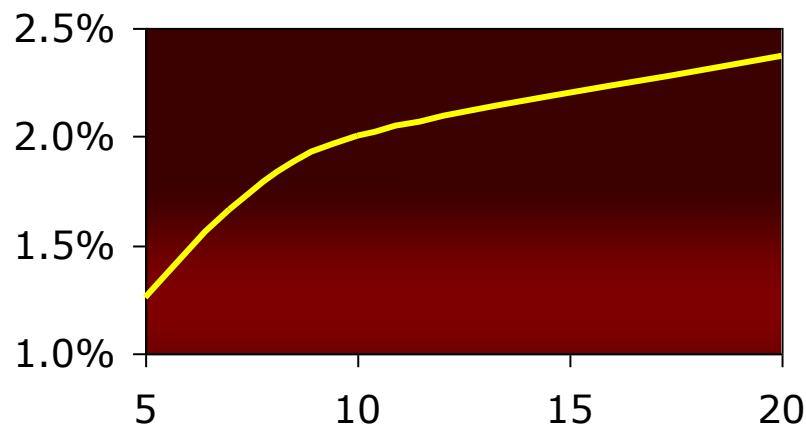
- Bond prices carry all the information on intertemporal rates of substitution,
 - primarily affected by expectations, and
 - only indirectly by risk considerations.
- Collection of interest rates for different times to maturity is a meaningful predictor of future economic developments.
 - More optimistic expectations produce an upward-sloping term structure of interest rates.



Term structure

- The price of a risk-free discount bond which matures in period t is $\beta_t = E[\mathbf{M}_t] = 1/(1+y_t)^t$
- The corresponding (gross) yield is
$$1+y_t = (\beta_t)^{-1/t} = \delta^{-1} [E[u'(w_t)] / u'(w_0)]^{-1/t}.$$
- Collection of interest rates is the term structure, $(y_1, y_2, y_3, \dots)$.
- Note that these are real yield rates (net of inflation), as are all prices and returns.

Term structure



- Left figure is an example of the term structure of **real** interest rates, measured with U.S. Treasury Inflation Protected Securities (TIPS), on August 2, 2004.

Source: www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/real_yield-hist.html

- Right hand shows nominal yield curve

Source: www.bloomberg.com



Term structure

$$1+y_t = \delta^{-1} \{E[u'(w_t)] / u'(w_0)\}^{-1/t}.$$

- Let g_t be the (state dependent) growth rate per period between period t and period 0, so $(1+g_t)^t = w_t / w_0$.
- Assume further that the representative agent has CRRA utility and a first-order approximations yields $y_t \approx \gamma E\{g_t\} - \ln \delta$. (Homework! Note $\ln(1+y) \approx y$)
- The yield curve measures expected growth rates over different horizons.



Term structure

$$y_t \approx \gamma E\{g_t\} - \ln \delta$$

- Approximation ignores second-order effects of uncertainty
- ...but we know that more uncertainty depresses interest rates if the representative agent is prudent.
- Thus, if long horizon uncertainty about the per capita growth rate is smaller than about short horizons (for instance if growth rates are mean reverting), then the term structure of interest rates will be upward sloping.



The expectations hypothesis

- *cross section of prices:*
The term structure are bond prices *at a particular point in time*. This is a cross section of prices.
- *time series properties:*
how do interest rates evolve as time goes by?
- Time series view is the relevant view for an investor
how tries to decide what kind of bonds to invest into,
or what kind of loan to take.



The expectations hypothesis

- Suppose you have some spare capital that you will not need for 2 years.
- You could invest it into 2 year discount bonds, yielding a return rate of $y_{0,2}$.
- Of course, since bonds are continuously traded, you could alternatively invest into 1-year discount bonds, and then roll over these bonds when they mature. The expected yield is $(1+y_{0,1})E[(1+y_{1,2})]$.
- Or you could buy a 3-year bond and sell it after 2 years.
- Which of these possibilities is the best?



The expectations hypothesis

- Only the first strategy is truly free of risk.
- The other two strategies are risky, since
 - price of 3-year bond in period 2 is unknown today, &
 - tomorrow's yield of a 1-year bond is not known today.
- **Term premia:**
the possible premium that these risky strategies have over the risk-free strategy are called *term premia*. (special form of risk premium).



The expectations hypothesis

- Consider a t -period discount bond. The price of this bond

$$\beta_{0,t} = E[\mathbf{M}_t] = E[m_1 \cdots m_t].$$

- one has to invest $\beta_{0,t}$ in $t=0$ in order to receive one consumption unit in period t .
- Alternatively, one could buy 1-period discount bonds and roll them over t -times. The investment that is necessary today to get one consumption unit (in expectation) in period t with this strategy is

$$E[m_1] \cdots E[m_t].$$

(to see this for $t=2$: buying $\beta_{1,2}$ bonds at $t=0$ costing $\beta_{0,1} \beta_{1,2}$ pays in expectations at $t=1$, which allows to pay one bond which ultimately pays \$1 at $t=2$)



The expectations hypothesis

- Two strategies yield same expected return rate if and only if

$$E[m_1 \cdots m_t] = E[m_1] \cdots E[m_t],$$

which holds if m_t is serially uncorrelated.

- In that case, there are no term premia — an assumption known as the expectations hypothesis.
- Whenever m_t is serially correlated (for instance because the growth process is serially correlated), then expectations hypothesis may fail.



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Futures contracts

- Exchange-traded “forward contracts”
- Typical features of futures contracts
 - Standardized, specified delivery dates, locations, procedures
 - A clearinghouse
 - Matches buy and sell orders
 - Keeps track of members’ obligations and payments
 - After matching the trades, becomes counterparty
- Differences from forward contracts
 - Settled daily through mark-to-market process → low credit risk
 - Highly liquid → easier to offset an existing position
 - Highly standardized structure → harder to customize

Example: S&P 500 Futures

- WSJ listing:

	OPEN	HIGH	LOW	SETTLE	CHANGE	LIFETIME		OPEN
						HIGH	LOW	INT.
S&P 500 Index (CME)-\$250 times Index								
Mar	112350	112370	109100	109530	- 2810	134960	94100	474,811
June	111950	111950	109350	109730	- 2830	170550	95030	17,224
Dec	111580	111580	110020	110390	- 2930	150070	96130	304
Est vol 79,914; vol Fri 65,250; open int 502,626, -701.								
Idx prf: Hi 1122.20; Lo 1092.25; Close 1094.44, -27.76.								

- Contract specifications:

FIGURE 5.2

Specifications for the S&P 500 index futures contract.

Underlying	S&P 500 index
Where traded	Chicago Mercantile Exchange
Size	\$250 × S&P 500 index
Months	Mar, Jun, Sep, Dec
Trading ends	Business day prior to determination of settlement price
Settlement	Cash-settled, based upon opening price of S&P 500 on third Friday of expiration month



Example: S&P 500 Futures (cont.)

