# An Alternative Sense of Asymptotic Efficiency

Ulrich K. Müller
Princeton University

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# Introduction

Asymptotically efficient tests for nonstandard problems

- Unit root/cointegration tests: Elliott, Rothenberg and Stock (1996), Elliott (1999), Müller and Elliott (2003), Elliott, Jansson and Pesavento (2005), Jansson (2005)

- Tests about cointegrating vector with local-to-unity stochastic trend: Stock and Watson (1996), Jansson and Moreira (2006)

- Structural breaks: Nyblom (1989), Andrews and Ploberger, Elliott and Müller (2006)

- Weak IV: Andrews, Moreira and Stock (2006, 2007)

# Standard Asymptotic Efficiency I

1. Consider first canonical parametric version of model, typically with Gaussian i.i.d. disturbances

   Unit Root Test: $y_t = \rho y_{t-1} + \varepsilon_t, \quad y_0 = 0, \quad \varepsilon_t \sim$i.i.d.$\mathcal{N}(0,1)$

   $H_0 : \rho = 1 \quad$ against $\quad H_1 : \rho = \rho_1 < 1$

2. Derive small sample efficient test in that canonical model

   $\text{LR}_T = \exp[-\frac{1}{2}(1-\rho_1)(y_T^2 - \sum(\Delta y_t)^2) - \frac{1}{2}(1-\rho_1)^2 \sum y_{t-1}^2]$

3. Take limits of small sample efficient test against local alternatives

   $\rho = \rho_T = 1 - \theta/T$

   $H_0 : \theta = 0 \quad$ against $\quad H_1 : \theta = \theta_1 > 0$

   $\text{LR}_T \rightsquigarrow \exp[-\frac{1}{2}\theta_1(J_\theta(1)^2 - 1) - \frac{1}{2}\theta_1^2 \int J_\theta(s)^2 ds]$

   $J_\theta(s) = \int_0^s e^{-\theta(s-r)} dW(r)$

# Standard Asymptotic Efficiency II

4. Construct robustified version of test statistic to obtain correct coverage also for non-canonical versions of the model

$$\widehat{\mathsf{LR}}_T = \exp[-\tfrac{1}{2}\theta_1(\hat{J}_T(1)^2 - 1) - \tfrac{1}{2}\theta_1^2 \int \hat{J}_T(s)^2 ds]$$

$$\hat{J}_T(\cdot) = T^{-1/2}\hat{\omega}_T^{-1} y_{\lfloor \cdot T \rfloor} \rightsquigarrow J_\theta(\cdot)$$

End-product is test that

i. is asymptotically efficient in canonical model

ii. has same asymptotic rejection probabilities for all models where robustified version yields same weak limits

whenever $\hat{\omega}_T$ and $\varepsilon_t$ are such that $T^{-1/2}\hat{\omega}_T^{-1} y_{\lfloor \cdot T \rfloor} \rightsquigarrow J_\theta(\cdot)$

But: There could exist a test that is also efficient in the canonical model, with higher asymptotic power against some non-canonical model

# Semiparametric Efficiency

- Model has infinitely dimensional nuisance parameter, such as distribution of disturbance

- Well developed only for standard problem with locally asymptotic normal likelihood ratios

- Jansson (2007)

  - Unit root test for model $y_t = \rho y_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim iidF(0, 1)$, and $F$ is nuisance parameter

  - Semiparametric power envelope with $F$ known

  - Adaption only possible for symmetric $F$. Otherwise, still asymptotic power gains over $\widehat{\mathsf{LR}}_T$

$\Rightarrow$ $\widehat{\mathsf{LR}}_T$ test asymptotically inadmissible in this set-up

# This Paper

1. For test of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ on double-array data $Y_T \in \mathbb{R}^{nT}$, consider typical weak convergence $X_T = h(Y_T) \rightsquigarrow X \sim P(\theta)$

   $X_T = \hat{J}_T(\cdot) = T^{-1/2}\hat{\omega}_T^{-1} y_{T,\lfloor \cdot T \rfloor} \rightsquigarrow X = J_\theta(\cdot)$

2. Derive best test in limiting problem with $X$ observed

   $L(X) = \exp[-\frac{1}{2}\theta_1(X(1)^2 - 1) - \frac{1}{2}\theta_1^2 \int X(s)^2 ds]$ is RN-derivative

   by Neyman-Pearson: reject for large values of $L(X)$

3. Robustness requirement: Tests in original problem must have correct asymptotic rejection probability whenever $X_T \rightsquigarrow X \sim P(\theta_0)$

