3.4 Kepler's laws

You are familiar with the idea that one can solve some mechanics problems using only conservation of energy and (linear) momentum. Thus, some of what we see as objects move around in the world is a direct consequence of these conservation laws rather than being the result of some detailed "mechanism." It is nice to give an example of how conservation of angular momentum has similarly powerful (and perhaps more famous) consquences.

We recall that roughly 500 years ago, Kepler made one of the great breakthroughs (not just in physics, but in human thought), providing evidence that planet motions as describe by Tycho Brahe are much more simply described in a world model with the sun (rather than the earth) at the center. I don't think we can overstate the importance of realizing that we are not at the center of the universe.

Quantitatively, Kepler noticed several things (Kepler's laws): The orbits of planets around the sun are elliptical, the periods of the orbits are related to their radii, and as the orbit proceeds it sweeps out equal area in equal times. Of these, the equal area law is the one which is related to conservation of angular momentum.

If the orbits were circular it would be trivial that they sweep out area at a constant rate. The equal area law is, in a sense, all that is left of the 'perfection' that people had sought with circular orbits.

If we are at a distance r from the center of our coordinate system (the sun), and we move by an angle $\Delta\theta$, then for small angles the area that is swept out is

$$\Delta A = \frac{1}{2}r^2\Delta\theta = \left(\frac{1}{2}r^2\frac{d\theta}{dt}\right)\Delta t. \tag{3.59}$$

The equal area law is the statement that the term in parentheses,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}, \tag{3.60}$$

is a constant, independent of time.

We know that angular momentum is conserved, so let's see if this has something to do with the equal area law. The vector position of the planet can always be written as $\vec{r} = r\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector pointing outward toward the current location. The velocity consists of components in the $\hat{\mathbf{r}}$ direction and in the $\hat{\theta}$ direction, around the curve,

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\theta}. \tag{3.61}$$

Hence

$$\vec{p} \equiv m\frac{d\vec{r}}{dt} = m\frac{dr}{dt}\hat{\mathbf{r}} + mr\frac{d\theta}{dt}\hat{\theta} \tag{3.62}$$

$$\vec{L} \equiv \vec{r}\times\vec{p} = (r\hat{\mathbf{r}})\times\left(m\frac{dr}{dt}\hat{\mathbf{r}}\right) + (r\hat{\mathbf{r}})\times\left(mr\frac{d\theta}{dt}\hat{\theta}\right). \tag{3.63}$$

To finish the calculation we pull all the scalars out of the cross porducts,

$$\vec{L} = \left(rm\frac{dr}{dt}\right)(\hat{\mathbf{r}}\times\hat{\mathbf{r}}) + \left(mr^2\frac{d\theta}{dt}\right)(\hat{\mathbf{r}}\times\hat{\theta}), \tag{3.64}$$

and then we note that

$$\hat{\mathbf{r}}\times\hat{\mathbf{r}} = 0, \tag{3.65}$$

$$\hat{\mathbf{r}}\times\hat{\theta} = \hat{\mathbf{z}}. \tag{3.66}$$

Thus we find that the angular momentum is given by

$$\vec{L} = \left(mr^2\frac{d\theta}{dt}\right)\hat{\mathbf{z}}. \tag{3.67}$$

Comparing Eq. (3.60) with Eq. (3.67), we see that

$$\frac{dA}{dt} = \frac{1}{2m}(\vec{L}\cdot\hat{\mathbf{z}}), \tag{3.68}$$

so that conservation of angular momentum ($\vec{L}$ = constant) implies that dA/dt is a constant—the equal area law.

To go further in deriving Kepler's laws we need to know about the actual forces between the sun and the planets. You probably know that one of Newton's great triumphs was to realize that if gravity obeys the "inverse square law," then the rate at which the moon is falling toward the earth as it orbits is consistent with the rate at which objects we can hold in our hands fall toward the ground. In modern language we say that the potential energy for two masses M and m separated by a distance r is given by

$$V(r) = -\frac{GMm}{r}, \tag{3.69}$$

where G is (appropriately enough) known as Newton's constant. We are interested in the case where m is the mass of a planet and M is the mass of the sun. We choose a coordinate system in which the sun is fixed at the origin.

To understand what happens it is useful to write down the total energy of the system. We have the potential energy explicitly, so we need the kinetic energy. We know that the velocity has two components, one in the radial direction and one in the angular direction,

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\theta}, \tag{3.70}$$

so that

$$\frac{1}{2}mv^2 \equiv \frac{1}{2}m\left|\frac{d\vec{r}}{dt}\right|^2 = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right]. \tag{3.71}$$

So the total energy of the system, kinetic plus potential, is given by

$$E = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{GMm}{r}. \tag{3.72}$$

But we know that angular momentum is conserved, so we can say something about the term that has $d\theta/dt$ in it:

$$\begin{aligned} L_z &= mr^2\frac{d\theta}{dt} && (3.73)\\ \frac{d\theta}{dt} &= \frac{L_z}{mr^2}. && (3.74) \end{aligned}$$

Substituting into our expression for the total energy this becomes

$$\begin{aligned} E &= \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{L_z}{mr^2}\right)^2\right] - \frac{GMm}{r} && (3.75)\\ &= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L_z^2}{2mr^2} - \frac{GMm}{r} && (3.76)\\ &= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V_{\text{eff}}(r), && (3.77) \end{aligned}$$

where in the last step we have introduced an "effective potential"

$$V_{\text{eff}}(r) = \frac{L_z^2}{2mr^2} - \frac{GMm}{r}. \tag{3.78}$$

Notice that by doing this our expression for the total energy comes to look like the energy for motion in *one* dimension (r), with a potential energy that has one part from gravity and one part from the indirect effect of the angular momentum.