- Notional value: $\$250 \times \text{Index}$
- Cash-settled contract
- Open interest: total number of buy/sell pairs
- Margin and mark-to-market
 - Initial margin
 - Maintenance margin (70-80% of initial margin)
 - Margin call
 - Daily mark-to-market
- Futures prices vs. forward prices
 - The difference negligible especially for short-lived contracts
 - Can be significant for long-lived contracts and/or when interest rates are correlated with the price of the underlying asset



Example: S&P 500 Futures (cont.)

- Mark-to-market proceeds and margin balance for 8 long futures:

TABLE 5.8 Mark-to-market proceeds and margin balance over 10 weeks from long position in 8 S&P 500 futures contracts. The final row represents expiration of the contract.

Week	Multiplier (\$)	Futures Price	Price Change	Margin Balance(\$)
0	2000.00	1100.00	—	220,000.00
1	2000.00	1027.99	−72.01	76,233.99
2	2000.00	1037.88	9.89	96,102.01
3	2000.00	1073.23	35.35	166,912.96
4	2000.00	1048.78	−24.45	118,205.66
5	2000.00	1090.32	41.54	201,422.13
6	2000.00	1106.94	16.62	234,894.67
7	2000.00	1110.98	4.04	243,245.86
8	2000.00	1024.74	−86.24	71,046.69
9	2000.00	1007.30	−17.44	36,248.72
10	2000.00	1011.65	4.35	44,990.57



Forwards versus futures pricing

- Price of *Forward* using EMM is

$$0 = E_t^*[\rho_T (F_{0,T} - S_T)] = E_t^*[\rho_T] (F_{0,T} - E_t^*[S_T]) - \text{Cov}_t^*[\rho_T, S_T]$$
 for fixed interest rate

$$F_{0,T} = E_t^*[S_T]$$

- Price of *Futures* contract is always zero.
 Each period there is a “dividend” stream of $\phi_t - \phi_{t-1}$
 and $\phi_T = S_T$

$$0 = E_t^*[\rho_{t+1}(\phi_{t+1} - \phi_t)] \text{ for all } t$$

since ρ_{t+1} is known at t

$$\phi_t = E_t^*[\phi_{t+1}] \text{ and } \phi_T = S_T$$

$$\phi_t = E_t^*[S_T]$$

Futures price process is always a martingale



Example: S&P 500 Futures (cont.)

- S&P index arbitrage: comparison of formula prices with actual prices:

TABLE 5.9

S&P 500 index futures prices and interest rate information from the *Wall Street Journal*, February 1, 2002. The closing S&P 500 spot price was 1130.20. Treasury-bill yields are reported yields on Treasury bills expiring in the same month as the futures contract. LIBOR rates are constructed from Eurodollar prices. The theoretical forward prices are constructed for each maturity from equation (5.7) using the interest rate in the preceding row and assuming a 1.3% dividend yield.

Expiration Month:	March	June	December
Days to Expiration:	42	140	322
S&P 500 Index Futures Price	1130.4	1132.5	1140.3
Treasury-Bill Yield	0.0167	0.017	0.0218
Theoretical Forward Price	1130.68	1131.93	1139.01
LIBOR	0.0187	0.0201	0.0240
Theoretical Forward Price	1130.94	1133.28	1141.22



Uses of index futures

- Why buy an index futures contract instead of synthesizing it using the stocks in the index? Lower transaction costs
- Asset allocation: switching investments among asset classes
- Example: Invested in the S&P 500 index and temporarily wish to temporarily invest in bonds instead of index. What to do?
 - Alternative #1: Sell all 500 stocks and invest in bonds
 - Alternative #2: Take a short forward position in S&P 500 index

TABLE 5.10

Effect of owning the stock and selling forward, assuming that $S_0 = \$100$ and $F_{0,1} = \$110$.

Transaction	Cash Flows		
	Today	1 year, $S_1 = \$80$	1 year, $S_1 = \$130$
Own Stock @ \$100	−\$100	\$80	\$130
Short Forward @ \$110	0	$110 - \$80$	$110 - \$130$
Total	−\$100	\$110	\$110



Uses of index futures (cont.)

- \$100 million portfolio with β of 1.4 and $r_f = 6\%$
 1. Adjust for difference in \$ amount
 - 1 futures contract $\$250 \times 1100 = \$275,000$
 - Number of contracts needed $\$100\text{mill}/\$0.275\text{mill} = 363.636$
 2. Adjust for difference in β
 - $363.636 \times 1.4 = 509.09$ contracts

Uses of index futures (cont.)

- Cross-hedging with perfect correlation

TABLE 5.11

Results from shorting 509.09 S&P 500 index futures against a \$100m portfolio with a beta of 1.4.

S&P 500 Index	Gain on 509 Futures	Portfolio Value	Total
900	33.855	72.145	106.000
950	27.491	78.509	106.000
1000	21.127	84.873	106.000
1050	14.764	91.236	106.000
1100	8.400	97.600	106.000
1150	2.036	103.964	106.000
1200	-4.327	110.327	106.000

- Cross-hedging with imperfect correlation
- General asset allocation: futures overlay
- Risk management for stock-pickers



Currency contracts

- Widely used to hedge against changes in exchange rates
- WSJ listing:

	OPEN	HIGH	LOW	SETTLE	CHANGE	LIFETIME		OPEN
						HIGH	LOW	INT.
CURRENCY								
Japan Yen (CME)-12.5 million yen; \$ per yen (.00)								
Mar	.7528	.7600	.7508	.7576	+ .0050	.8760	.7416	105,563
June	.7570	.7635	.7547	.7611	+ .0050	.8776	.7453	20,834
Est vol 11,091; vol Fri 25,220; open int 126,860, -1,352.								
Canadian Dollar (CME)-100,000 dls.; \$ per Can \$								
Mar	.6282	.6287	.6264	.6266	- .0016	.6725	.6170	60,355
June	.6280	.6287	.6263	.6264	- .0016	.6700	.6180	3,952
Sept	.6265	.6282	.6265	.6266	- .0016	.6590	.6175	1,331
Dec	.6274	.6280	.6265	.6269	- .0016	.6555	.6190	1,075
Est vol 5,343; vol Fri 7,699; open int 66,818, -652.								
British Pound (CME)-62,500 pds.; \$ per pound								
Mar	1.4112	1.4206	1.4106	1.4186	+ .0058	1.4700	1.3810	33,978
June	1.4066	1.4140	1.4038	1.4102	+ .0058	1.4550	1.3910	89
Est vol 5,315; vol Fri 4,859; open int 34,067, -174.								
Swiss Franc (CME)-125,000 francs; \$ per franc								
Mar	.5836	.5902	.5825	.5892	+ .0061	.6370	.5540	47,600
June	.5866	.5906	.5830	.5895	+ .0061	.6320	.5813	238
Est vol 5,676; vol Fri 6,330; open int 47,871, -587.								
Australian Dollar (CME)-100,000 dls.; \$ per A.\$								
Mar	.5076	.5102	.5074	.5098	+ .0031	.5300	.4810	21,597
June	.5050	.5069	.5046	.5070	+ .0031	.5218	.4885	456
Est vol 944; vol Fri 1,871; open int 22,079, -518.								
Mexican Peso (CME)-500,000 new Mex. peso, \$ per MP								
Mar	.10843	.10850	.10810	.10835	+ .00005	.10940	.09770	28,070
June				.10645	+ .00010	.10750	.09730	1,509
Sept	.10450	.10450	.10450	.10453	+ .00010	.10500	.09930	508
Est vol 1,940; vol Fri 2,817; open int 30,163, +626.								
Euro FX (CME)-Euro 125,000; \$ per Euro								
Mar	.8601	.8694	.8593	.8686	+ .0085	.9630	.8336	103,910
June	.8663	.8663	.8564	.8657	+ .0085	.9275	.8365	1,492
Dec	.8600	.8600	.8600	.8616	+ .0085	.9175	.8390	236
Est vol 14,904; vol Fri 17,547; open int 105,729, -560.								