   Robust unit root tests do not overreject whenever $\hat{J}_T(\cdot) \rightsquigarrow W(\cdot)$

   further step in progression to weaker assumptions about disturbances

   Stock (1994), White (2001), Breitung (2002), Davidson (2007) *define* I(1) property in terms of $T^{-1/2} y_{T,\lfloor \cdot T \rfloor} \rightsquigarrow \omega W(\cdot)$

# Main Result

4. Asymptotically best robust test is optimal test in limiting problem, evaluated at sample analogues

   Rejecting for large values of

   $$L(X_T) = \widehat{\mathrm{LR}}_T = \exp[-\tfrac{1}{2}\theta_1 \hat{J}_T((1)^2 - 1) - \tfrac{1}{2}\theta_1^2 \int \hat{J}_T(s)^2 ds]$$

   identical to ERS

$\Rightarrow$ For any test that has higher asymptotic power than best robust test, there exists model with $X_T \rightsquigarrow P(\theta_0)$ where test overrejects asymptotically

there exists $T^{-1/2}\hat{\omega}_T^{-1} y_{T,\lfloor \cdot T \rfloor} \rightsquigarrow W(\cdot)$ for which Jansson's (2007) test

has asymptotic rejection probability larger than nominal level

# Scope and Applications

- Generic result whenever null and alternative hypotheses induce weak convergences. Very weak regularity conditions beyond a.e. continuity of best test in limiting problem.

- General set-up allows for
  - composite hypotheses
  - additional restrictions on tests: unbiasedness, (conditional) similarity, invariance

- Applications
  - Broader sense of asymptotic efficiency of tests mentioned in introductory slide
  - Müller and Watson (2007, 2008) and Ibragimov and Müller (2007) that take weak convergence as a starting point

# Plan of Talk

# Set-Up

- Suppose we observe double array $Y_T \in \mathbb{R}^{nT}$ whose distribution $F_T(m, \theta)$ in model $m$ depends on a parameter $\theta$. We are interested in testing

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta = \theta_1.$$

- Consider function $h_T : \mathbb{R}^{nT} \mapsto S$, where $S$ is a (complete and separable) metric space. Let $P_T(m, \theta)$ be the distribution of $X_T = h_T(Y_T)$. Assume that for the typical model $m$,

$$X_T = h_T(Y_T) \rightsquigarrow X \sim P(\theta)$$

where $P(\theta_0)$ and $P(\theta_1)$ are mutually absolutely continuous.

- Tests $\varphi_T$ are $\mathbb{R}^{nT} \mapsto [0, 1]$ functions, so that $\varphi_T(y_T)$ is the probability of rejection conditional on observing $Y_T = y_T$. Null and alternative rejection probabilities are $\int \varphi_T dF_T(m, \theta_0)$ and $\int \varphi_T dF_T(m, \theta_1)$.

# Robustness and Limiting Problem

- A test is robust if it has correct asymptotic null rejection probability for large set of models $m$.

  Let $\mathcal{M}_0$ be the set of models $m$ for which $h_T(Y_T) \rightsquigarrow X$ with $\theta = \theta_0$, i.e. $P_T(m, \theta_0) \rightsquigarrow P(\theta_0)$. Then call a test is *robust* if

  $$\limsup_{T\to\infty} \int \varphi_T dF_T(m, \theta_0) \leq \alpha \quad \text{for all } m \in \mathcal{M}_0.$$

  Let $\mathcal{M}_1$ be the set of models $m$ for which $P_T(m, \theta_1) \rightsquigarrow P(\theta_1)$.

- Limiting problem

  $$H_0^{lp} : X \sim P(\theta_0) \quad \text{against} \quad H_1^{lp} : X \sim P(\theta_1)$$

  with tests $\varphi_S : S \mapsto [0, 1]$.

- Let $L : S \mapsto \mathbb{R}$ be the Radon-Nikodym derivative $L = dP(\theta_1)/dP(\theta_0)$.

  By Neyman-Pearson the best test in the limiting problem, $\varphi_S^*$, rejects for large values of $L$.