Notice that the contribution from angular momentum is positive, and varies as $\sim 1/r^2$. This means that the corresponding force $F = -\partial V/\partial r \sim$

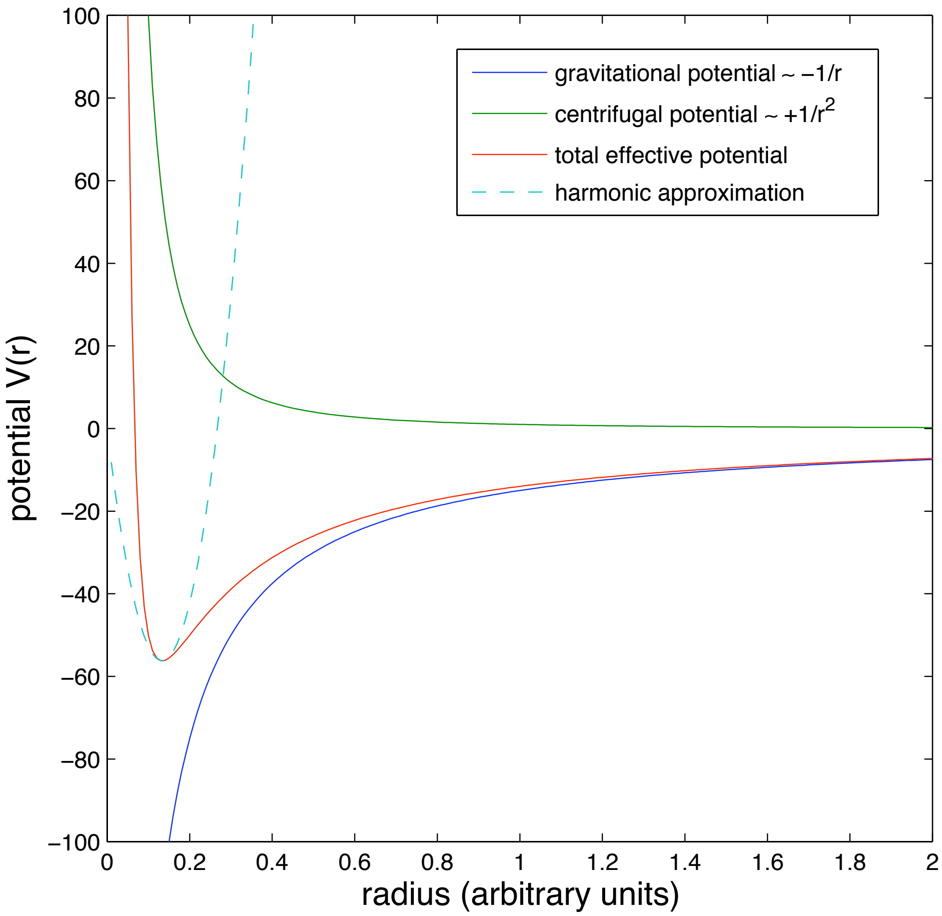

Figure 3.1: Effective potential energy for planetary motion, from Eq (3.78).

$1/r$ is positive—it pushes outward along the radius. This force is what we experience when we sit in a car going around a curve: the "centrifugal" force. Imagine that we tie a weight on the end of a string and swing it in a circle over our heads. The string will stay taught, and this must be because there is a force pulling outward; again this is the centrifugal force, and is generated by this special term in the effective potential. Notice that we have eliminated any mention of the angle θ, and in the process have changed the potential energy for motion along the radial direction r. This is a much more general idea.

We often eliminate coordinates in the hope of simplifying things, and try to take account of their effects through an effective potential for the coordinates that we do keep in our description. This is very important in big molecules, for example, where we don't want to keep track of every atom but hope that we can just think about a few things such as the distance between key residues or the angle between two big "arms" of the molecule. It's not at all obvious that this should work, even as an approximation, although in the present case it's actually exact.

Recall that the total energy is the sum of kinetic and potential, and this total is conserved or constant over time. There is a minimum effective potential energy for radial motion, as can be seen in Fig 3.1, If the total energy is equal to this minimum, then there can be no kinetic energy associated with the coordinate r, hence $dr/dt = 0$. Thus for minimum energy orbits, the radius is constant—the planet moves in a circular orbit.

If we look at orbits that have energies just a bit larger than the minimum, we can approximate $V_{\text{eff}}(r)$ as being like a harmonic oscillator. Then the radius should oscillate in time, but time is being marked by going around the orbit, so really the radius will be a sine or cosine function of the angle, and this is the description of an ellipse if it is not too eccentric. In fact if you work harder you can show that the orbits are exactly ellipses for any value of the energy up to some maximum. This is another of Kepler's laws. Once you have the ellipse you can relate its size (the analog of radius for a circle) to the period of the orbit, and this is the last of Kepler's laws.

Notice that if the energy is positive then it is possible for the planet to escape toward $r \to \infty$ at finite velocity, and then the orbit is not bound. But if the total energy is negative, there is no escape, and the radius r moves between two limiting values, namely the points where the total energy intersects the effective potential. We really should say more about all this, but it is treated in many standard texts.