Currency contracts: pricing

- Currency prepaid forward
 - Suppose you want to purchase ¥1 one year from today using \$s
 - $F_{0,T}^P = x_0 e^{-r_y T}$ (*price of prepaid forward*)
 - where x_0 is current (\$/ ¥) exchange rate, and r_y is the yen-denominated interest rate
 - Why? By deferring delivery of the currency one loses interest income from bonds denominated in that currency
- Currency forward
 - $F_{0,T} = x_0 e^{(r-r_y)T}$
 - r is the \$-denominated domestic interest rate
 - $F_{0,T} > x_0$ if $r > r_y$ (domestic risk-free rate exceeds foreign risk-free rate)



Currency contracts: pricing (cont.)

- Example 5.3:
 - ¥-denominated interest rate is 2% and current (\$/ ¥) exchange rate is 0.009. To have ¥1 in one year one needs to invest today:
 - $0.009/\text{¥} \times \text{¥}1 \times e^{-0.02} = \$0.008822$
- Example 5.4:
 - ¥-denominated interest rate is 2% and \$-denominated rate is 6%. The current (\$/ ¥) exchange rate is 0.009. The 1-year forward rate:
 - $0.009e^{0.06-0.02} = 0.009367$



Currency contracts: pricing (cont.)

- Synthetic currency forward: borrowing in one currency and lending in another creates the same cash flow as a forward contract
- Covered interest arbitrage: offset the synthetic forward position with an actual forward contract

TABLE 5.12

Synthetically creating a yen forward contract by borrowing in dollars and lending in yen. The payoff at time 1 is $¥1 - \$0.009367$.

Transaction	Cash Flows			
	Year 0		Year 1	
	\$	¥	\$	¥
Borrow $x_0 e^{-r_y}$ Dollar at 6% (\$)	+0.008822	—	−0.009367	—
Convert to Yen @ 0.009 \$/¥	−0.008822	+0.9802	—	—
Invest in Yen-Denominated Bill (¥)	—	−0.9802	—	1
Total	0	0	−0.009367	1

Eurodollar futures

- WSJ listing

	OPEN	HIGH	LOW	SETTLE	CHANGE	YIELD	CHANGE	OPEN INT.
Treasury Bills (CME)-\$1 mil.; pts of 100%								
Mar				98.26	+	.01	1.74	749
Est vol 17; vol Fri 3; open int 749, -10.								
Eurodollar (CME)-\$1 Million; pts of 100%								
Feb	98.08	98.09	98.08	98.09	+	.01	1.91	37,493
Mar	98.02	98.05	98.01	98.04	+	.02	1.96	758,791
Apr	97.94	97.96	97.94	97.96	+	.04	2.04	4,565
June	97.67	97.74	97.65	97.73	+	.06	2.27	684,053
Sept	97.17	97.30	97.17	97.28	+	.09	2.72	629,125
Dec	96.58	96.74	96.58	96.72	+	.12	3.28	709,233
Mr03	96.05	96.15	96.05	96.14	+	.12	3.86	399,549
June	95.57	95.66	95.57	95.64	+	.11	4.36	260,328
Sept	95.19	95.26	95.19	95.26	+	.11	4.74	232,148
Dec	94.85	94.92	94.84	94.90	+	.10	5.10	168,299
Mr04	94.64	94.70	94.63	94.69	+	.09	5.31	112,715
June	94.42	94.47	94.22	94.46	+	.09	5.54	116,246
Sept	94.22	94.28	94.22	94.27	+	.09	5.73	109,477
Dec	94.03	94.08	94.02	94.07	+	.09	5.93	70,835
Mr05	93.97	94.02	93.97	94.01	+	.08	5.99	75,902
June	93.86	93.90	93.85	93.90	+	.08	6.10	67,857
Sept	93.77	93.82	93.76	93.81	+	.08	6.19	79,848
Dec	93.64	93.69	93.63	93.67	+	.08	6.33	54,328
Mr06	93.63	93.68	93.62	93.67	+	.07	6.33	46,271
June	93.59	93.61	93.58	93.61	+	.07	6.39	35,319
Sept	93.54	93.56	93.54	93.55	+	.06	6.45	44,724
Dec	93.43	93.45	93.42	93.44	+	.06	6.56	32,917
Ju07	93.38	93.42	93.38	93.42	+	.05	6.58	18,263
Sept	93.35	93.39	93.35	93.38	+	.05	6.62	14,107
Dec	93.24	93.28	93.24	93.27	+	.04	6.73	13,227
Ju08	93.23	93.26	93.23	93.25	+	.04	6.75	11,239
Ju09	93.04	93.09	93.04	93.08	+	.02	6.92	2,406
Est vol 568,992; vol Fri 1,183,059; open int 4,864,582, +85,457.								

- Contract specifications

FIGURE 5.7

Specifications for the
Eurodollar futures
contract.

Where traded	Chicago Mercantile Exchange
Size	3-month Eurodollar time deposit, \$1 million principal
Months	Mar, Jun, Sep, Dec, out 10 years, plus 2 serial months and spot month
Trading ends	5 A.M. (11 A.M. London) on the second London bank business day immediately preceding the third Wednesday of the contract month.
Delivery	Cash settlement
Settlement	100 – British Banker's Association Futures Interest Settlement Rate for 3-Month Eurodollar Interbank Time Deposits. (This is a 3-month rate annualized by multiplying by 360/90.)