# Main Result

**Theorem:** Suppose $\varphi_S^* : S \mapsto [0,1]$ is the best level $\alpha$ test of $H_0^{lp}$ against $H_1^{lp}$, and $\varphi_S^*$ is $P(\theta_0)$ almost everywhere continuous. Then $\hat{\varphi}_T^* : \mathbb{R}^{nT} \mapsto [0,1]$ with $\hat{\varphi}_T^*(y_T) = (\varphi_S^* \circ h_T)(y_T) = \varphi_S^*(X_T)$ satisfies

(i) $\lim_{T\to\infty} \int \hat{\varphi}_T^* dF_T(m,\theta_0) = \alpha$ for all $m \in \mathcal{M}_0$, and $\lim_{T\to\infty} \int \hat{\varphi}_T^* dF_T(m,\theta_1) = \int \varphi_S^* dP(\theta_1)$ for all $m \in \mathcal{M}_1$.

(ii) For any level $\alpha$ robust test $\varphi_T : \mathbb{R}^{nT} \mapsto [0,1]$,

$$\limsup_{T\to\infty} \int \varphi_T dF_T(m,\theta_1) \leq \lim_{T\to\infty} \int \hat{\varphi}_T^* dF_T(m,\theta_1) = \int \varphi_S^* dP(\theta_1)$$

for all $m \in \mathcal{M}_1$.

Application to Unit Root testing:

- ERS's test is asymptotically point optimal among all robust tests.
- Jansson's (2007) semiparametric test is not robust.

# Heuristic Proof I

- Reconsider unit root testing problem, where

$$X_T = \hat{J}_T(\cdot) = T^{-1/2}\hat{\omega}_T^{-1} y_{T,\lfloor \cdot T \rfloor} \rightsquigarrow X = J_\theta(\cdot)$$

and in limiting problem with $X$ observed, Neyman-Pearson test of $H_0$ : $\theta = 0$ against $H_1 : \theta = \theta_1 > 0$ rejects for large values of

$$L(X) = \exp[-\tfrac{1}{2}\theta_1(X(1)^2 - 1) - \tfrac{1}{2}\theta_1^2 \int X(s)^2 ds]$$

- Idea of proof: For *any* $X_T \sim Q_T \rightsquigarrow J_{\theta_1}(\cdot)$, one can construct $X_T \sim P_T \rightsquigarrow J_0(\cdot)$ such that the best small sample test of

$$H_{T,0} : X_T \sim P_T \quad \text{against} \quad H_{T,1} : X_T \sim Q_T$$

rejects for $L(X_T)$. Robust tests must control asymptotic size under $H_{0,T}$, and no test can have a better asymptotic level and power trade-off than a sequence of small sample optimal tests.

# Heuristic Proof II

- For simplicity, pretend that $\frac{1}{L}$ is bounded and continuous.

- Let $Q_T$ be any given probability measure of $X_T \rightsquigarrow J_{\theta_1}(\cdot) \sim Q$. Construct measure $P_T$ as

$$\begin{aligned} \int_A dP_T &= \kappa_T^{-1} \int_A \frac{1}{L} dQ_T \quad \text{for all measurable } A \subset \mathbb{R}^T \\ \kappa_T &= \int \frac{1}{L} dQ_T \end{aligned}$$

where $L : D_{[0,1]} \mapsto \mathbb{R}$ is Radon-Nikodym derivative of $Q$ with respect to $P$ $(L = dQ/dP)$. By construction $dQ_T/dP_T = \kappa_T L$.

- Note that $\kappa_T = \int \frac{1}{L} dQ_T \to \int \frac{1}{L} dQ = \int \frac{1}{L} L dP = 1$. Also, for any bounded and continuous function $\vartheta : D_{[0,1]} \mapsto \mathbb{R}$, $P_T$ satisfies

$$\int \vartheta dP_T = \kappa_T^{-1} \int \frac{\vartheta}{L} dQ_T \to \int \frac{\vartheta}{L} dQ = \int \frac{\vartheta}{L} L dP = \int \vartheta dP$$

so that $P_T \rightsquigarrow J_0(\cdot) \sim P$.

# Generalizations

- Composite null and alternative hypothesis, using weighted average power as criterion for efficiency of tests

- Additional restrictions on tests in limiting problem, with analogous restrictions in original problem:

  $\Rightarrow$ Invariance, unbiasedness, (conditional) similarity

- Consistently estimable parameters that affect limiting distribution

# Discussion

- Complete class of tests in limiting problem that are continuous a.e., evaluated at sample analogues, form an "asymptotically essentially complete class of robust tests".

- Appeal of efficiency property of depends on appropriateness of robustness constraint

  - Weak convergences as regularity condition. Much more natural in time series context.