Introduction to Commodity Forwards

- *Commodity* forward prices can be described by the same formula as that for *financial* forward prices:

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

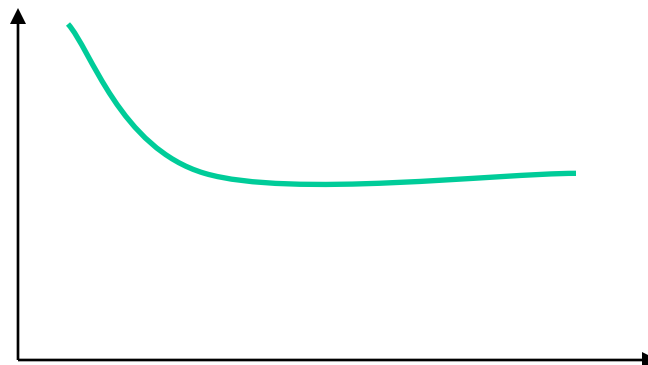
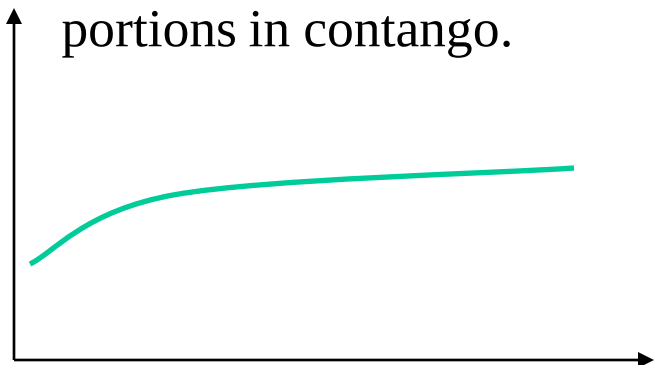
- For financial assets, δ is the dividend yield. For commodities, δ is the commodity lease rate. The lease rate is the return that makes an investor willing to buy and then lend a commodity.
- The lease rate for a commodity can typically be estimated only by observing the forward prices.



Introduction to Commodity Forwards

- The set of prices for different expiration dates for a given commodity is called the **forward curve** (or the **forward strip**) for that date.
- If on a given date the forward curve is upward-sloping, then the market is in **contango**. If the forward curve is downward sloping, the market is in **backwardation**.

➤ Note that forward curves can have portions in backwardation and portions in contango.





Forward rate agreements

- FRAs are over-the-counter contracts that guarantee a borrowing or lending rate on a given notional principal amount
- Can be settled at maturity (in arrears) or the initiation of the borrowing or lending transaction
 - FRA settlement in arrears: $(r_{qtrly} - r_{FRA}) \times \text{notional principal}$
 - At the time of borrowing: $\text{notional principal} \times (r_{qtrly} - r_{FRA}) / (1 + r_{qtrly})$
- FRAs can be synthetically replicated using zero-coupon bonds



Forward rate agreements (cont.)

TABLE 7.2

Investment strategy undertaken at time 0, resulting in net cash flows of $-\$1$ on day t , and receiving the implied forward rate, $1 + r_0(t, t + s)$ at $t + s$. This synthetically creates the cash flows from entering into a forward rate agreement on day 0 to lend at day t .

Transaction	Cash Flows		
	0	t	$t + s$
Buy $1 + r_0(t, t + s)$ Zeros Maturing at $t + s$	$-P(0, t + s) \times (1 + r_0(t, t + s))$	—	$1 + r_0(t, t + s)$
Short 1 Zero Maturing at t	$+P(0, t)$	-1	—
Total	0	-1	$1 + r_0(t, t + s)$

TABLE 7.3

Example of synthetic FRA. The transactions in this table are exactly those in Table 7.2, except that all bonds are sold at time t .

Transaction	Cash Flows	
	0	t
Buy $1 + r_0(t, t + s)$ Zeros Maturing at $t + s$	$-P(0, t + s) \times [1 + r_0(t, t + s)]$	$\frac{1 + r_0(t, t + s)}{1 + r_t(t, t + s)}$
Short 1 Zero Maturing at t	$+P(0, t)$	-1
Total	0	$\frac{r_0(t, t + s) - r_t(t, t + s)}{1 + r_t(t, t + s)}$



Eurodollar futures

- Very similar in nature to an FRA with subtle differences
 - The settlement structure of Eurodollar contracts favors borrowers
 - Therefore the rate implicit in Eurodollar futures is greater than the FRA rate => Convexity bias
- The payoff at expiration: $[\text{Futures price} - (100 - r_{\text{LIBOR}})] \times 100 \times \$25$
- Example: Hedging \$100 million borrowing with Eurodollar futures:

TABLE 7.4

Results from hedging \$100m in borrowing with 98.23 short Eurodollar futures.

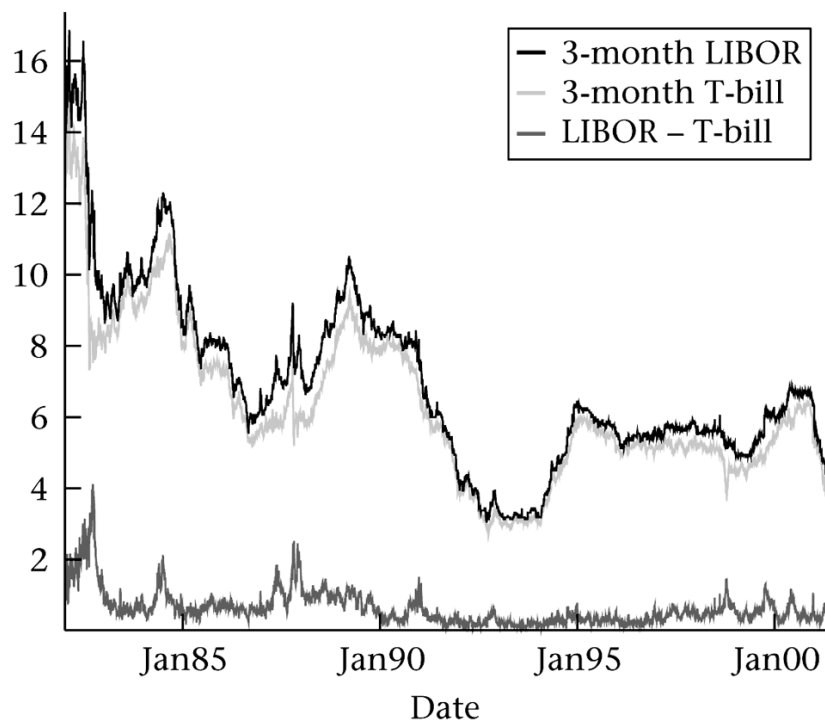
	Cash Flows			
	June		September	
	1.5%	2%	1.5%	2%
Borrowing Rate:				
Borrow \$100m	+100m	+100m	−101.5m	−102.0m
Gain on 98.23 Short Eurodollar Contracts	−0.294695m	0.196464m		
Gain Plus Interest			−0.299115m	0.200393m
Net			−101.799m	−101.799m



Eurodollar futures (cont.)

- Recently Eurodollar futures took over T-bill futures as the preferred contract to manage interest rate risk.
- **LIBOR** tracks the corporate borrowing rates better than the T-bill rate

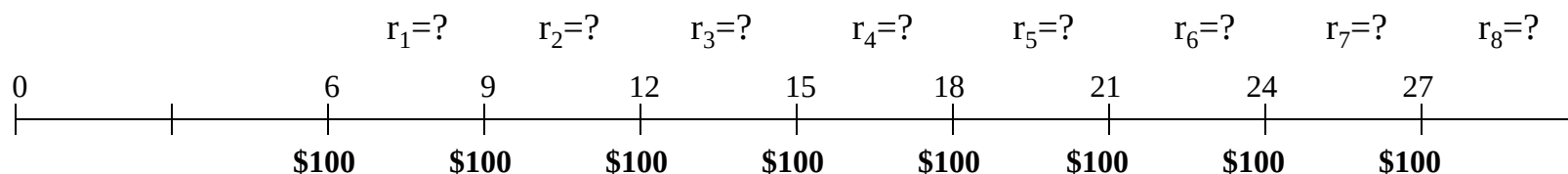
3 Month Rate (%)





Interest rate strips and stacks

- Suppose we will borrow \$100 million in 6 months for a period of 2 years by rolling over the total every 3 months



- Two alternatives to hedge the interest rate risk:
 - Strip: Eight separate \$100 million FRAs for each 3-month period
 - Stack: Enter 6-month FRA for ~\$800 million. Each quarter enter into another FRA decreasing the total by ~\$100 each time
 - Strip is the best alternative but requires the existence of FRA far into the future. Stack is more feasible but suffers from basis risk

Treasury bond/note futures

- WSJ listings for T-bond and T-note futures

INTEREST RATE

Treasury Bonds (CBT)-\$100,000; pts 32nds of 100%

Mar 103-11 104-08 103-11 104-05 + 26 111-16 97-11 457,263
 June 102-17 103-02 102-14 103-00 + 26 110-00 96-30 41,817
 Est vol 122,000; vol Fri 193,638; open int 499,310, +10,880.

Treasury Notes (CBT)-\$100,000; pts 32nds of 100%

Mar 106-09 106-31 106-08 06-285 + 20.5 11-085 101-23 532,232
 June 105-07 05-205 105-04 105-19 + 20.5 107-10 101-10 61,297
 Est vol 220,000; vol Fri 272,886; open int 593,529, +1,178.

10 Yr Agency Notes (CBT)-\$100,000; pts 32nds of 100%

Mar 102-04 102-14 102-01 102-13 + 18.5 106-04 96-27 36,659
 Est vol 1,000; vol Fri 1,519; open int 36,659, +69.

5 Yr Treasury Notes (CBT)-\$100,000; pts 32nds of 100%

Mar 06-145 106-27 06-115 106-25 + 12.5 109-08 04-015 531,946
 June 105-23 105-29 05-205 105-29 + 14.0 06-105 103-30 30,569
 Est vol 107,000; vol Fri 152,108; open int 562,515, +2,338.

2 Yr Treasury Notes (CBT)-\$200,000; pts 32nds of 100%

Mar 04-282 105-02 04-282 105-02 + 6.5 105-27 03-255 104,180
 Est vol 6,500; vol Fri 11,586; open int 104,180, +858.

- Contract specifications

FIGURE 7.6

Specifications for the Treasury-note futures contract.

Where traded	CBOT
Underlying	6% 10-year Treasury note
Size	\$100,000 Treasury note
Months	Mar, Jun, Sep, Dec, out 15 months
Trading ends	Seventh business day preceding last business day of month. Delivery until last business day of month.
Delivery	Physical T-note with at least 6.5 years to maturity and not more than 10 years to maturity. Price paid to the short for notes with other than 6% coupon is determined by multiplying futures price by a conversion factor. The conversion factor is the price of the delivered note (\$1 par value) to yield 6%. Settlement until last business day of the month.



Treasury bond/note futures (cont.)

- Long T-note futures position is an obligation to buy a 6% bond with maturity between 6.5 and 10 years to maturity
- The short party is able to choose from various maturities and coupons: the “cheapest-to-deliver” bond
- In exchange for the delivery the long pays the short the “invoice price.”

Price of the bond if it were to yield 6%

Invoice price = (Futures price x conversion factor) + accrued interest

TABLE 7.5

Prices, yields, and the conversion factor for two bonds. The futures price is 97.583. The short would break even delivering the 8-year 7% bond, and lose money delivering the 7-year 5% bond. Both bonds make semiannual coupon payments.

Description	8-Year 7% Coupon, 6.4% Yield	7-Year 5%, 6.3% Yield
Market Price	103.71	92.73
Price at 6% (Conversion Factor)	106.28	94.35
Invoice Price (Futures × Conversion Factor)	103.71	92.09
Invoice – Market	0	–0.66



Overview

1. Bond basics
2. Duration
3. Term structure of the real interest rate
4. Forwards and futures
 - Forwards versus futures price
 - Currency futures
 - Commodity futures: backwardation and contango
5. Repos
6. Swaps



Repurchase agreements

- A repurchase agreement or a repo entails selling a security with an agreement to buy it back at a fixed price
- The underlying security is held as collateral by the counterparty => A repo is collateralized borrowing
- Used by securities dealers to finance inventory
- A “haircut” is charged by the counterparty to account for credit risk



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Introduction to Swaps

- A **swap** is a contract calling for an exchange of payments, on one or more dates, determined by the difference in two prices.
- A swap provides a means to hedge a *stream* of risky payments.
- A single-payment swap is the same thing as a cash-settled forward contract.



An example of a commodity swap

- An industrial producer, IP Inc., needs to buy 100,000 barrels of oil 1 year from today and 2 years from today.
- The forward prices for deliver in 1 year and 2 years are \$20 and \$21/barrel.
- The 1- and 2-year zero-coupon bond yields are 6% and 6.5%.



An example of a commodity swap

- IP can guarantee the cost of buying oil for the next 2 years by entering into long forward contracts for 100,000 barrels in each of the next 2 years. The PV of this cost per barrel is

$$\frac{\$20}{1.06} + \frac{\$21}{1.065^2} = \$37.383$$

- Thus, IP could pay an oil supplier \$37.383, and the supplier would commit to delivering one barrel in each of the next two years.



An example of a commodity swap

- With a prepaid swap, the buyer might worry about the resulting credit risk. Therefore, a better solution is to defer payments until the oil is delivered, while still fixing the total price.