  - Conservative. How sure are we about conventional primitive conditions, such as mixing?

  - Quality of small sample approximation.

- Construction of reasonable tests for complicated models.

# Low-Frequency Unit Root Tests

- Standard unit root test literature assumes $T^{-1/2}u_{T,\lfloor \cdot T \rfloor} \rightsquigarrow \omega J_\theta(\cdot)$.
- Müller and Watson (2007) instead consider point-optimal scale invariant unit root test under the strictly weaker assumption

$$\left\{ T^{-1/2} \int_0^1 \psi_l(s) u_{T,\lfloor sT \rfloor} ds \right\}_{l=1}^q \rightsquigarrow \left\{ \omega \int_0^1 \psi_l(s) J_\theta(s) ds \right\}_{l=1}^q \qquad (1)$$

where $\psi_l(s) = \sqrt{2}\cos(\pi l s)$ and $q$ is chosen so that the frequency of the weight functions $\psi_l$, $l = 1, \cdots, q$ are below business cycle frequency for the span of the data under study (so that $q = 13$ for data spanning 50 years).

- Results here imply that low-frequency unit root test is point-optimal among all tests that controls asymptotic size whenever (1) holds with $\theta = 0$.

# Robustified Efficient Test in Canonical Model I

- Consider the hypothesis test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ in the canonical model $m^*$, so that the best test rejects for large values of $\mathrm{LR}_T$

$$\mathrm{LR}_T = \exp[-\tfrac{1}{2}(1-\rho_1)(y_T^2 - \sum(\Delta y_t)^2) - \tfrac{1}{2}(1-\rho_1)^2 \sum y_{t-1}^2]$$

- Write $\mathrm{LR}_T = L(X_T^*) + o_p(1)$, where $L$ is a continuous function of $X_T^* = h_T^*(Y_T)$, and $X_T^* \rightsquigarrow X$ with $X \sim P(\theta)$. By CMT, $\mathrm{LR}_T \rightsquigarrow L(X)$.

$$X_T^* = T^{-1/2} y_{\lfloor \cdot T \rfloor} \rightsquigarrow X = J_\theta(\cdot)$$

$$L(x) = \exp[-\tfrac{1}{2}\theta_1(x(1)^2 - 1) - \tfrac{1}{2}\theta_1^2 \int x(s)^2 ds]$$

- Robustified test is based on $L(X_T)$, where $X_T \rightsquigarrow X \sim P(\theta)$ in many models $m$ of interest, including $m^*$

$$X_T(\cdot) = T^{-1/2} \hat{\omega}_T^{-1} y_{\lfloor \cdot T \rfloor} \rightsquigarrow J_\theta(\cdot)$$

# Robustified Efficient Test in Canonical Model II

- LeCam's Third Lemma: If the models $Y_T \sim F_T(m^*, \theta_0)$ and $Y_T \sim F_T(m^*, \theta_1)$ are contiguous with likelihood ratio statistic $\mathrm{LR}_T$, and under $Y_T \sim F_T(m^*, \theta_0)$, $(\mathrm{LR}_T, X_T^*) \rightsquigarrow (L(X), X)$ with $X \sim P(\theta_0)$ and some function $L : S \mapsto \mathbb{R}$, then under $Y_T \sim F_T(m^*, \theta_1)$, $X_T^* \rightsquigarrow X \sim Q$, and the Radon-Nikodym derivative of $Q$ with respect to $P(\theta_0)$ is equal to $L$.

- In canonical alternative model $Y_T \sim F_T(m^*, \theta_1)$, $X_T^* \rightsquigarrow X \sim P(\theta_1)$, so that $Q = P(\theta_1)$, and $L(X)$ is best test in limiting problem.

# Comparison to Traditional Semiparametric Efficiency via Limit of Experiments (TSE)

- Both approaches yield a semiparametric efficiency bound via consideration of a simpler 'limit experiment'.

- But different focus and advantages

  – TSE typically describes underlying models in terms of moment conditions (and regularity to ensure weak convergence of the likelihood ratio process), whereas this paper starts with weak convergence assumption for particular function of $h(Y_T)$ to a parametric model

  – TSE has many (often i.i.d.) observations of underlying model, whereas in this paper, there is only one approximate parametric model for all data $Y_T$, and an i.i.d. structure is not easily imposed

  – TSE well developed and understood for models with LAN likelihood ratio, whereas results here are straightforward to apply to nonstandard testing problems

# Uniformity

- With $\mathcal{M}_0$ the set of all models where $P_T(m, \theta_0) \rightsquigarrow P(\theta_0)$, no uniform claim is possible.