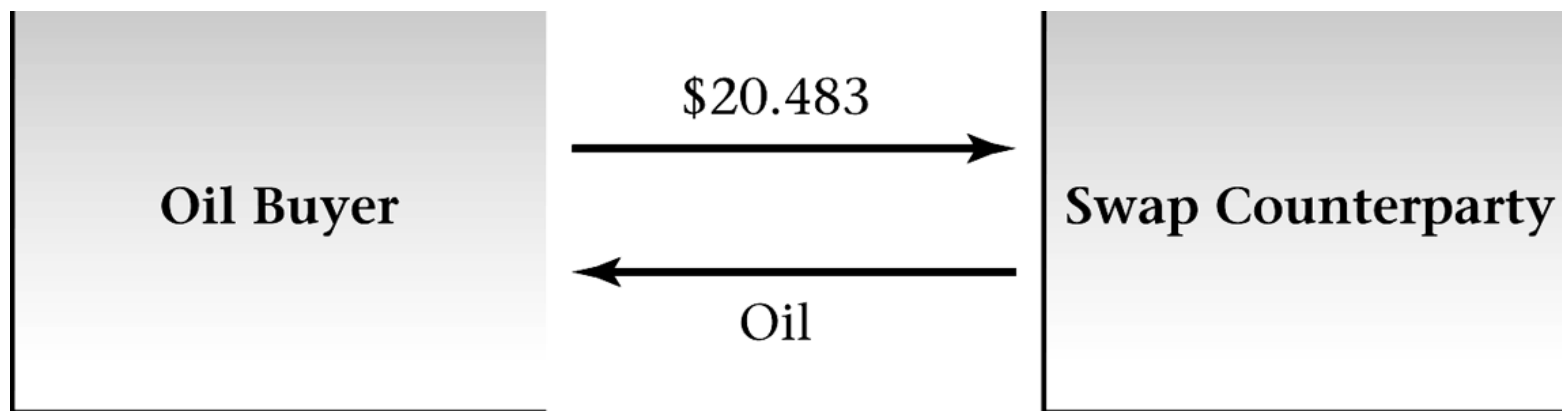
$$\frac{x}{1.06} + \frac{x}{1.065^2} = \$37.383$$

- Any payment stream with a PV of \$37.383 is acceptable. Typically, a swap will call for equal payments in each year.
 - For example, the payment per year per barrel, x , will have to be \$20.483 to satisfy the following equation:



Physical versus financial settlement

- **Physical settlement of the swap:**



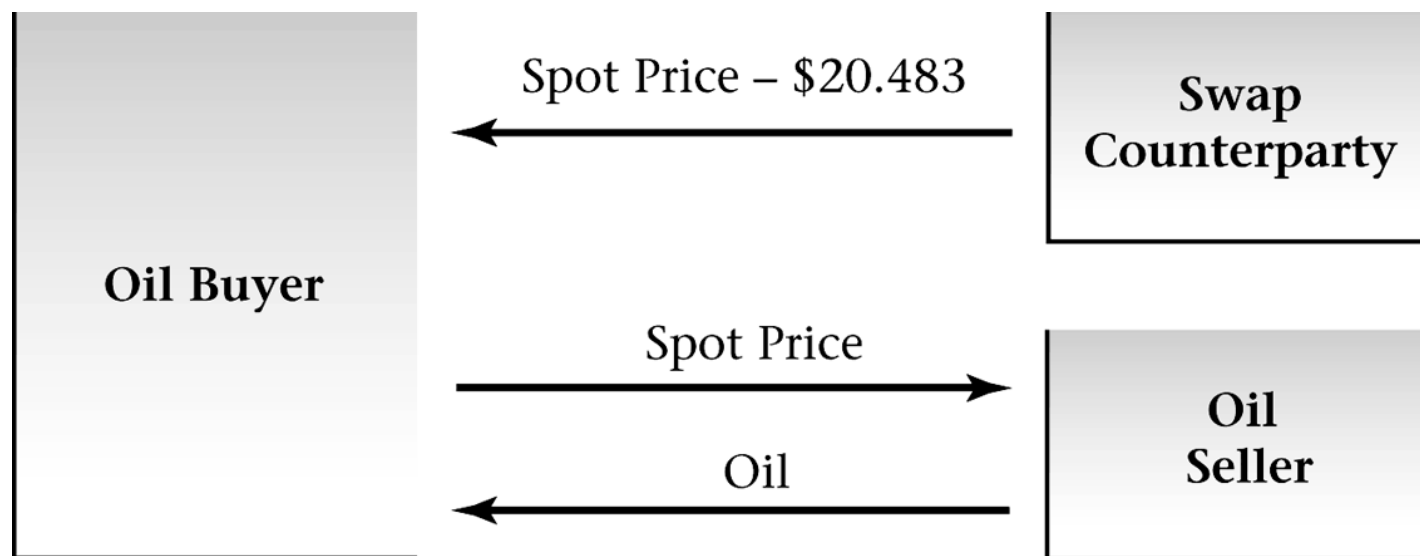


Physical versus financial settlement

- **Financial settlement** of the swap:
 - The oil buyer, IP, pays the swap counterparty the difference between \$20.483 and the spot price, and the oil buyer then buys oil at the spot price.
 - If the difference between \$20.483 and the spot price is negative, then the swap counterparty pays the buyer.

Physical versus financial settlement

- The results for the buyer are the same whether the swap is settled physically or financially. In both cases, the net cost to the oil buyer is \$20.483.





- Swaps are nothing more than forward contracts coupled with borrowing and lending money.
 - Consider the swap price of \$20.483/barrel. Relative to the forward curve price of \$20 in 1 year and \$21 in 2 years, we are overpaying by \$0.483 in the first year, and we are underpaying by \$0.517 in the second year.
 - Thus, by entering into the swap, we are lending the counterparty money for 1 year. The interest rate on this loan is
$$0.517 / 0.483 - 1 = 7\%.$$
 - Given 1- and 2-year zero-coupon bond yields of 6% and 6.5%, 7% is the 1-year implied forward yield from year 1 to year 2.



The market value of a swap

- The market value of a swap is zero at interception.
- Once the swap is struck, its market value will generally no longer be zero because:
 - the forward prices for oil and interest rates will change over time;
 - even if prices do not change, the market value of swaps will change over time due to the implicit borrowing and lending.
- A buyer wishing to exit the swap could enter into an offsetting swap with the original counterparty or whomever offers the best price.
- The market value of the swap is the difference in the PV of payments between the original and new swap rates.



Interest Rate Swaps

- The notional principle of the swap is the amount on which the interest payments are based.
- The life of the swap is the **swap term** or **swap tenor**.
- If swap payments are made at the end of the period (when interest is due), the swap is said to be settled in arrears.

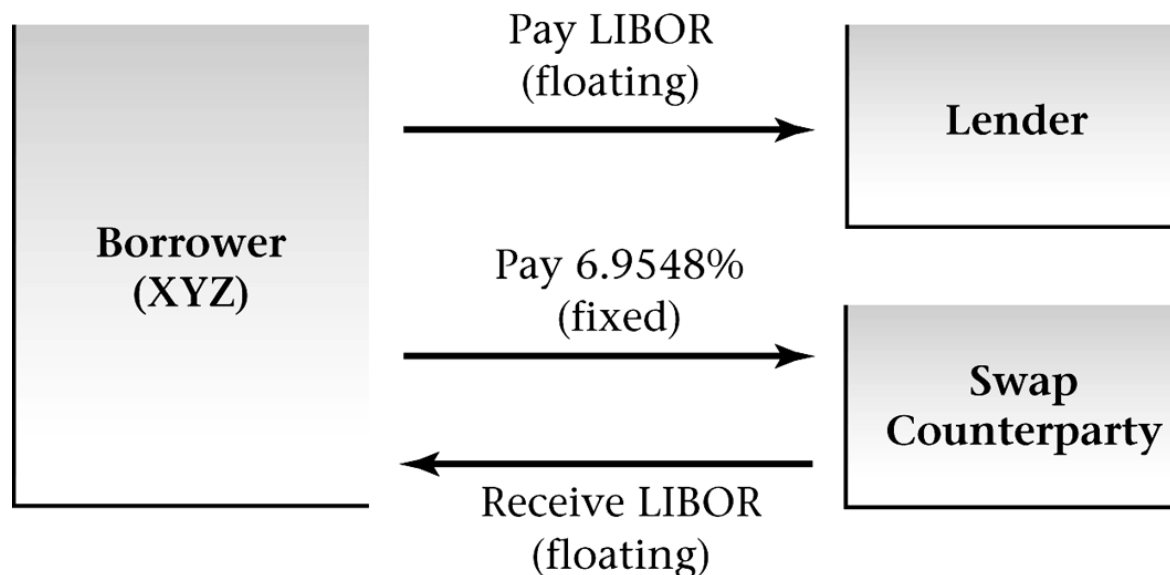


An example of an interest rate swap

- XYZ Corp. has \$200M of floating-rate debt at LIBOR, i.e., every year it pays that year's current LIBOR.
- XYZ would prefer to have fixed-rate debt with 3 years to maturity.
- XYZ could enter a swap, in which they receive a floating rate and pay the fixed rate, which is 6.9548%.



An example of an interest rate swap



On net, XYZ pays 6.9548%:

$$\text{XYZ net payment} = \underbrace{-\text{LIBOR}}_{\text{Floating Payment}} + \underbrace{\text{LIBOR} - 6.9548\%}_{\text{Swap Payment}} = -6.9548\%$$

Floating Payment

Swap Payment



Computing the swap rate

- Suppose there are n swap settlements, occurring on dates t_i , $i = 1, \dots, n$.
 - The implied forward interest rate from date t_{i-1} to date t_i , known at date 0, is $r_0(t_{i-1}, t_i)$.
 - The price of a zero-coupon bond maturing on date t_i is $P(0, t_i)$.
 - The fixed swap rate is R .
-
- The market-maker is a counterparty to the swap in order to earn fees, not to take on interest rate risk. Therefore, the market-maker will hedge the floating rate payments by using, for example, forward rate



Computing the swap rate

- The requirement that the hedged swap have zero net PV is

$$\sum_{i=1}^n P(0, t_i) [R - r_0(t_{i-1}, t_i)] = 0 \quad (8.1)$$

- Equation (8.1) can be rewritten as

$$R = \frac{\sum_{i=1}^n P(0, t_i) r_0(t_{i-1}, t_i)}{\sum_{i=1}^n P(0, t_i)} \quad (8.2)$$

where $\sum_{i=1}^n P(0, t_i) r_0(t_{i-1}, t_i)$ is the PV of interest payments implied by the strip of forward rates, and $\sum_{i=1}^n P(0, t_i)$ is the PV of a \$1 annuity when interest rates vary over time.



Computing the swap rate

- We can rewrite equation (8.2) to make it easier to interpret:

$$R = \sum_{i=1}^n \left[\frac{P(0, t_i)}{\sum_{j=1}^n P(0, t_j)} \right] r_0(t_{i-1}, t_i)$$

Thus, the fixed swap rate is as a weighted average of the implied forward rates, where zero-coupon bond prices are used to determine the weights.



Computing the swap rate

- Alternative way to express the swap rate is

$$R = \frac{1 - P(0, t_n)}{\sum_{i=1}^n P(0, t_i)} \quad (8.3)$$

- using $r_0(t_1, t_2) = P(0, t_1)/P(0, t_2) - 1$

This equation is equivalent to the formula for the coupon on a par coupon bond.

Thus, the swap rate is the coupon rate on a par coupon bond.

(firm that swaps floating for fixed ends up with economic equivalent of a fixed-rate bond)



The swap curve

- A set of swap rates at different maturities is called the *swap curve*.
- The swap curve should be consistent with the interest rate curve implied by the Eurodollar futures contract, which is used to hedge swaps.
- Recall that the Eurodollar futures contract provides a set of 3-month forward LIBOR rates. In turn, zero-coupon bond prices can be constructed from implied forward rates. Therefore, we can use this information to compute swap rates.



The swap curve

TABLE 8.4

3-month LIBOR forward rates implied by Eurodollar futures prices with maturity dates given in the first column. Prices are from June 7, 2000.