- For some sequence $\delta_T \to 0$, let $\mathcal{M}_0^u(\delta)$ be the set of models such that

$$\Delta(P_T(m, \theta_0), P(\theta_0)) \leq \delta_T$$

where $\Delta$ is a metric on the space of distributions on $S$ which metrizes weak convergence.

- For suitable choice of $\Delta$ (such as the Prohorov metric), one can show that under the conditions of Theorem 1

$$\limsup_{T\to\infty} \sup_{m\in\mathcal{M}_0^u(\delta)} \int \hat{\varphi}_T^* dF_T(m, \theta_0) \leq \alpha. \tag{2}$$

- Does not settle question of efficiency of $\hat{\varphi}_T^*$ in class of tests that satisfy (2). Paper provides limited result of asymptotic efficiency under stronger smoothness assumptions on the limiting problem.

# Weak IV Regression I

- Structural and reduced form equation

$$\begin{aligned} y_{1,t} &= y_{2,t}\beta + u_{t,1} \\ y_{2,t} &= z_t'\pi + v_{t,2} \qquad\qquad z_t : k \times 1 \end{aligned}$$

- Reduced form

$$y_{1,t} = z_t'\pi\beta + v_{t,1}$$

- AMS consider small sample efficient tests of

$$H_0 : \beta = \beta_0$$

for nonstochastic $z_t$ and $v_t = (v_{1,t}, v_{2,t})' \sim$i.i.d.$\mathcal{N}(0, \Omega)$ with $\Omega$ known. By sufficiency, tests may be restricted to functions of

$$\sum_{t=1}^{T} \begin{pmatrix} z_t y_{1,t} \\ z_t y_{2,t} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} S_z\pi\beta \\ S_z\pi \end{pmatrix}, \Omega \otimes S_z\right), \quad S_z = \sum_{t=1}^{T} z_t z_t'$$

# Weak IV Regression II

- AMS derive weighted average power (WAP) maximizing similar tests that are invariant to the group of transformations

$$\{z_t\}_{t=1}^T \to \{Oz_t\}_{t=1}^T \text{ for any orthogonal matrix } O.$$

- AMS then consider Staiger and Stock (1997) weak instrument asymptotics, where $\pi = T^{-1/2}C$ for some fixed $C$

- AMS derive test that

  1. maximizes WAP among all asymptotically invariant and asymptotically similar tests when $v_t \sim$i.i.d.$\mathcal{N}(0, \Omega)$ independent of $\{z_t\}_{t=1}^T$

  2. yields correct asymptotic null rejection probability under much broader conditions, including heteroskedastic and autocorrelated $v_t$

# Weak IV Regression III

- Typical weak convergence under weak IV asymptotics

$$\hat{D}_z = T^{-1}\sum_{t=1}^{T} z_t z_t' \xrightarrow{p} D_z \qquad \hat{\Sigma} \xrightarrow{p} \Sigma$$

$$X_T = T^{-1/2}\sum_{t=1}^{T} \begin{pmatrix} z_t y_{1,t} \\ z_t y_{2,t} \end{pmatrix} \rightsquigarrow X \sim \mathcal{N}\left( \begin{pmatrix} D_z C\beta \\ D_z C \end{pmatrix}, \Sigma \right)$$

- In just-identified case ($D_z$ a scalar), rely on AMS small sample result for uniformly most powerful similar test in limiting problem (actually Moreira (2001))

  $\Rightarrow$ reject for large values of Anderson-Rubin statistic $(b_0' X_T)^2 / b_0' \hat{\Sigma} b_0$, where $b_0 = (1, -\beta_0)'$

  $\Rightarrow$ by results here, uniformly most asymptotically powerful similar robust test

- In overidentified case, no general known solution for best limiting test (AMS results apply only when $\Sigma = \Omega \otimes D_z$ for some $\Omega$)

# Weak IV Regression IV

- Robustness constraint is large, since weak convergence

$$T^{-1/2}\sum_{t=1}^{T}\begin{pmatrix} z_t y_{1,t} \\ z_t y_{2,t} \end{pmatrix} \rightsquigarrow \mathcal{N}\left(\begin{pmatrix} D_z C\beta \\ D_z C \end{pmatrix}, \mathbf{\Sigma}\right)$$

can hold for many 'weird' data generating processes.