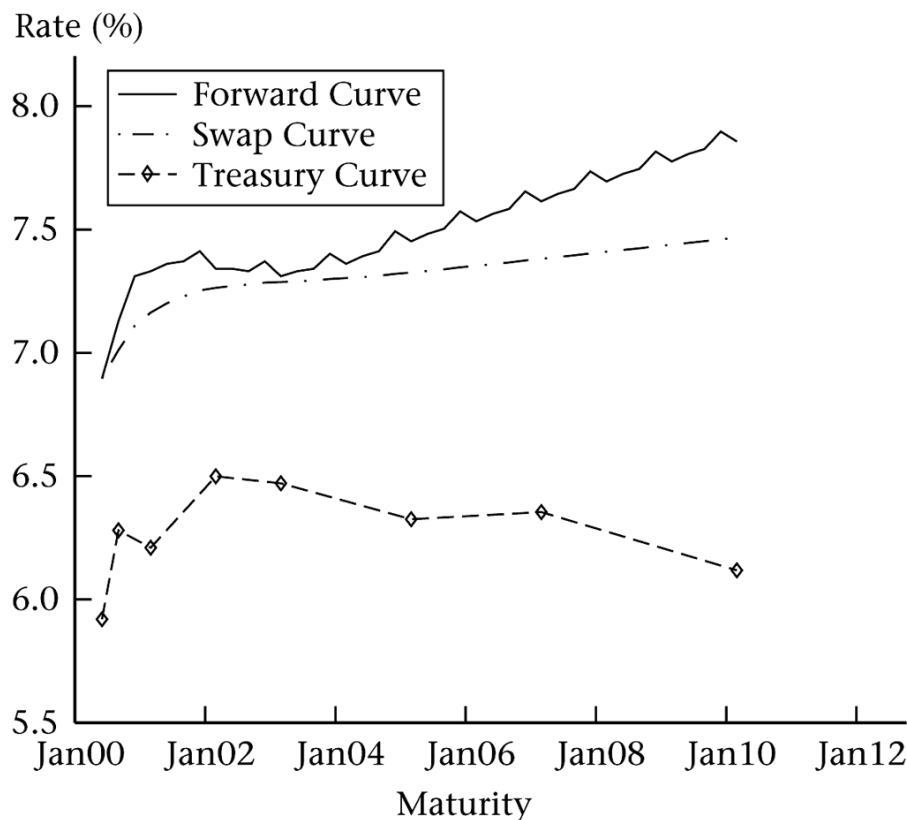
Maturity Date, t_i	Eurodollar Futures Price	Implied Quarterly Rate, $r(t_i, t_{i+1})$	Implied June 2000 Price of \$1	
			Paid on Maturity Date, t_i , $P(0, t_i)$	Swap Rate
Jun-00	93.18	0.01724	—	6.895%
Sep-00	92.95	0.01782	0.9830	7.011%
Dec-00	92.77	0.01827	0.9658	7.109%
Mar-01	92.75	0.01832	0.9485	7.162%
Jun-01	92.72	0.01840	0.9314	7.201%
Sep-01	92.70	0.01842	0.9146	7.228%
Dec-01	92.67	0.01852	0.8980	7.252%
Mar-02	92.74	0.01835	0.8817	7.263%

For example, the December swap rate can be computed using equation (8.3): $(1 - 0.9485) / (0.9830 + 0.9658 + 0.9485) = 1.778\%$. Multiplying 1.778% by 4 to annualize the rate gives the December swap rate of 7.109%.



The swap curve

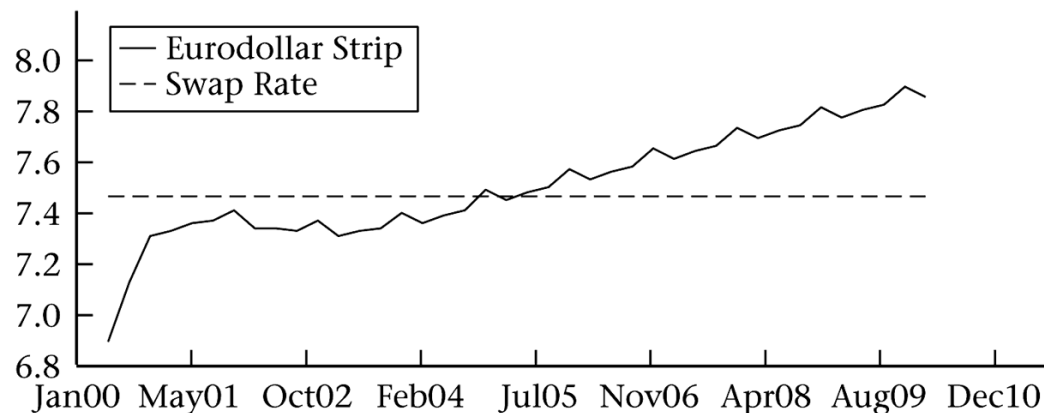
- The **swap spread** is the difference between swap rates and Treasury-bond yields for comparable maturities.



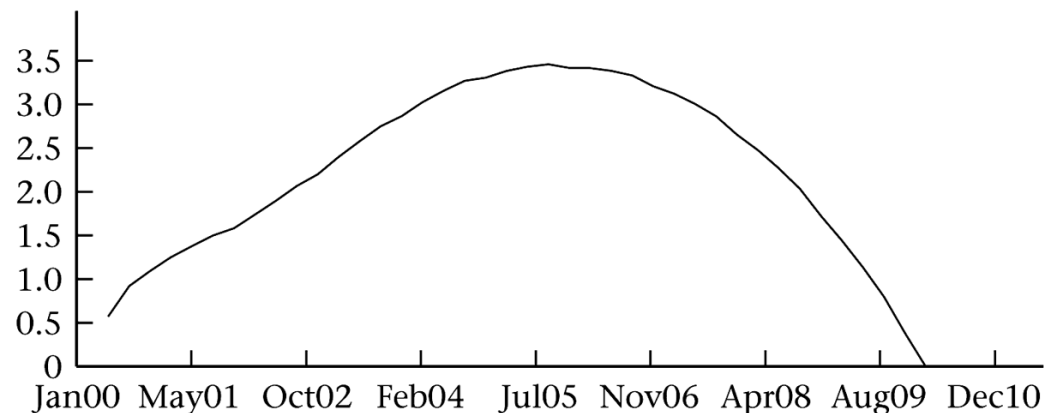
The swap's implicit loan balance

- Implicit borrowing and lending in a swap can be illustrated using the following graph, where the 10-year swap rate is 7.4667%:

Interest Rates (%)



Implicit Loan





The swap's implicit loan balance

- In the above graph,
 - Consider an investor who pays fixed and receives floating. This investor is paying a high rate in the early years of the swap, and hence is lending money. About halfway through the life of the swap, the Eurodollar forward rate exceeds the swap rate and the loan balance declines, falling to zero by the end of the swap.
 - Therefore, the credit risk in this swap is borne, at least initially, by the fixed-rate payer, who is lending to the fixed-rate recipient.



Deferred swap

- A **deferred swap** is a swap that begins at some date in the future, but its swap rate is agreed upon today.

$$R = \frac{\sum_{i=k}^T P(0, t_i) r_0(t_{i-1}, t_i)}{\sum_{i=k}^T P(0, t_i)}$$

- The fixed rate on a deferred swap beginning in k periods is computed as

(8.4)



Why swap interest rates?

- Interest rate swaps permit firms to separate credit risk and interest rate risk.
 - By swapping its interest rate exposure, a firm can pay the short-term interest rate it desires, while the long-term bondholders will continue to bear the credit risk.



Amortizing and accreting swaps

- An **amortizing swap** is a swap where the notional value is *declining* over time (e.g., floating rate mortgage).
- An **accreting swap** is a swap where the notional value is *growing* over time.
- The fixed swap rate is still a weighted average of implied forward rates, but now the weights also involve changing notional principle, Q_t :

$$R = \frac{\sum_{i=1}^n Q_{t_i} P(0, t_i) r(t_{i-1}, t_i)}{\sum_{i=1}^n Q_{t_i} P(0, t_i)} \quad (8.7)$$



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