- To rule out at least some 'weird' DGPs, assume in addition

$$\begin{aligned} T^{-1/2}\sum_{t=1}^{\lfloor \cdot T \rfloor}\begin{pmatrix} z_t y_{1,t} \\ z_t y_{2,t} \end{pmatrix} &\rightsquigarrow G(\cdot) \\ G(s) &= s\begin{pmatrix} D_z C\beta \\ D_z C \end{pmatrix} + \mathbf{\Sigma}^{1/2} W(s) \end{aligned}$$

Since $G(1)$ is sufficient for $(C,\beta)$, does not change best test in limiting problem.

# GMM Stability Test I

- GMM framework with parameter $\beta \in \mathbb{R}^k$. Parametrize

$$\beta_{T,t} = \beta_0 + T^{-1/2}\theta(t/T)$$

where $\theta \in D^k_{[0,1]}$ and $\theta(0) = 0$.
Parameter stability test

$$H_0 : \theta = 0 \quad \text{against} \quad H_1 : \theta \neq 0.$$

- Let $g_{T,t}(\beta) \in \mathbb{R}^p$ with $p \geq k$ be the sample moment condition for $y_{T,t}$ evaluated at $\beta$. Under standard conditions

$$\begin{aligned} G_T(\cdot) &= T^{-1/2} \sum_{t=1}^{\lfloor \cdot T \rfloor} g_{T,t}(\hat{\beta}_T) \rightsquigarrow G(\cdot), \qquad \hat{H}_T \xrightarrow{p} H, \quad \hat{V}_T \xrightarrow{p} V \\ G(s) &= V^{1/2}W(s) - sH(H'V^{-1}H)^{-1}H'V^{-1/2}W(1) \\ &\quad + H\left(\int_0^s \theta(l)dl - s\int_0^1 \theta(l)dl\right) \end{aligned}$$

# GMM Stability Test II

- Sowell (1996) derives WAP maximizing test in limiting problem (with $G$ assumed observed), that is $\varphi_S^*$, and calls this test, evaluated at sample analogues (that is, $\hat{\varphi}_T^*$) an "optimal" test for structural change

- Counterexample to unqualified optimality:
  - Let $y_{T,t} = \beta + \theta(t/T) + \varepsilon_t$, where $\varepsilon_t$ is i.i.d. with $P(\varepsilon_t = -1) = P(\varepsilon_t = 1) = 1/2$
  - Let $\varphi_T^{**}$ be the test that rejects whenever any one of $\{y_{T,t} - y_{T,t-1}\}_{t=2}^T$ is not $-2$, 0 or 2
  - Then $\varphi_T^{**}$ has level zero for any $T \geq 2$ and asymptotic power equal to one against any local alternative

Results here provide precise sense in which $\hat{\varphi}_T^*$ is an asymptotically efficient test

# t-statistic Based Correlation Robust Inference

- Problem of testing $H_0 : \beta = \beta_0$ where $\beta$ is first element of $k \times 1$ GMM parameter $\theta$ from data set $Y_T$ with correlations of largely unknown form.

- Partition data in $q$ groups, estimate the model $q$ times using data from each group only, and assume that resulting estimators satisfy

$$\{n^{1/2}(\hat{\beta}_j - \beta_0)\}_{j=1}^q \Rightarrow \{X_j\}_{j=1}^q \tag{3}$$

  where $X_j \sim \mathcal{N}(b, \sigma_j^2)$ independent of $X_i$ for $i \neq j$, and $b = n^{1/2}(\beta - \beta_0)$.

- Ibragimov and Müller (2007) show that usual 5% level t-test based on $\{X_j\}_{j=1}^q$ is uniformly most powerful scale invariant test in limiting problem $H_0^{lp} : b = 0$ (using the result of Bakirov and Székely (2005) that this test if of size 5% over $\{\sigma_j^2\}_{j=1}^q$).

- Results here imply that if there is a corresponding small sample scale invariance group, then the 5% level t-test based on $\{\hat{\beta}_j\}_{j=1}^q$ is the asymptotically most powerful scale invariant robust test relative to (3).

# Conclusion

- Alternative sense of asymptotic efficiency

  1. for a number of tests that are efficient in canonical models

  2. for nonstandard methods that start with weak convergence assumption

- Stringent robustness constraint

  $\Rightarrow$ Most natural in time series context

- Asymptotic efficient robust tests are simply best tests of limiting problem, evaluated at sample analogues